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On existence and uniqueness theorems for singular nonlinear partial differential equations

By

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Abstract

We present in this survey article all the known results on the existence and uniqueness of solutions for singular nonlinear partial differential equations of the form

$$t\partial u/\partial t = F\left(t, x, u, \partial u/\partial x\right),$$

with independent variables $(t, x) \in \mathbb{C} \times \mathbb{C}$ or $(t, x) \in \mathbb{R} \times \mathbb{C}$, and where $F(t, x, u, v)$ is either holomorphic in (t, x, u, v) or holomorphic in (x, u, v) and only continuous in t.

§ 1. Introduction

One of the most fundamental results in the theory of partial differential equations in the complex domain is the Cauchy-Kowalewski theorem. Let $(t, x) \in \mathbb{C}^2$ and consider the Cauchy problem

$$\frac{\partial u}{\partial t} = F\Big(t, x, u, \frac{\partial u}{\partial x}\Big), \quad u(0, x) = \varphi(x), \tag{1.1}$$

where $F(t, x, u, v)$ is a holomorphic function in a neighborhood of $(0, 0, a, b) \in \mathbb{C}_t \times \mathbb{C}_x \times \mathbb{C}_u \times \mathbb{C}_v$, and $\varphi(x)$ is a holomorphic function in a neighborhood of $x = 0 \in \mathbb{C}$ that satisfies $\varphi(0) = a$ and $\varphi'(0) = b$. The theorem asserts that the Cauchy problem (1.1) has a unique holomorphic solution $u(t, x)$ in a neighborhood of $(0, 0) \in \mathbb{C}_t \times \mathbb{C}_x$.

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Afterwards, Nagumo [9], Nirenberg [10] and Ovsjannikov [12] extended the study of the equation (1.1) to the case where $F(t,x,u,v)$ is holomorphic in (x,u,v) and only continuous in t.

In this survey article, we present all the known results on the existence and uniqueness of solutions of the equation

$$t\frac{\partial u}{\partial t} = F\Big(t,x,u,\frac{\partial u}{\partial x}\Big) \tag{1.2}$$

that fall under the following two categories:

(1) **(Holomorphic Category)** $F(t,x,u,v)$ is holomorphic in (t,x,u,v);

(2) **(Continuous Category)** $F(t,x,u,v)$ is holomorphic in (x,u,v) but only continuous in t.

The first category corresponds to the Cauchy-Kowalewski theorem, while the second corresponds to the works of [9], [10] and [12].

§ 2. Holomorphic category

Denote by $\mathbb{N}$ and $\mathbb{N}^*$ the sets $\{0,1,2,3,\dots\}$ and $\{1,2,\dots\}$, respectively. Let $(t,x) \in \mathbb{C}^2$ and consider the singular nonlinear partial differential equation

$$t\frac{\partial u}{\partial t} = F\Big(t,x,u,\frac{\partial u}{\partial x}\Big). \tag{2.1}$$

Suppose that $F(t,x,u,v)$ is a holomorphic function in a neighborhood of the origin $(0,0,0,0) \in \mathbb{C}^4$ and $F(0,x,0,0) \equiv 0$ near $x=0$. Then, using the Taylor expansion of $F(t,x,u,v)$ with respect to the variables (t,u,v), we can write the right-hand side of equation (2.1) as

$$F\Big(t,x,u,\frac{\partial u}{\partial x}\Big) = a(x)t + \lambda(x)u + b(x)\frac{\partial u}{\partial x} + R_2\Big(t,x,u,\frac{\partial u}{\partial x}\Big),$$

where $R_2(t,x,u,v)$ is the sum of all the terms in the Taylor expansion whose degrees with respect to (t,u,v) are at least 2. In this situation, solving (2.1) can be divided into three cases:

(C_1) $b(x) \equiv 0$;

(C_2) $b(0) \neq 0$;

(C_3) $b(x) = x^p\gamma(x)$ where $\gamma(0) \neq 0$ and $p \in \mathbb{N}^*$.

In the case (C_1), the equation (1.2) is called *a Briot-Bouquet type* partial differential equation with respect to t. Gérard-Tahara [6] proved the existence and uniqueness of holomorphic solution of this equation when $\lambda(0) \notin \mathbb{N}^*$; Yamazawa [14] then solved the case $\lambda(0) \in \mathbb{N}^*$. Their results are stated precisely in the following theorem.

