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Characterization of Stokes graphs and Voros coefficients of hypergeometric differential equations with a large parameter

By

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Abstract

Characterization of types of Stokes curves is given in terms of parameters for hypergeometric differential equations. Voros coefficients of these equations are defined and an explicit form of one of them is given.

§ 1. Introduction

The purpose of this paper is to characterize the types of Stokes geometry and to give an explicit form of Voros coefficients for hypergeometric differential equations with a large parameter. The notion of Voros coefficients, or Voros symbols, had been introduced by [14] and effectively used to describe Stokes phenomena with respect to parameters for WKB solutions of Schrödinger equations with polynomial potentials ([4], [5]). Here the parameters mean those contained in the potentials or the energy parameters. Recently explicit forms of Voros coefficients have been obtained for Weber equations ([11], [12]) and for Whittaker equations ([9]). Note that those literatures studied differential equations with irregular singularities. We define and study Voros coefficients for hypergeometric differential equations and we give an explicit form of them for a special case. Voros coefficients play a role in analyzing Stokes phenomena of WKB solutions with respect to parameters in differential equations. As a matter of

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fact, Voros coefficients undertake such Stokes phenomena through Borel resummation method. To analyze Stoke phenomena of Voros coefficients, we must know how the types of Stokes graphs depend on the parameters. Hence we give a characterization of types Stokes graphs in terms of the parameters. This characterization had been originally observed by numerical experiments in [13]. We give a proof of this observation in this paper. Next we define Voros coefficients for hypergeometric equations and give an explicit form of them for a special configuration of Stokes curves.

§ 2. Hypergeometric differential equations with a large parameter

We consider the hypergeometric differential equation:

$$x(1-x)\frac{d^2w}{dx^2}+(c-(a+b+1)x)\frac{dw}{dx}-abw=0, \tag{2.1}$$

where a, b and c are complex parameters. We introduce a large parameter η by setting $a=1/2+\eta\alpha$, $b=1/2+\eta\beta$, $c=1+\eta\gamma$ with complex parameters α,β and γ. We regard η being real positive and large. Next we eliminate the first-order term by taking

$$\psi=x^{\frac{1}{2}+\frac{\eta\gamma}{2}}(1-x)^{\frac{1}{2}+\frac{\eta(\alpha+\beta-\gamma)}{2}}w \tag{2.2}$$

as unknown function. Then we have

$$\Big(-\frac{d^2}{dx^2}+\eta^2Q\Big)\psi=0 \tag{2.3}$$

with

$$Q=Q_0+\eta^{-2}Q_1, \tag{2.4}$$

$$Q_0=\frac{(\alpha-\beta)^2x^2+2(2\alpha\beta-\alpha\gamma-\beta\gamma)x+\gamma^2}{4x^2(x-1)^2}, \tag{2.5}$$

$$Q_1=-\frac{x^2-x+1}{4x^2(x-1)^2}. \tag{2.6}$$

Equation (2.3) has formal solutions of the form

$$\psi_\pm=\frac{1}{\sqrt{S_{\mathrm{odd}}}}\exp(\pm\int_{a_h}^{x}S_{\mathrm{odd}}\,dx), \tag{2.7}$$

which are called WKB solutions. Here a_h $(h=0,1)$ is one of the turning points of (2.3), that is, zeros of Q_0 and S_{odd} denotes the odd-order part of the formal solution $S=\sum_{j=-1}^{\infty}\eta^{-j}S_j$ of the Riccati equation

$$\frac{dS}{dx}+S^2=\eta^2Q \tag{2.8}$$

associated with (2.3). We define involutions ι_j $(j = 0, 1, 2)$ in the space of parameters as follows:

$$
\begin{aligned}
\iota_0 :\quad & (\alpha, \beta, \gamma) & \mapsto (\beta, \alpha, \gamma) \\
\iota_1 :\quad & (\alpha, \beta, \gamma) & \mapsto (\gamma - \alpha, \gamma - \beta, \gamma) \\
\iota_2 :\quad & (\alpha, \beta, \gamma) & \mapsto (-\alpha, -\beta, -\gamma)
\end{aligned}
\tag{2.9}
$$

The potential Q is invariant under those involutions. A Stokes curve emanating from the turning point a_h $(h = 0, 1)$ is a curve defined by

$$
\mathrm{Im} \int_{a_h}^{x} \sqrt{Q_0}\, dx = 0 \tag{2.10}
$$

with

$$
\sqrt{Q_0} = \frac{(\alpha - \beta)\sqrt{(x - a_0)(x - a_1)}}{2x(x - 1)}. \tag{2.11}
$$

A Stokes curve flows into a singular point or a turning point. We say that the Stokes geometry of (2.3) is non-degenerate if any Stokes curve does not flow into a turning point. If the Stokes geometry is non-degenerate, we can associate a graph with it. The Stokes graph ([1]) of (2.3) is, by definition, a two-colored sphere graph consisting of all Stokes curves (emanating from a_0 and a_1) as edges, $\{a_0, a_1\}$ as vertices of the first color and $b_0 = 0, b_1 = 1, b_2 = \infty$ as vertices of the second color. The Stokes graph of (2.1) is, by definition, that of (2.3).

