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**B40** (2013), 031–040

# On oscillatory integrals with $C^\infty$ phases

By

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## § 1. Introduction

The purpose of this article is to announce a part of recent results in [10] (see also [14]). We consider the asymptotic behavior of oscillatory integrals, that is, integrals of the form

$$I(\tau) = \int_{\mathbb{R}^n} e^{i\tau f(x)} \varphi(x) dx, \tag{1.1}$$

for large values of the real parameter $\tau$. Here $f$ is a real valued $C^\infty$ smooth function defined on $\mathbb{R}^n$ and $\varphi$ is a real-valued $C^\infty$ smooth function whose support is contained in a small neighborhood of the origin in $\mathbb{R}^n$. Here $f$ and $\varphi$ are called the *phase* and the *amplitude*, respectively.

Suppose that $f$ has a critical point at the origin, that is $\nabla f(0) = (0, \dots, 0)$. By using famous Hironaka's resolution of singularities, the following deep results can be obtained (c.f. [13]). If $f$ is real analytic on a neighborhood of the origin and the support of $\varphi$ is contained in a sufficiently small neighborhood of the origin, then the integral $I(\tau)$ has an asymptotic expansion of the form

$$I(\tau) \sim e^{i\tau f(0)} \sum_{\alpha} \sum_{k=1}^{n} C_{\alpha k}(\varphi) \tau^{\alpha} (\log \tau)^{k-1} \quad \text{as } \tau \to +\infty, \tag{1.2}$$

where $\alpha$ runs through a finite number of arithmetic progressions, not depending on the amplitude $\varphi$, which consist of negative rational numbers.

Since we are interested in the largest $\alpha$ occurring in the asymptotic expansion (1.2), let us recall important quantities: oscillation index and its multiplicity. Let $S(f)$ be the

Received March 30, 2012, Accepted August 8, 2012.
2010 Mathematics Subject Classification(s): 58K55 (14M25, 42B20).
*Key Words*: oscillatory integrals, Newton polyhedra, toric resolution, the class $\hat{\mathcal{E}}(U)$.
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set of pairs $(\alpha, k)$ such that for each neighborhood of the origin in $\mathbb{R}^n$, there exists a $C^\infty$ function $\varphi$ with support in this neighborhood for which $C_{\alpha k}(\varphi) \neq 0$ in the asymptotic expansion (1.2). We denote by $(\beta(f), \eta(f))$ the maximum of the set $S(f)$ under the lexicographic ordering, i.e., $\beta(f)$ is the maximum of values $\alpha$ for which we can find $k$ so that $(\alpha, k)$ belongs to $S(f)$; $\eta(f)$ is the maximum of integers $k$ satisfying that $(\beta(f), k)$ belongs to $S(f)$. We call $\beta(f)$ the *oscillation index* of $f$ and $\eta(f)$ the *multiplicity* of its index.

There have been many interesting studies about the above oscillation index and its multiplicity ([17],[4],[15],[5],[6],[7],[8],[9], [3], etc.). In particular, an important work of Varchenko [17] shows that under a certain nondegeneracy condition of the real analytic phase, the oscillation index can be estimated or exactly determined by using the theory of toric varieties based on the geometry of the Newton polyhedra of $f$ (see Theorem 3.1 in Section 3). Since his study, the investigation of the behavior of oscillatory integrals has been more closely linked with the theory of singularities. Refer to the excellent expositions [1], [12] for studies in this direction.

The purpose of our study is to generalize the above results of Varchenko [17] to the case that the phase is contained in a certain class of $C^\infty$ functions including the real analytic functions. This class is denoted by $\hat{\mathcal{E}}(U)$. In the above investigation of Varchenko, the function *$\gamma$-part*, which corresponds to each face $\gamma$ of the Newton polyhedron, plays an important role. In the real analytic case, the $\gamma$-part is easily defined as a function for every face $\gamma$, but this definition cannot be directly generalized to the case of noncompact face in the $C^\infty$ case. From the convex geometrical points of view (c.f. [18]), we give another definition of the $\gamma$-part, which is a natural generalization of the definition in the real analytic case. We remark that not all smooth functions admit the $\gamma$-part for every face $\gamma$ of their Newton polyhedra in our sense. The class $\hat{\mathcal{E}}(U)$ is defined to be the set of $C^\infty$ functions admitting the $\gamma$-part for every face $\gamma$ of $\Gamma_+(f)$. Many kinds of $C^\infty$ functions are contained in $\hat{\mathcal{E}}(U)$. In particular, it contains the Denjoy-Carleman quasianalytic classes, which are interesting classes of $C^\infty$ functions and have been studied from various points of view (c.f. [2],[16]). We can construct a toric resolution of singularities in the class $\hat{\mathcal{E}}(U)$ and we succeed to generalize the above result of Varchenko.