Theorem 2.1. *Assume that* (C_1) *holds. Then, we have the following:*

(1) *(Gérard-Tahara [6]) If* $\lambda(0) \notin \mathbb{N}^*$*, the equation (2.1) has a unique holomorphic solution* $u(t,x)$ *in a neighborhood of* $(0,0) \in \mathbb{C}_t \times \mathbb{C}_x$ *that satisfies* $u(0,x) \equiv 0$ *near* $x = 0 \in \mathbb{C}$.

(2) *(Yamazawa [14]) Also in the case* $\lambda(0) \in \mathbb{N}^*$ *but* $\lambda(x) \neq \lambda(0)$*, the equation (2.1) has a solution* $u(t,x)$ *in the class* $\widetilde{\mathcal{O}}_+$*, which is not necessarily holomorphic at* $(0,0)$.

The class $\widetilde{\mathcal{O}}_+$ in (2) of Theorem 2.1 is defined as follows. Let $\mathcal{R}(\mathbb{C} \setminus \{0\})$ be the universal covering space of $\mathbb{C} \setminus \{0\}$. Denote by S_θ and D_r the sets $\{t \in \mathcal{R}(\mathbb{C} \setminus \{0\}) : |\arg\, t| < \theta\}$ and $\{x \in \mathbb{C} : |x| \leq r\}$, respectively. Then $\widetilde{\mathcal{O}}_+$ is the set of all holomorphic functions $u(t,x)$ on $\{t \in \mathcal{R}(\mathbb{C} \setminus \{0\}) \,:\, 0 < |t| < \rho(\arg\, t)\} \times D_r$ for some positive-valued continuous function $\rho(s)$ and $r > 0$ that satisfy the condition that for any $\theta > 0$ and compact subset K of D_r, there exists $a > 0$ such that

$$\max_{x \in K} |u(t,x)| = O(|t|^a) \quad (\text{as } t \longrightarrow 0 \text{ in } S_\theta).$$

For the second case (C_2), by the implicit function theorem we can rewrite (2.1) into the form

$$\frac{\partial u}{\partial x} = G\Big(t, x, u, t\frac{\partial u}{\partial t}\Big)$$

and so we can apply the Cauchy-Kowalewski theorem to this equation with data on $x = 0$ and arrive at the following result.

Theorem 2.2. *Assume that* (C_2) *holds. For any holomorphic function* $\phi(t)$ *with* $\phi(0) = 0$*, the equation (2.1) has a unique holomorphic solution* $u(t,x)$ *in a neighborhood of* $(0,0) \in \mathbb{C}_t \times \mathbb{C}_x$ *that satisfies* $u(t,0) = \phi(t)$ *and* $u(0,x) = 0$.

The equation (2.1) is said to be *a totally characteristic type* partial differential equation if it satisfies (C_3). In the case where $p=1$, Chen-Tahara [4] and Tahara [13] established the solvability of the equation when $\gamma(0) \in \mathbb{C} \setminus [0,\infty)$.

Theorem 2.3. *Assume that (C_3) holds and $p=1$. Then we have the following:*

(1) (Chen-Tahara [4]) If $\gamma(0) \in \mathbb{C}\setminus[0,\infty)$ and $i-\lambda(0)-j\gamma(0) \neq 0$ for any $(i,j) \in \mathbb{N}^* \times \mathbb{N}$, then the equation (2.1) has a unique holomorphic solution $u(t,x)$ in a neighborhood of $(0,0) \in \mathbb{C}_t \times \mathbb{C}_x$ that satisfies $u(0,x) \equiv 0$ near $x=0$.

(2) (Tahara [13]) If $\gamma(0) \in \mathbb{C} \setminus [0,\infty)$, then the equation (2.1) has a solution $u(t,x)$ in the class $\widetilde{\mathcal{O}}_+$, which is not necessarily holomorphic at $(0,0)$.