§ 3. Characterization of Stokes graphs of hypergeometric differential equations

In this section we give a topological classification of Stokes graphs of (2.3) in term of the parameters (α, β, γ). We define that the sets E_j $(j = 0, 1, 2)$ of the parameters α, β, γ as follows:

$$
E_0 = \{(\alpha, \beta, \gamma) \in \mathbb{C}^3 \mid \alpha \cdot \beta \cdot \gamma \cdot (\alpha - \beta) \cdot (\alpha - \gamma) \cdot (\beta - \gamma) \cdot (\alpha + \beta - \gamma) = 0\}, \tag{3.1}
$$

$$
E_1 = \{(\alpha, \beta, \gamma) \in \mathbb{C}^3 \mid \mathrm{Re}\alpha \cdot \mathrm{Re}\beta \cdot \mathrm{Re}(\gamma - \alpha) \cdot \mathrm{Re}(\gamma - \beta) = 0\}, \tag{3.2}
$$

$$
E_2 = \{(\alpha, \beta, \gamma) \in \mathbb{C}^3 \mid \mathrm{Re}(\alpha - \beta) \cdot \mathrm{Re}(\alpha + \beta - \gamma) \cdot \mathrm{Re}\gamma = 0\}. \tag{3.3}
$$

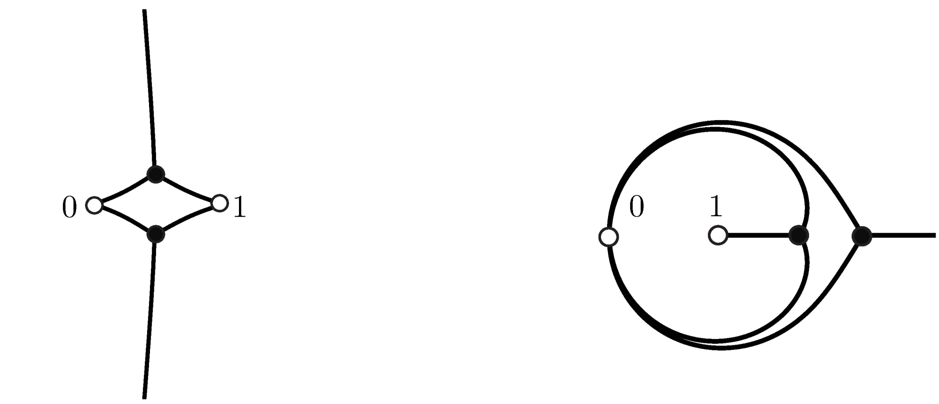
If (α, β, γ) is not contained in E_0, the turning points and the singular points of (2.1) are mutually distinct. Moreover, if (α, β, γ) is not contained in $E_1 \cup E_2$, then the Stokes geometry is non degenerate as the second author has proved the following theorem.

Theorem 3.1. ([13], Theorem 3.1) *We assume that (α, β, γ) is not contained in E_0.*

(i) *If two distinct turning points are connected by a Stokes curve, then* (α, β, γ) *belongs to* E_1*.*

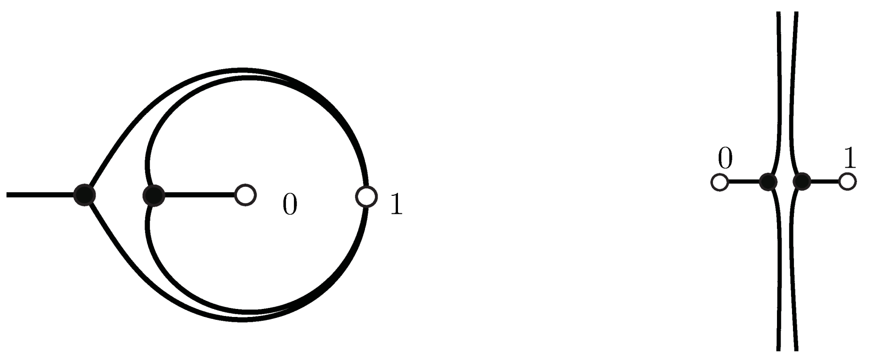
(ii) *If a Stokes curve forms a closed curve with a single turning point as the base point, then* (α, β, γ) *belongs to* E_2*.*

We assume that (α, β, γ) is not contained in the sets $E_0 \cup E_1 \cup E_2$. We note that the topological type of a Stokes graph is characterized by its order sequence $\hat{n} = (n_0, n_1, n_2)$, where n_j is the number of Stokes curves which flow into the singularity b_j $(j = 0, 1, 2)$. By the definition, we have $n_0 + n_1 + n_2 = 6$. Some examples are given in Fig. 3.1 and Fig. 3.2.



$(\alpha, \beta, \gamma) = (0.1, 2, 1)$, $\hat{n} = (2, 2, 2)$ $(\alpha, \beta, \gamma) = (0.5, 0.995, 1)$, $\hat{n} = (4, 1, 1)$

Fig. 3.1



$(\alpha, \beta, \gamma) = (1.01, 2, 1)$, $\hat{n} = (1, 1, 4)$ $(\alpha, \beta, \gamma) = (-0.03, 2, 1)$, $\hat{n} = (1, 4, 1)$

Fig. 3.2

Here bullets and white bullets designate turning points and singular points, respectively.