It is known (c.f. [1], [12]) that the asymptotic analysis of oscillatory integral (1.1) can be reduced to an investigation of the poles of the local zeta function

$$Z(s) = \int_{\mathbb{R}^n} |f(x)|^s \varphi(x) dx,$$

where $f$, $\varphi$ are the same as in (1.1) with $f(0) = 0$. Our substantial analysis is to investigate the properties of poles of the local zeta function $Z(s)$ where $f$ belongs to the class $\hat{\mathcal{E}}(U)$.

*Notation and symbols.*

(i) We denote by $\mathbb{Z}_+, \mathbb{Q}_+, \mathbb{R}_+$ the subsets consisting of all nonnegative numbers in $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, respectively.

(ii) We use the multi-index as follows. For $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$, define

$$
\begin{aligned}
&|x| = \sqrt{|x_1|^2 + \cdots + |x_n|^2}, \quad \langle x, y\rangle = x_1 y_1 + \cdots + x_n y_n, \\
&x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad \partial^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}, \\
&\langle \alpha \rangle = \alpha_1 + \cdots + \alpha_n, \quad \alpha! = \alpha_1! \cdots \alpha_n! \quad (0! = 1! = 1).
\end{aligned}
$$

(iii) For $A, B \subset \mathbb{R}^n$ and $c \in \mathbb{R}$, we set

$$
A + B = \{a + b \in \mathbb{R}^n; a \in A \text{ and } b \in B\}, \quad c \cdot A = \{ca \in \mathbb{R}^n; a \in A\}.
$$

## § 2. Newton polyhedra and the classes $\hat{\mathcal{E}}[P](U)$ and $\hat{\mathcal{E}}(U)$

### § 2.1. Polyhedra

Let us explain fundamental notions in the theory of convex polyhedra, which are necessary for our study. Refer to [18], etc. for general theory of convex polyhedra.

For $(a, l) \in \mathbb{R}^n \times \mathbb{R}$, let $H(a, l)$ and $H^+(a, l)$ be a hyperplane and a closed half space in $\mathbb{R}^n$ defined by

$$
\begin{aligned}
H(a, l) &:= \{x \in \mathbb{R}^n; \langle a, x\rangle = l\}, \\
H^+(a, l) &:= \{x \in \mathbb{R}^n; \langle a, x\rangle \geq l\},
\end{aligned}
$$

respectively. A (*convex rational*) *polyhedron* is an intersection of closed halfspaces: a set $P \subset \mathbb{R}^n$ presented in the form $P = \bigcap_{j=1}^N H^+(a^j, l_j)$ for some $a^1, \ldots, a^N \in \mathbb{Z}^n$ and $l_1, \ldots, l_N \in \mathbb{Z}$.

Let $P$ be a polyhedron in $\mathbb{R}^n$. A pair $(a, l) \in \mathbb{Z}^n \times \mathbb{Z}$ is said to be *valid* for $P$ if $P$ is contained in $H^+(a, l)$. A *face* of $P$ is any set of the form $F = P \cap H(a, l)$, where $(a, l)$ is valid for $P$. Since $(0, 0)$ is always valid, we consider $P$ itself as a trivial face of $P$; the other faces are called *proper faces*. Conversely, it is easy to see that any face is a polyhedron. Considering the valid pair $(0, -1)$, we see that the empty set is always a face of $P$. Indeed, $H^+(0, -1) = \mathbb{R}^n$, but $H(0, -1) = \emptyset$. The *dimension* of a face $F$ is the dimension of its affine hull of $F$ (i.e., the intersection of all affine flats that contain $F$), which is denoted by $\dim(F)$. The faces of dimensions $0, 1$ and $\dim(P) - 1$ are called *vertices*, *edges* and *facets*, respectively. The *boundary* of a polyhedron $P$, denoted by $\partial P$, is the union of all proper faces of $P$. For a face $F$, $\partial F$ is similarly defined.

### § 2.2. Newton polyhedra

Let $f$ be a real-valued $C^\infty$ function defined on a neighborhood of the origin in $\mathbb{R}^n$. Denote by $\hat{f}(x)$ the Taylor series of $f$ at the origin, i.e.,

$$\hat{f}(x) = \sum_{\alpha \in \mathbb{Z}_+^n} c_\alpha x^\alpha \quad \text{with } c_\alpha = \frac{\partial^\alpha f(0)}{\alpha!}.$$

The *Newton polyhedron* of $f$ is the integral polyhedron:

$$\Gamma_+(f) = \text{the convex hull of the set } \bigcup \{\alpha + \mathbb{R}_+^n; c_\alpha \neq 0\} \text{ in } \mathbb{R}_+^n$$

(i.e., the intersection of all convex sets which contain $\bigcup\{\alpha + \mathbb{R}_+^n; c_\alpha \neq 0\}$). It is known (cf. [18]) that the Newton polyhedron $\Gamma_+(f)$ is a polyhedron. The union of the compact faces of the Newton polyhedron $\Gamma_+(f)$ is called the *Newton diagram* $\Gamma(f)$ of $f$, while the boundary of $\Gamma_+(f)$ is denoted by $\partial\Gamma_+(f)$. The *principal part* of $f$ is defined by $f_0(x) = \sum_{\alpha \in \Gamma(f) \cap \mathbb{Z}_+^n} c_\alpha x^\alpha$.