On the other hand, for $p \geq 2$, Chen-Luo-Tahara [5] studied Gevrey type estimates of formal solutions, and Luo-Chen-Zhang [8] showed the solvability in a sectorial domain by using the summability theory. Before we state their results, let us recall what is meant by the formal Gevrey class. Let us write the equation (2.1) in the form

$$t\frac{\partial u}{\partial t} = a(x)t + \lambda(x)u + b(x)\frac{\partial u}{\partial x} + \sum_{i+j+\alpha\geq 2} a_{i,j,\alpha}(x)t^i u^j \Big(\frac{\partial u}{\partial x}\Big)^{\alpha}.$$

Set $J = \{(i,j,\alpha)\,;\, i+j+\alpha \geq 2, \alpha > 0, a_{i,j,\alpha}(0) \neq 0\}$ and denote by $\mathbb{C}[[t,x]]$ the ring of formal power series in (t,x). For $s \geq 1$ and $\sigma \geq 1$, the formal Gevrey class, denoted by $\mathbb{C}\{t,x\}_{(s,\sigma)}$, is defined to be the set of all formal power series $u(t,x) = \sum_{i\geq 0, j\geq 0} u_{i,j} t^i x^j$ in $\mathbb{C}[[t,x]]$ that satisfy the property:

$$\sum_{i\geq 0, j\geq 0} \frac{u_{i,j}}{(i!)^{s-1}(j!)^{\sigma-1}} t^i x^j \ \text{ is convergent.}$$

Theorem 2.4. *Assume that (C_3) holds for $p \geq 2$. Then we have the following:*

(1) (Chen-Luo-Tahara [5]) If $\lambda(0) \notin \mathbb{N}^*$, the equation (2.1) has a unique formal solution $u(t,x)$ with $u(0,x) \equiv 0$ that belongs to the formal Gevrey class $\mathbb{C}\{t,x\}_{(s,\sigma)}$ for any (s,σ) satisfying

$$s \geq 1 + \max\Big[0, \sup_{(i,j,\alpha)\in J} \Big(\frac{1}{(p-1)(i+j+\alpha-1)}\Big)\Big] \tag{2.2}$$

and $\sigma \geq p/(p-1)$.

(2) In addition, if

$$a_{i,j,\alpha}(0) = 0 \text{ for all } \alpha, \tag{2.3}$$

then we have $J = \emptyset$, and hence, we can take $s = 1$ and $\sigma = p/(p-1)$. In this case, the formal solution $u(t,x)$ is convergent in t and divergent in x of Gevrey order $\sigma = 1 + 1/k$ (with $k = p - 1$).

(3) (Luo-Chen-Zhang [8]) In (2), the formal solution $u(t,x)$ is k-summable in all directions in the x-plane except at most a countable directions belonging to the set

$$\bigcup_{\nu=0}^{k-1} \Big\{ \frac{\arg(z) + 2\nu\pi}{k} \,;\, z \in \Big\{ \frac{1}{\gamma}, \frac{1-\lambda}{\gamma}, \frac{2-\lambda}{\gamma}, \frac{3-\lambda}{\gamma}, \dots \Big\} \Big\}, \tag{2.4}$$

where $\lambda = \lambda(0)$ and $\gamma = \gamma(0)$. (As to the meaning of k-summability, refer to [2].)

In [8], the existence of analytic solution of (2.1) in a sectorial domain is established without the condition (2.3), but this result is only an application of (3) of Theorem 2.4. At present, it is still open whether it is possible to get a summability result of the formal solution without assuming (2.3).

§ 3. Continuous category

In this section, we consider the equation

$$t\frac{\partial u}{\partial t} = F\Big(t, x, u, \frac{\partial u}{\partial x}\Big), \tag{3.1}$$

where $(t,x) \in \mathbb{R} \times \mathbb{C}$ and $F(t,x,u,v)$ is holomorphic with respect to the variables (x,u,v) but only continuous in t. We shall state all the known existence and uniqueness results for the equation (3.1) that correspond to each of the three cases discussed in section 2.

The first attempt for the case (C_1) under the continuous category was done by Baouendi-Goulaouic [3]. They formulated existence and uniqueness theorems for the equation

$$t\frac{\partial u}{\partial t} = \lambda(x)u + f(t,x) + tG\Big(t, x, u, \frac{\partial u}{\partial x}\Big)$$

in an abstract setting and gave several applications. Their result was then extended to a wider class of equations by Lope-Roque-Tahara [7] using a concept of weight functions.