Next we define the sets ω_k $(k = 1, 2, 3, 4)$ of the parameters α, β, γ as follows:

$$\omega_1 = \{(\alpha, \beta, \gamma) \in \mathbb{C}^3 \mid 0 < \mathrm{Re}\alpha < \mathrm{Re}\gamma < \mathrm{Re}\beta\}, \tag{3.4}$$

$$\omega_2 = \{(\alpha, \beta, \gamma) \in \mathbb{C}^3 \mid 0 < \mathrm{Re}\alpha < \mathrm{Re}\beta < \mathrm{Re}\gamma < \mathrm{Re}\alpha + \mathrm{Re}\beta\}, \tag{3.5}$$

$$\omega_3 = \{(\alpha, \beta, \gamma) \in \mathbb{C}^3 \mid 0 < \mathrm{Re}\gamma < \mathrm{Re}\alpha < \mathrm{Re}\beta\}, \tag{3.6}$$

$$\omega_4 = \{(\alpha, \beta, \gamma) \in \mathbb{C}^3 \mid \mathrm{Re}\gamma - \mathrm{Re}\beta < \mathrm{Re}\alpha < 0\} \tag{3.7}$$

and Π_k by

$$\Pi_k = \bigcup_{r \in G} r(\omega_k) \qquad (k = 1, 2, 3, 4), \tag{3.8}$$

where G is the group generated by ι_j $(j = 0, 1, 2)$. We characterize the types of Stokes graphs in terms of the parameters. The following theorem had been claimed as a conjecture based on numerical experiments in [13] and announced in [2].

Theorem 3.2. *Let $\hat{n}$ denote the order sequence of the Stokes graph with parameters (α, β, γ).*
(1) *If $(\alpha, \beta, \gamma) \in \Pi_1$, then $\hat{n} = (2, 2, 2)$.*
(2) *If $(\alpha, \beta, \gamma) \in \Pi_2$, then $\hat{n} = (4, 1, 1)$.*
(3) *If $(\alpha, \beta, \gamma) \in \Pi_3$, then $\hat{n} = (1, 4, 1)$.*
(4) *If $(\alpha, \beta, \gamma) \in \Pi_4$, then $\hat{n} = (1, 1, 4)$.*

Remark. For a fixed Re $\gamma > 0$, configurations of ω_k's and Π_k's in the real α-β plane are shown in Fig. 3.3.

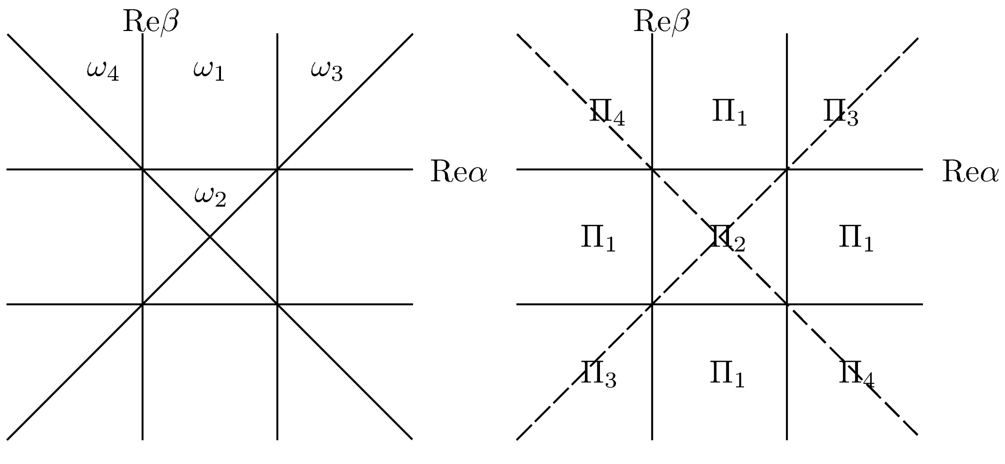


Fig. 3.3

Proof. First we note that $\hat{n}$ is constant on each Π_k since any Stokes curve depends on (α, β, γ) continuously there. This can be seen as follows. The turning points are written explicitly in the forms

$$a_0, a_1 = \frac{\beta\gamma + \gamma\alpha - 2\alpha\beta \pm 2\sqrt{\alpha\beta(\gamma - \alpha)(\gamma - \beta)}}{(\alpha - \beta)^2}. \tag{3.9}$$

The denominator and the real part of inside of the square root of (3.9) never vanishes in Π_k. Hence they are continuous functions of (α, β, γ) on Π_k. We take a segment connecting a_0 and a_1 as a branch cut and fix a branch of $\sqrt{Q_0}$. We may write

$$\sqrt{Q_0} = \sqrt{x - a_0}\, R_0(x), \tag{3.10}$$

where we set

$$R_0(x) = \frac{(\alpha - \beta)\sqrt{x - a_1}}{2x(x-1)}. \tag{3.11}$$

There are three Stokes curves emanating from a_0 and the initial angles $\theta_0, \theta_1, \theta_2$ of these curves are given by

$$\theta_j = \frac{2}{3}(j\pi - \phi_0) \quad (j = 0, 1, 2). \tag{3.12}$$

Here we set $\phi_0 = \arg R_0(a_0) = \tan^{-1} \operatorname{Im} R_0(a_0)/\operatorname{Re} R_0(a_0)$. Since $\operatorname{Re} R_0(a_0)$ never vanishes on Π_k, ϕ_0 is a continuous function. Therefore θ_j are continuous functions of (α, β, γ) in Π_k. Each Stokes curve emanating from a_0 flows into one of regular singularities. It follows from Lemmas 4.7 and 4.9 in [10] that the destination of each Stokes curves do not change under any small perturbation of the parameters. This is also true for Stokes curves emanating from a_1. Hence $\hat{n}$ does not change under small perturbations. Since Π_k is an open set, $\hat{n}$ is constant on the set.