A function $f$ is said to be *convenient* if the Newton polyhedron $\Gamma_+(f)$ intersects all the coordinate axes.

Let $q_0$ be the point at which the line $\alpha_1 = \cdots = \alpha_n$ in $\mathbb{R}^n$ intersects the boundary of $\Gamma_+(f)$. The coordinate of $q_0$ is called the *Newton distance* of $\Gamma_+(f)$, which is denoted by $d(f)$, i.e., $q_0 = (d(f), \ldots, d(f))$. The face whose relative interior contains $q_0$ is called the *principal face* of $\Gamma_+(f)$, which is denoted by $\tau_0$. The codimension of $\tau_0$ is called the *Newton multiplicity* of $\Gamma_+(f)$, which is denoted by $m(f)$. Here, when $q_0$ is a vertex of $\Gamma_+(f)$, we consider that $\tau_0$ is the point $q_0$ and $m(f) = n$.

### § 2.3. The $\gamma$-part

Let $f$ be a real-valued $C^\infty$ function on a neighborhood $V$ of the origin in $\mathbb{R}^n$, $P \subset \mathbb{R}_+^n$ a nonempty polyhedron containing $\Gamma_+(f)$ and $\gamma$ a face of $P$. Note that the condition: $\Gamma_+(f) \subset P$ implies that $P + \mathbb{R}_+^n \subset P$ and, moreover, every valid pair for $P$ is contained in the set $\mathbb{Z}_+^n \times \mathbb{Z}_+$. We say that $f$ *admits the $\gamma$-part* on an open neighborhood $U \subset V$ of the origin if for any $x \in U$ the limit:

$$\lim_{t \to 0} \frac{f(t^{a_1} x_1, \ldots, t^{a_n} x_n)}{t^l} \tag{2.1}$$

exists for *all* valid pairs $(a, l) = (a_1, \ldots, a_n, l) \in \mathbb{Z}_+^n \times \mathbb{Z}_+$ defining $\gamma$ (i.e., $H(a, l) \cap P = \gamma$). We remark that when $f$ admits the $\gamma$-part, the above limits take the same value for any $(a, l)$, which is denoted by $f_\gamma(x)$. We consider $f_\gamma$ as the function on $U$, which is called the $\gamma$-part of $f$ on $U$.

*Remark* 1. We give many remarks on the $\gamma$-part.

(i) The readers might feel that "all" is too strict in the above definition of the admission of the $\gamma$-part. Actually, even if "all" is replaced by "some" in the definition, this exchange does not affect our analysis.

(ii) The above $\gamma$-part $f_\gamma$ is a $C^\infty$ function defined on $U$.

(iii) When $\gamma = P$, $f$ always admits the $\gamma$-part on $V$ and $f_P = f$. (Consider the case when $(a, l) = (0, 0)$.)

(iv) When $\gamma$ is contained in some coordinate plane, $f$ always admits the $\gamma$-part on $U$. Indeed, for any pair $(a, l)$ defining $\gamma$, we have $l = 0$ and the limit (2.1) exists.

(v) For a face $\gamma$ of $P$, the condition $\gamma \cap \Gamma_+(f) = \emptyset$ does not always mean that $f$ admits the $\gamma$-part: $f_\gamma \equiv 0$. (See Section 2.5.)

(vi) If $f$ admits the $\gamma$-part $f_\gamma$ on $U$, then $f_\gamma$ has the quasihomogeneous property:

$$f_\gamma(t^{a_1}x_1, \dots, t^{a_n}x_n) = t^l f_\gamma(x) \quad \text{for} \quad 0 < t < 1 \text{ and } x \in U,$$

where $(a, l)$ is a valid pair defining $\gamma$.

(vii) For a compact face $\gamma$ of $\Gamma_+(f)$, $f$ always admits the $\gamma$-part near the origin and $f_\gamma(x)$ equals the polynomial $\sum_{\alpha \in \gamma \cap \mathbb{Z}_+^n} c_\alpha x^\alpha$, which coincides with the definition of the $\gamma$-part of $f$ in [17].