A function $\mu(t)$ is said to be a *weight function* on $(0, T]$ if it is a continuous, nonnegative, increasing function on $(0, T]$, and

$$\int_0^T \frac{\mu(s)}{s} ds < +\infty. \tag{3.2}$$

Examples of such functions are t^η, $1/(-\log t)^{1+\eta}$ and $1/(-\log t)(\log(-\log t))^\eta$ for any $\eta > 0$. Note that for any such weight function $\mu(t)$, $\lim_{t \to 0} \mu(t) = 0$. Moreover, the condition (3.2) allows us to define the function

$$\varphi(t) = \int_0^t \frac{\mu(s)}{s} ds, \quad 0 \le t \le T.$$

For any $r > 0$, $T > 0$ and $R > 0$, we define the sets D_R, W_r, $X_0(W_r)$ and $X_1(W_r)$ as follows:

$$\begin{aligned} D_R &= \{x \in \mathbb{C}\,;\ |x| < R\}; \\ W_r &= \Big\{(t,x) : 0 \le t \le T \text{ and } |x| + \tfrac{\varphi(t)}{r} < R\Big\}; \\ X_0(W_r) &= \{w(t,x) \in C^0(W_r) \text{: } w \text{ is holomorphic in } x \text{ for any fixed } t\}; \\ X_1(W_r) &= X_0(W_r) \cap C^1(W_r \cap \{t > 0\}). \end{aligned}$$

Let $F(t, x, u, v)$ be a function on $\Delta_0 = [0, T_0] \times D_{R_0} \times D_{\rho_0} \times D_{\rho_0}$ for some $T_0 > 0$, $R_0 > 0$ and $\rho_0 > 0$. Let $0 < T < T_0$ and $0 < R < R_0$. Consider the equation (3.1) under the following conditions:

(A_1) $F(t, x, u, v)$ is continuous in t and holomorphic in (x, u, v);
(A_2) $F(t, x, 0, 0) = O(\mu(t))$ uniformly on D_R (as $t \longrightarrow +0$);
(A_3) $F_v(t, x, 0, 0) = O(\mu(t))$ uniformly on D_R (as $t \longrightarrow +0$);
(A_4) $\mathrm{Re}\, F_u(t, x, 0, 0) < -L$ on $[0, T] \times D_R$ for some $L > 0$.

Set $a(t, x) = F(t, x, 0, 0)$, $\lambda(t, x) = F_u(t, x, 0, 0)$ and $b(t, x) = F_v(t, x, 0, 0)$. Then, using the Taylor expansion of $F(t, x, u, v)$ with respect to the variables (u, v), we may write equation (3.1) as

$$t\frac{\partial u}{\partial t} = a(t,x) + \lambda(t,x)u + b(t,x)\frac{\partial u}{\partial x} + G_2\Big(t, x, u, \frac{\partial u}{\partial x}\Big),$$

where $G_2(t, x, u, v)$ is the sum of all the terms in the Taylor expansion whose degrees with respect to (u, v) are at least 2.

Theorem 3.1 (Lope-Roque-Tahara [7]). *Assume that (A1)-(A4) hold. Then there exist $R > 0$, $r > 0$, $M > 0$ and $T > 0$ with $M\mu(T) < \rho_0$ such that the equation (3.1) has a unique solution $u(t,x)$ in $X_1(W_r)$ that satisfies*

$$|u(t,x)| \le M\mu(t) \ \ and \ \ \left|\frac{\partial u}{\partial x}(t,x)\right| \le M\mu(t) \quad on\ W_r.$$

Remark 3.2. The equation discussed by Baouendi and Goulaouic in [3] corresponds to the case $\mu(t) = t$. Since they were interested mainly in getting a solution of C^∞ class with respect to t, this restricted case was sufficient for their purpose.

In the case (C_2), if we apply the implicit function theorem to equation (3.1), then the equation can be written in the form

$$\frac{\partial u}{\partial x} = G\Big(t, x, u, t\frac{\partial u}{\partial t}\Big),$$

where $G(t,x,u,v)$ is holomorphic in (x,u,v) but only continuous in t. At present, we have no good references that deal with such type of equation.

In the case (C_3) with $p = 1$, the authors succeeded in getting a good result. We state this result and give an outline of its proof.