Next we look at a point on the boundary of Π_k, say, $(\alpha, \beta, \gamma) = (0, 2, 1)$, which is located on the boundary between Π_1 and Π_4. For these values of parameters, we have

$$Q_0(x) = \frac{(x - \frac{1}{2})^2}{(x(x-1))^2}. \tag{3.13}$$

Then $a := a_0 = a_2 = 1/2$ is a double turning point. Stokes curves are defined by the equation

$$\operatorname{Im} \int_{\frac{1}{2}}^{x} \frac{x - \frac{1}{2}}{x(x-1)} dx = \frac{1}{2} \operatorname{Im} \log(4x(1-x)) = 0. \tag{3.14}$$

Setting $x = u + iv$ $(u, v \in \mathbb{R})$, we have

$$x(1-x) = v^2 - u^2 - u + iv(1 - 2u). \tag{3.15}$$

Hence (3.14) yields the semialgebraic set

$$\mathcal{S} = \{u + iv \,|\, 0 < u < 1,\, v = 0\} \cup \left\{ u + iv \,\middle|\, u = \frac{1}{2} \right\}, \tag{3.16}$$

which consider of one straight line and one segment (cf. Fig. 3.4).

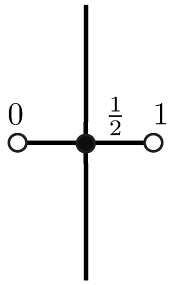


Fig. 3.4

More precisely, we have four Stokes curves:

$$s_0 = \left\{ u + iv \;\middle|\; \frac{1}{2} \le u < 1,\; v = 0 \right\}, \tag{3.17}$$

$$s_1 = \left\{ u + iv \;\middle|\; u = \frac{1}{2}, v \ge 0 \right\}, \tag{3.18}$$

$$s_2 = \left\{ u + iv \;\middle|\; 0 < u \le \frac{1}{2},\; v = 0 \right\}, \tag{3.19}$$

$$s_3 = \left\{ u + iv \;\middle|\; u = \frac{1}{2}, v \le 0 \right\}. \tag{3.20}$$

Note that we have taken

$$\sqrt{Q_0} = \frac{x - \frac{1}{2}}{x(x-1)}. \tag{3.21}$$

Let f and g denote the real part and the imaginary part of (3.21), respectively. It is easy to see that s_0 and s_2 are the integral curves of the system of differential equations

$$\frac{du}{dt} = -\frac{f}{\sqrt{f^2 + g^2}}, \tag{3.22}$$

$$\frac{dv}{dt} = \frac{g}{\sqrt{f^2 + g^2}} \tag{3.23}$$

with the initial condition $(u(0), v(0)) = (1/2, 0)$ and the boundary conditions

$$\lim_{t \to +0} \left(\frac{du}{dt}, \frac{dv}{dt} \right) = (1, 0) \tag{3.24}$$

and

$$\lim_{t \to +0} \left(\frac{du}{dt}, \frac{dv}{dt} \right) = (-1, 0), \tag{3.25}$$

respectively. On the other hand, s_1 and s_3 are integral curves of

$$\frac{du}{dt} = \frac{f}{\sqrt{f^2 + g^2}}, \tag{3.26}$$

$$\frac{dv}{dt} = -\frac{g}{\sqrt{f^2 + g^2}} \tag{3.27}$$

with the same initial condition and the boundary conditions

$$\lim_{t\to+0}\left(\frac{du}{dt},\frac{dv}{dt}\right)=(0,1) \tag{3.28}$$

and

$$\lim_{t\to+0}\left(\frac{du}{dt},\frac{dv}{dt}\right)=(0,-1), \tag{3.29}$$

respectively.

Now we consider the Stokes curve s_2 and analyze the behavior of the curve under small perturbations of α. Let ε be a small positive number and we set $(\alpha,\beta,\gamma)=(\varepsilon,2,1)$. Then the potential turns out to be

$$Q_{0,\varepsilon}(x)=\frac{(\varepsilon-2)^2x^2+2(3\varepsilon-2)x+1}{4(x(x-1))^2}. \tag{3.30}$$

There are two distinct turning simple points $a_{0,\varepsilon}$ and $a_{1,\varepsilon}$. We take $a_{0,\varepsilon}$ so that the imaginary part is positive:

$$a_{0,\varepsilon}=\frac{2-3\varepsilon+2\sqrt{2}\sqrt{\varepsilon(1-\varepsilon)}i}{(2-\varepsilon)^2} \tag{3.31}$$

$$=\frac{1}{2}+\frac{i}{\sqrt{2}}\sqrt{\varepsilon}+O(\varepsilon) \tag{3.32}$$

It is clear that $a_{1,\varepsilon}$ has a negative imaginary part and

$$a_{1,\varepsilon}=\frac{1}{2}-\frac{i}{\sqrt{2}}\sqrt{\varepsilon}+O(\varepsilon). \tag{3.33}$$

We see that $(\varepsilon,2,1)\in\Pi_1$ and hence the Stokes geometry is non-degenerate. To fix a branch of $\sqrt{Q_{0,\varepsilon}(x)}$, we take two straight segments respectively connecting $a_{0,\varepsilon}$ and $a=1/2$, $a_{1,\varepsilon}$ and a. The union of these segments will be a branch cut, which will de denoted by κ. We take the branch so that $\sqrt{Q_{0,\varepsilon}(x)}\sim(1-\varepsilon/2)/x$ for sufficiently large positive x. Then $\sqrt{Q_{0,\varepsilon}(x)}$ converges locally uniformly to $\sqrt{Q_0(x)}$ as $\varepsilon\to 0$ outside the branch cut and the singularities. We denote the real part and the imaginary part of $\sqrt{Q_{0,\varepsilon}(x)}$ by f_ε and g_ε, respectively.

There are three Stokes curves emanating from $a_{0,\varepsilon}$ (resp. $a_{1,\varepsilon}$). The initial directions of the curves are computed as follows. Let $R_{0,\varepsilon}(x)$ denote the function

$$R_{0,\varepsilon}(x)=\frac{(2-\varepsilon)\sqrt{x-a_{1,\varepsilon}}}{2x(x-1)}. \tag{3.34}$$

Then we have

$$\sqrt{Q_{0,\varepsilon}(x)}=\sqrt{x-a_{0,\varepsilon}}\,R_{0,\varepsilon}(x). \tag{3.35}$$

Here the branches of $\sqrt{x-a_{0,\varepsilon}}$ and $\sqrt{x-a_{1,\varepsilon}}$ are taken for which they are asymptotically the same as $\sqrt{x}$ for positive large x. Let $\phi_{\varepsilon,0}$ be the argument of $R_{0,\varepsilon}(a_{0,\varepsilon})$. Since

$$R_{0,\varepsilon}(a_{0,\varepsilon}) = -2^{\frac{7}{4}}(1+i)\varepsilon^{\frac{1}{4}} + O(\varepsilon^{\frac{5}{4}}), \tag{3.36}$$

we observe that $\phi_{\varepsilon,0} = -\frac{3}{4}\pi + O(\varepsilon)$. An initial angle θ of Stokes curves satisfies

$$\sin\left(\frac{3}{2}\theta + \phi_{\varepsilon,0}\right) = 0. \tag{3.37}$$

Hence we have three initial angles $\theta = \theta_j$ $(j = 0, 1, 2)$ satisfying

$$\theta_0 = -\frac{1}{6}\pi + O(\varepsilon), \tag{3.38}$$

$$\theta_1 = \frac{1}{2}\pi + O(\varepsilon), \tag{3.39}$$

$$\theta_2 = -\frac{5}{6}\pi + O(\varepsilon). \tag{3.40}$$

These three initial angles yield three Stokes curves $s_{0,\varepsilon}$, $s_{1,\varepsilon}$ and $s_{2,\varepsilon}$: $s_{0,\varepsilon}$ and $s_{2,\varepsilon}$ are solution curves of

$$\frac{du}{dt} = -\frac{f_\varepsilon}{\sqrt{f_\varepsilon^2 + g_\varepsilon^2}}, \tag{3.41}$$

$$\frac{dv}{dt} = \frac{g_\varepsilon}{\sqrt{f_\varepsilon^2 + g_\varepsilon^2}} \tag{3.42}$$

with the initial condition $(u(0), v(0)) = (\operatorname{Re} a_{0,\varepsilon}, \operatorname{Im} a_{0,\varepsilon})$ and the boundary conditions

$$\lim_{t\to+0}\left(\frac{du}{dt}, \frac{dv}{dt}\right) = (\cos\theta_0, \sin\theta_0) \tag{3.43}$$

and

$$\lim_{t\to+0}\left(\frac{du}{dt}, \frac{dv}{dt}\right) = (\cos\theta_2, \sin\theta_2), \tag{3.44}$$

respectively. On the other hand, $s_{1,\varepsilon}$ is the solution curve of

$$\frac{du}{dt} = \frac{f_\varepsilon}{\sqrt{f_\varepsilon^2 + g_\varepsilon^2}}, \tag{3.45}$$

$$\frac{dv}{dt} = -\frac{g_\varepsilon}{\sqrt{f_\varepsilon^2 + g_\varepsilon^2}} \tag{3.46}$$

with the same initial condition as above and the boundary condition

$$\lim_{t\to+0}\left(\frac{du}{dt}, \frac{dv}{dt}\right) = (\cos\theta_1, \sin\theta_1). \tag{3.47}$$

Intuitively, it is clear that $s_{2,\varepsilon}$ tends to s_2 as $\varepsilon \to +0$. To see this, we investigate the behavior of $s_{2,\varepsilon}$ near $a_{0,\varepsilon}$. Recall that the Stokes curves emanating from $a_{0,\varepsilon}$ are defined by

$$\operatorname{Im} \int_{a_{0,\varepsilon}}^{x} \sqrt{Q_{0,\varepsilon}(x)}dx = 0. \tag{3.48}$$

We may write $R_{0,\varepsilon}(s)$ in the form

$$R_{0,\varepsilon}(x) = R_{0,\varepsilon}(a_{0,\varepsilon}) + (x - a_{0,\varepsilon})\tilde{R}(x). \tag{3.49}$$

Here we set

$$\begin{aligned}\tilde{R}(x) = \frac{2-\varepsilon}{2x(x-1)} \Bigg\{ &\frac{1}{\sqrt{x - a_{1,\varepsilon}} + \sqrt{a_{0,\varepsilon} - a_{1,\varepsilon}}} \\ &- \frac{\sqrt{a_{0,\varepsilon} - a_{1,\varepsilon}}}{a_{0,\varepsilon}(a_{0,\varepsilon} - 1)}(x - 1 + a_{0,\varepsilon}) \Bigg\}.\end{aligned} \tag{3.50}$$

Using (3.49), we can rewrite the left-hand side of (3.48) to

$$\operatorname{Im} \left(\frac{2}{3} R_{0,\varepsilon}(a_{0,\varepsilon})(x - a_{0,\varepsilon})^{\frac{3}{2}} + (x - a_{0,\varepsilon})^{\frac{5}{2}} \int_0^1 t^{\frac{3}{2}} \tilde{R}(a_{0,\varepsilon} + t(x - a_{0,\varepsilon}))dt \right). \tag{3.51}$$

We use the polar coordinates: $x = a_{0,\varepsilon} + re^{i\theta}$. Then we have a defining equation of the Stokes curves emanating from $a_{0,\varepsilon}$ of the form

$$\sin \left(\frac{3}{2}\theta + \phi_{\varepsilon,0} \right) + r\Phi(r,\theta) = 0. \tag{3.52}$$