Again, we let $0 < T < T_0$, $0 < R < R_0$, and $F(t,x,u,v)$ be a function on Δ_0. Consider equation (3.1) under the following assumptions:

- (B_1) $F(t,x,u,v)$ is continuous in t and holomorphic in (x,u,v);
- (B_2) $F(t,x,0,0) = O(\mu(t))$ uniformly on D_R (as $t \longrightarrow +0$);
- (B_3) $F_v(t,0,0,0) = O(\mu(t))$ (as $t \longrightarrow +0$);
- (B_4) $F_v(0,x,0,0) = x\gamma(x)$ and $\mathrm{Re}\,\gamma(x) < -\delta$ on D_R for some $\delta > 0$;
- (B_5) $\mathrm{Re}\,F_u(t,x,0,0) < -L$ on $[0,T] \times D_R$ for some $L > 0$.

Set $a(t,x) = F(t,x,0,0)$, $\lambda(t,x) = F_u(t,x,0,0)$, $b(t) = F_v(t,0,0,0)$, and $c(t,x) = (F_v(t,x,0,0) - F_v(t,0,0,0))/x$. Then, using the Taylor expansion of $F(t,x,u,v)$ with respect to the variables (u,v), the equation (3.1) may be written as

$$t\frac{\partial u}{\partial t} = a(t,x) + \lambda(t,x)u + (b(t) + xc(t,x))\frac{\partial u}{\partial x} + G_2\left(t, x, u, \frac{\partial u}{\partial x}\right), \tag{3.3}$$

where $G_2(t,x,u,v)$ represents the sum of all the terms in the Taylor expansion whose degrees with respect to (u,v) are at least 2.

Theorem 3.3 (Bacani-Tahara [1]). *Assume that* $(B_1) - (B_5)$ *hold. Then there exist* $R > 0$, $r > 0$, $M > 0$ *and* $T > 0$ *with* $M\mu(T) < \rho_0$ *such that the equation (3.1) has a unique solution* $u(t,x)$ *in* $X_1(W_r)$ *that satisfies*

$$|u(t,x)| \leq M\mu(t) \ \ \textit{and} \ \ \left|\frac{\partial u}{\partial x}(t,x)\right| \leq M\mu(t) \quad \textit{on } W_r.$$

In [1], Theorem 3.3 is proved using the Banach fixed point theorem (also known as the contraction mapping principle). Here, we show briefly that this result can also be proved using the method of Nirenberg [10] and Nishida [11], in a similar manner as it was used in [7].

Proof of Theorem 3.3. For simplicity, we set

$$a = a(t,x),$$

$$\mathcal{P} = t\frac{\partial}{\partial t} - \lambda(t,x) - xc(t,x)\frac{\partial}{\partial x}$$

and

$$\Phi[u] = b(t)\frac{\partial u}{\partial x} + G_2\left(t, x, u, \frac{\partial u}{\partial x}\right).$$

Then equation (3.3) may be expressed as

$$\mathcal{P}u = a + \Phi[u]. \tag{3.4}$$

To solve equation (3.4), we construct a sequence of approximate solutions $u_k(t,x)$ $(k = 0, 1, 2, \ldots)$, where $u_0(t,x)$ is a solution of

$$\mathcal{P}u_0 = a, \tag{3.5}$$

and $u_k(t,x)$ $(k \geq 1)$ is a solution of

$$\mathcal{P}u_k = a + \Phi[u_{k-1}]. \tag{3.6}$$

The following result guarantees that we can solve equations (3.5) and (3.6).

Proposition 3.4. *Let* $\Omega = \{(s,t,x) : 0 < s \leq t \ \ \text{and} \ \ (t,x) \in (0,T] \times D_R\}$. *For any given* $g(t,x) \in X_0(W_r)$, *the equation* $\mathcal{P}u = g$ *has a unique solution* $u(t,x)$ *in* $X_1(W_r)$, *and it is given by*

$$w(t,x) = \int_0^t \exp\left[\int_s^t \lambda(\tau, \phi(\tau,t,x))\frac{d\tau}{\tau}\right] g(s, \phi(s,t,x))\frac{ds}{s},$$

where $\phi(s,t,x)$ is the unique solution of

$$(3.7)\qquad \begin{cases} t\dfrac{\partial \phi}{\partial t} - xc(t,x)\dfrac{\partial \phi}{\partial x} = 0 & \text{on } \Omega, \\ \phi(s,s,x) = x & \text{on } (0,T]\times D_R. \end{cases}$$

Moreover, the following estimates hold on $W_{r,R}$ given any nondecreasing, nonnegative function $\psi(t)$:

(a) *If $|g(t,x)| \le K\psi(t)\mu(t)$, then $|w(t,x)| \le K\psi(t)\varphi(t)$.*

(b) *If $|g(t,x)| \le K\psi(t)$ and $|g_x(t,x)| \le K_1\psi(t)$, then for some $H > 0$, we have*

$$|w(t,x)| \le \frac{K}{L}\psi(t) \ \text{ and } \ \left|\frac{\partial w}{\partial x}(t,x)\right| \le \left(\frac{K_1}{L} + \frac{\Lambda K}{L^2}\right) H\psi(t).$$

(c) *If $|g(t,x)| \le \dfrac{K\psi(t)\mu(t)}{R-|x|-\varphi(t)/r}$, then for some $Q > 0$, we have*

$$|w(t,x)| \le \frac{K\psi(t)r}{R-|x|-\varphi(t)/r} \ \text{ and } \ \left|\frac{\partial w}{\partial x}(t,x)\right| \le \frac{(4+\Lambda Q)HK\psi(t)r}{R-|x|-\varphi(t)/r}.$$

Now, let $0 < \rho < \rho_0$ and $\Delta = [0,T_0]\times D_R \times D_\rho \times D_\rho$. Suppose that $|b(t,x)| \le B\mu(t)$ on $[0,T]$ for some $B > 0$, and the second order partial derivatives $F_{uu}(t,x,u,v)$, $F_{uv}(t,x,u,v)$ and $F_{vv}(t,x,u,v)$ are bounded by B_0, B_1 and B_2, respectively, on Δ. Set $C = C_1\cdot C_2$ where $C_1 = (B+B_0+2B_1+B_2)/\mu(T)$ and $C_2 = \max\{1,(4+\Lambda Q)H\}$. Then, choose a sequence $\{r_k\}_{k=0}^{\infty}$ that satisfies $0 < 2Cr_0 < 1$ and $r_k = r_{k-1}(1-(2Cr_0)^k)$ for $k \ge 1$; this is a decreasing sequence of numbers converging to a positive limit r_∞.

To show the existence of a solution, we first apply (b) of Proposition 3.4 to (3.5) to obtain a unique solution $u_0 \in X_1(W_{r_0})$ satisfying

$$|u_0(t,x)| \le \frac{A}{L}\mu(t) \ \text{ and } \ \left|\frac{\partial u_0}{\partial x}(t,x)\right| \le \left(\frac{1}{L} + \frac{\Lambda}{L^2}\right) AH\mu(t)\,,$$

for some $A > 0$. Next, we choose $T > 0$ sufficiently small so that

$$(3.8)\qquad \max\left\{|u_0|, \left|\frac{\partial u_0}{\partial x}\right|\right\} \le \frac{M\mu(t)}{2} \quad \text{on } W_{r_0},$$

where $M = \rho/\mu(T)$. Then, we use the following result, which is analogous to Proposition 3.1 in [7], to prove that the approximate solutions converge to a desired solution of (3.1) on W_{r_∞}.

Proposition 3.5. *For $k \geq 1$, the following statements hold:*

(a) There exists a unique u_k in $X_1(W_{r_{k-1}})$ that satisfies the equation (3.6).

(b) On $W_{r_{k-1}}$, we have

$$\max\left\{|u_k - u_{k-1}|, \left|\frac{\partial}{\partial x}(u_k - u_{k-1})\right|\right\} \leq \frac{1}{2} \cdot \frac{CM(Cr_0)^{k-1}\varphi(t)\mu(t)}{R - |x| - \varphi(t)/r_{k-1}}.$$

(c) On W_{r_k}, we have

$$\max\left\{|u_k - u_{k-1}|, \left|\frac{\partial}{\partial x}(u_k - u_{k-1})\right|\right\} \leq \frac{M\mu(t)}{2^{k+1}}.$$

(d) On W_{r_k}, we have

$$\max\left\{|u_k|, \left|\frac{\partial u_k}{\partial x}\right|\right\} \leq \frac{(2^{k+1} - 1)M\mu(t)}{2^{k+1}}.$$

The uniqueness of solution of (3.1) in $X_1(W_{r_\infty})$ can be proved in the same way as in [7]. □

In the case (C_3) with $p \geq 2$, we have no references at present.

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