Here we set

$$\Phi(r,\theta) = \frac{3}{2|R_{0,\varepsilon}(a_{0,\varepsilon})|} \operatorname{Im} \left(e^{i\frac{5}{2}\theta} \int_0^1 t^{\frac{3}{2}} \tilde{R}(a_{0,\varepsilon} + rte^{i\theta})dt \right). \tag{3.53}$$

Let $B(z,\delta)$ denote the open disk of radius $\delta > 0$ with center at z. We set

$$\tilde{B}_\varepsilon = \bigcup_{z \in \kappa} B\left(z, \sqrt{\frac{\varepsilon}{2}}\right) \tag{3.54}$$

and

$$D_\varepsilon = \left\{ x \,\middle|\, 0 \le \operatorname{Re} x \le \frac{1}{2},\ 0 \le \operatorname{Im} x \le \sqrt{\frac{\varepsilon}{2}} \right\} - B(0,\delta_0) \cup \tilde{B}_\varepsilon, \tag{3.55}$$

where κ is the branch cut we have taken and δ_0 a small positive number. By the definition of $\tilde{R}$ and (3.32), (3.33), we see that

$$\sup_{x \in D_\varepsilon} |\tilde{R}(x)| = O(\varepsilon^{-\frac{1}{4}}) \tag{3.56}$$

and hence, by using (3.36), we have

$$\sup_{x\in D_\varepsilon}|\Phi(r,\theta)| = O(\varepsilon^{-\frac{1}{2}}). \tag{3.57}$$

It follows from the implicit function theorem that equation (3.52) for θ has a solution $\theta = \vartheta_j(r)$ near $x = a_{0,\varepsilon}$. Clearly $\theta = \vartheta_j(r)$ and its continuation represent $s_{j,\varepsilon}$. Combining (3.52) and (3.57) yields

$$\theta_j - \vartheta_j(r) = O\left(\frac{r}{\varepsilon^{\frac{1}{2}}}\right). \tag{3.58}$$

We consider three points z_0, z_1 and z_2 defined by

$$z_0 = a_{0,\varepsilon} + r_0 e^{i\vartheta_2(r_0)}, \tag{3.59}$$

$$z_1 = a_{0,\varepsilon} + r_0 e^{i\theta_2}, \tag{3.60}$$

$$z_2 = \frac{1}{2} - r_0, \tag{3.61}$$

where r_0 is a positive small number. We can find r_0 of order $\sqrt{\varepsilon}$ so that $z_j \in D_\varepsilon$ $(j = 0, 1, 2)$. Note that $z_0 \in s_{2,\varepsilon}$ and $z_2 \in s_2$. Then we have

$$|z_0 - z_2| \le |z_0 - z_1| + |z_1 - z_2| = O(\sqrt{\varepsilon}). \tag{3.62}$$

On the other hand, $\sqrt{Q_0} - \sqrt{Q_{0,\varepsilon}}$ can be rewritten in the form

$$\frac{\varepsilon\left\{\left(1-\frac{\varepsilon}{4}\right)x - \frac{3}{2}\right\}}{(x-1)\left(x - \frac{1}{2} + \left(1-\frac{\varepsilon}{2}\right)\sqrt{(x-a_{0,\varepsilon})(x-a_{1,\varepsilon})}\right)} \tag{3.63}$$

and hence we have

$$\sup_{x\in D_\varepsilon}\left|\sqrt{Q_0(x)} - \sqrt{Q_{0,\varepsilon}(x)}\right| = O(\sqrt{\varepsilon}). \tag{3.64}$$

Hence the integral curve of the Cauchy problem (3.41), (3.42) with the initial condition $(u(0), v(0)) = (\operatorname{Re} z_0, \operatorname{Im} z_0)$ and that of (3.22), (3.23) with the initial condition $(u(0), v(0)) = (\operatorname{Re} z_2, \operatorname{Im} z_2)$ are very close. Hence $s_{2,\varepsilon}$ flows into the origin.

We can discuss the Stokes curves emanating from $a_{1,\varepsilon}$ in a similar way and we find that s_2 splits into two Stokes curves which flow into the origin under a small positive perturbation in α. Next we consider s_0 and the same arguments as above imply s_0 splits into two Stokes curves which flow into 1, while s_1 and s_3 are stable under the perturbation. Thus we conclude that $\hat{n} = (2, 2, 2)$ if $(\alpha, \beta, \gamma) \in \Pi_1$.

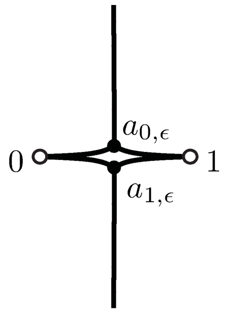


Fig. 3.5

Similarly, if we consider a negative perturbation in α, i.e., take $(-\varepsilon, 2, 1)$ as a perturbation of $(\alpha, \beta, \gamma) = (0, 2, 1)$ with $\varepsilon > 0$, we have $\hat{n} = (1, 1, 4)$ (cf. Fig. 3.6).

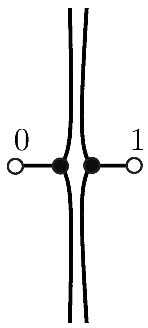


Fig. 3.6

The other cases can be discussed in a similar way if we take $(\alpha, \beta, \gamma) = (1/2, 1, 1)$ and $(\alpha, \beta, \gamma) = (1, 2, 1)$ for which the Stokes curves are

$$\{x \mid 1 < \operatorname{Re} x,\ \operatorname{Im} x = 0\} \cup \{x \mid (\operatorname{Re} x - 1)^2 + (\operatorname{Im} x)^2 = 1, 0 < \operatorname{Re} x\} \tag{3.65}$$

and

$$\{x \mid \operatorname{Re} x < 0,\ \operatorname{Im} x = 0\} \cup \{x \mid (\operatorname{Re} x)^2 + (\operatorname{Im} x)^2 = 1\}, \tag{3.66}$$

respectively (cf. Fig. 3.7 and Fig. 3.8).

$(\alpha, \beta, \gamma) = (\frac{1}{2}, 1 - \varepsilon, 1)$ $(\frac{1}{2}, 1, 1)$ $(\frac{1}{2}, 1 + \varepsilon, 1)$

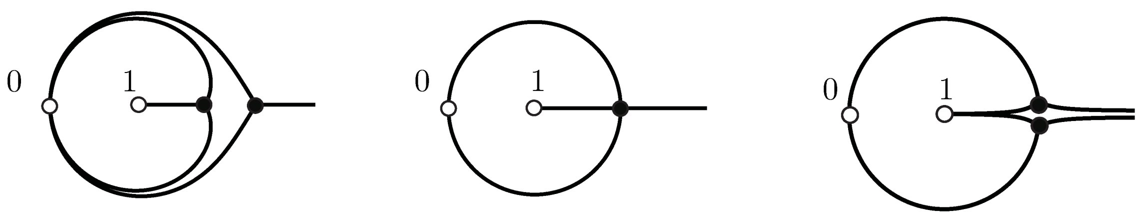


Fig. 3.7

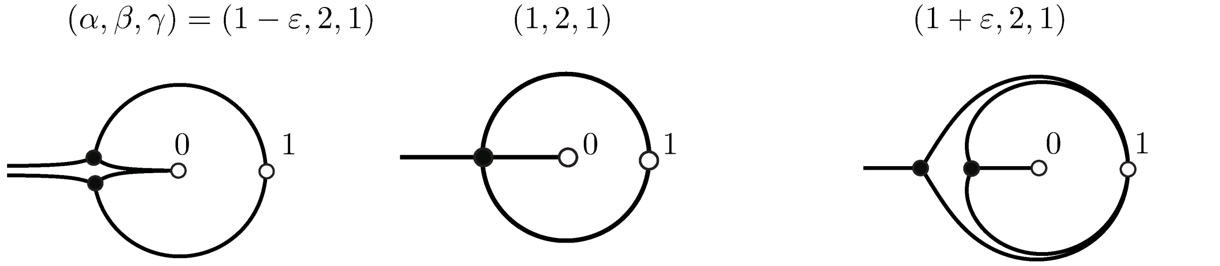


Fig. 3.8

That is, $(\alpha,\beta,\gamma)=(\frac{1}{2},2-\varepsilon,1)$ (resp. $(\frac{1}{2},2+\varepsilon,1)$) with a small $\varepsilon>0$ yields $\hat{n}=(4,1,1)$ (resp. $\hat{n}=(2,2,2)$) in left-hand side (resp. right-hand side) of Fig. 3.7 and $(\alpha,\beta,\gamma)=(1-\varepsilon,2,1)$ (resp. $(1+\varepsilon,2,1)$) yields $\hat{n}=(2,2,2)$ (resp. $\hat{n}=(1,4,1)$) in left-hand side (resp. right-hand side) of Fig. 3.8. This completes the proof of Theorem 3.1.

□

§ 4. Voros coefficients

We consider an integral of the form

$$V=V(\alpha,\beta,\gamma):=\int_0^{a_0}(S_{\mathrm{odd}}-\eta S_{-1})dx, \tag{4.1}$$

where one of the turning points, a_0 is $S_{-1}=\sqrt{Q_0}$ and the branch of $\sqrt{Q_0}$ is chosen as follows: we take a segment connecting the turning points as branch cut and the branch of $\sqrt{Q_0}$ so that $\sqrt{Q_0}\sim\frac{\beta-\alpha}{x}$ as $x\to\infty$. Since the residues of S_{odd} and ηS_{-1} at the origin coincide (See [7] for the computation of residues of S_{odd}.), this integral is well-defined for every homotopy class of the path of integration and we have a formal power series $V(\alpha,\beta,\gamma)$ in η^{-1}. We call $V(\alpha,\beta,\gamma)$ the Voros coefficient of (2.3) with respect to $(0,a_0)$. The Voros coefficient describes the discrepancy between WKB solutions normalized at a_0 and those normalized at the origin, that is, when we set

$$\psi_{\pm}=\frac{1}{\sqrt{S_{\mathrm{odd}}}}\exp\left(\pm\int_{a_0}^{x}S_{\mathrm{odd}}dx\right) \tag{4.2}$$

and

$$\psi_{\pm}^{(0)}=\frac{1}{\sqrt{S_{\mathrm{odd}}}}\exp\left(\pm\int_0^x(S_{\mathrm{odd}}-\eta S_{-1})dx\pm\eta\int_{a_0}^{x}S_{-1}dx\right), \tag{4.3}$$

we have

$$\psi_{\pm}^{(0)}=\exp(\pm V)\psi_{\pm}. \tag{4.4}$$

Here the paths of integration should be chosen suitably.

We consider the case where (α,β,γ) is sufficiently close to $(-0.1,2,1)$. In this case we have two distinct turning points a_0,a_1 on the segment $(0,1)$ in the real axis. We may assume $0 < \mathrm{Re}\, a_0 < \mathrm{Re}\, a_1 < 1$ when (α, β, γ) is sufficiently close to $(-0.1, 2, 1)$. Then we can take the segment connecting the origin and a_0 as the path of integration in (4.1). When the parameter α moves from negative to positive, Stokes phenomena occurs for $\psi_\pm$, while no Stokes phenomenon occurs for $\psi_\pm^{(0)}$ (cf. [8]). That is, V undertakes Stokes phenomena with respect to parameters.

Theorem 4.1. *The Voros coefficient with respect to* $(0, a_0)$ *has the following form:*

$$V(\alpha,\beta,\gamma) = \frac{1}{2}\sum_{n=2}^{\infty} \frac{B_n}{n(n-1)}\eta^{1-n}\left\{(1-2^{1-n})\left(\frac{1}{(\gamma-\alpha)^{n-1}} + \frac{1}{(\gamma-\beta)^{n-1}} + \frac{1}{\alpha^{n-1}} + \frac{1}{\beta^{n-1}}\right) + \frac{2}{\gamma^{n-1}}\right\}. \tag{4.5}$$

Here B_n $(n = 1, 2, \cdots)$ *are the Bernoulli numbers defined by*

$$\frac{te^t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n. \tag{4.6}$$

To derive (4.5), basically we use the method developed by Takei [12] and operator which give the contiguity relations of solution space of hypergeometric differential equations [6]:

$$H_1(a,b,c) = x\frac{d}{dx} + a : \mathcal{T}(a,b,c) \to \mathcal{T}(a+1,b,c), \tag{4.7}$$

$$H_2(a,b,c) = x\frac{d}{dx} + b : \mathcal{T}(a,b,c) \to \mathcal{T}(a,b+1,c), \tag{4.8}$$

$$B_3(a,b,c) = x\frac{d}{dx} + c - 1 : \mathcal{T}(a,b,c+1) \to \mathcal{T}(a,b,c). \tag{4.9}$$

Here $\mathcal{T}(a,b,c)$ denotes the solution space of (2.3). These operators yield the following system of difference equations for V:

$$
\begin{aligned}
(4.10)\quad & V(\alpha+\eta^{-1},\beta,\gamma)-V(\alpha,\beta,\gamma)\\
&=\frac{1}{2}\log\frac{\gamma-\alpha-\frac{1}{2}\eta^{-1}}{-\alpha-\frac{1}{2}\eta^{-1}}-\frac{\eta}{2}\Big\{\alpha\log(-\alpha)-(\alpha+\eta^{-1})\log(-\alpha-\eta^{-1})\\
&-(\alpha-\gamma)\log(\gamma-\alpha)+(\alpha+\eta^{-1}-\gamma)\log(\gamma-\alpha-\eta^{-1})\Big\},
\end{aligned}
$$

$$
\begin{aligned}
(4.11)\quad & V(\alpha,\beta+\eta^{-1},\gamma)-V(\alpha,\beta,\gamma)\\
&=\frac{1}{2}\log\frac{\beta+\frac{1}{2}\eta^{-1}-\gamma}{\beta+\frac{1}{2}\eta^{-1}}-\frac{\eta}{2}\Big\{\beta\log\beta-(\beta+\eta^{-1})\log(\beta+\eta^{-1})\\
&-(\beta-\gamma)\log(\beta-\gamma)+(\beta+\eta^{-1}-\gamma)\log(\beta+\eta^{-1}-\gamma)\Big\},
\end{aligned}
$$

$$
\begin{aligned}
(4.12)\quad & V(\alpha,\beta,\gamma+\eta^{-1})-V(\alpha,\beta,\gamma)\\
&=\frac{1}{2}\log\frac{\gamma(\gamma+\eta^{-1})}{(\gamma+\frac{1}{2}\eta^{-1}-\alpha)(\beta-\gamma-\frac{1}{2}\eta^{-1})}-\frac{\eta}{2}\Big\{(\alpha-\gamma-\eta^{-1})\log(-\alpha+\gamma+\eta^{-1})\\
&-(\alpha-\gamma)\log(-\alpha+\gamma)+(\beta-\gamma-\eta^{-1})\log(\beta-\gamma-\eta^{-1})\\
&-(\beta-\gamma)\log(\beta-\gamma)+(\gamma+\eta^{-1})\log(\gamma+\eta^{-1})^2-\gamma\log\gamma^2\Big\}.
\end{aligned}
$$

Since the above system contains three variables (α,β,γ), the method used in [12], [9] cannot be applied to solve the equation. Thus we employ an argument developed by Candelpergher-Coppo-Delabaere [3]. Main idea is to use a formal differential operator of infinite order of the form

$$
\eta^{-1}\partial_\alpha(e^{\eta^{-1}\partial_\alpha}-1)^{-1}=\sum_{n=0}^{\infty}\frac{(-1)^n B_n}{n!}\eta^{-n}\partial_\alpha^n, \tag{4.13}
$$

where $\partial_\alpha=\frac{\partial}{\partial\alpha}$. The following Lemma plays a role in solving the system.

Lemma 4.2. *We have the two formulas:*

$$
\partial_\alpha(e^{\eta^{-1}\partial_\alpha}-1)^{-1}\log\left(1+\frac{1}{\eta\alpha}\right)=\frac{1}{\alpha}, \tag{4.14}
$$

$$
\partial_\alpha(e^{\eta^{-1}\partial_\alpha/2}-1)^{-1}\log\left(1+\frac{1}{2\eta\alpha}\right)=\frac{1}{\alpha}. \tag{4.15}
$$

Detailed computation as well as Voros coefficients with respect other pair of singular point and turning point or those for other cases of parameters and analysis of Stokes phenomena for them will be given in our forthcoming paper.

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