(viii) Let $\gamma$ be a noncompact face of $\Gamma_+(f)$. If $f$ admits the $\gamma$-part $f_\gamma$ on $U$, then the Taylor series at the origin of $f_\gamma$ is $\sum_{\alpha \in \gamma \cap \mathbb{Z}_+^n} c_\alpha x^\alpha$.

(ix) If $f$ is real analytic near the origin and $\gamma$ is a face of $\Gamma_+(f)$, then there exists an open neighborhood $U$ of the origin such that $f$ admits the $\gamma$-part $f_\gamma$ on $U$ and, moreover, $f_\gamma$ is equal to a convergent power series $\sum_{\alpha \in \gamma \cap \mathbb{Z}_+^n} c_\alpha x^\alpha$ on $U$.

## § 2.4. The classes $\hat{\mathcal{E}}[P](U)$ and $\hat{\mathcal{E}}(U)$

Let $P \subset \mathbb{R}_+^n$ be a polyhedron (possibly an empty set) satisfying $P + \mathbb{R}_+^n \subset P$ if $P \neq \emptyset$ and $U$ an open neighborhood of the origin. Denote by $\mathcal{E}[P](U)$ the set of $C^\infty$ functions defined on $U$ whose Newton polyhedra are contained in $P$. Moreover, when $P \neq \emptyset$, we denote by $\hat{\mathcal{E}}[P](U)$ the set of the elements $f$ of $\mathcal{E}[P](U)$ for which $f$ admits the $\gamma$-part on $U$ for any face $\gamma$ of $P$. We set $\hat{\mathcal{E}}[\emptyset](U) = \{0\}$, i.e., the set consisting of only the function identically equaling zero on $U$. We define

$$\hat{\mathcal{E}}(U) = \{f \in C^\infty(U); f \in \hat{\mathcal{E}}[\Gamma_+(f)](U)\}.$$

*Remark* 2. In the definition of $\hat{\mathcal{E}}[P](U)$, "any face" can be replaced by "any noncompact facet".

*Remark* 3. The class $\hat{\mathcal{E}}(U)$ contains many kinds of $C^\infty$ functions. Here $U$ is some open neighborhood of the origin.

(i) The function identically equaling zero on $U$ is contained in $\hat{\mathcal{E}}(U)$. (This easily follows from the definition.)

(ii) Every real analytic function defined on $U$ belongs to $\hat{\mathcal{E}}(U)$. (From Remark 1 (ix).)

(iii) Every convenient $C^\infty$ function defined on $U$ belongs to $\hat{\mathcal{E}}(U)$. (From Remark 1 (iv),(vii).)

(iv) In the one-dimensional case, every $C^\infty$ function defined on $U$, whose Newton polyhedron is nonempty, belongs to $\hat{\mathcal{E}}(U)$. (This is a particular case of the above (iii).)

(v) The Denjoy-Carleman classes are contained in $\hat{\mathcal{E}}(U)$.

More detailed properties of the classes $\hat{\mathcal{E}}[P](U)$ and $\hat{\mathcal{E}}(U)$ are investigated in [10].

*Remark* 4. The classes $\hat{\mathcal{E}}[P](U)$ and $\hat{\mathcal{E}}(U)$ are useful for the investigation of the behavior of *weighted* oscillatory integrals:

$$\tilde{I}(t;\varphi) = \int_{\mathbb{R}^n} e^{itf(x)} g(x)\varphi(x)dx,$$

where $f, \varphi$ are the same as in (1.1) and $g$ is a *weight* function satisfying some conditions (see [11],[14]).

## § 2.5. Examples

Let us consider the following two-dimensional example.

$$\begin{aligned} &f(x) = f(x_1,x_2) = x_1^2 x_2^2 + x_1^k e^{-1/x_2^2} \quad k \in \mathbb{Z}_+; \\ &P = \{(\alpha_1,\alpha_2) \in \mathbb{R}_+^2; \alpha_1 \geq 1,\ \alpha_2 \geq 1\}. \end{aligned} \tag{2.2}$$

Here, the value of $e^{-1/x_2^2}$ at $x_2 = 0$ is defined by 0. The set of the proper faces of $\Gamma_+(f)$ and $P$ consists of $\gamma_1, \gamma_2, \gamma_3$ and $\tau_1, \tau_2, \tau_3$, where

$$\begin{aligned} &\gamma_1 = \{(2,\alpha_2); \alpha_2 \geq 2\},\ \gamma_2 = \{(2,2)\},\ \gamma_3 = \{(\alpha_1,2); \alpha_1 \geq 2\}, \\ &\tau_1 = \{(1,\alpha_2); \alpha_2 \geq 1\},\ \tau_2 = \{(1,1)\},\ \tau_3 = \{(\alpha_1,1); \alpha_1 \geq 1\}. \end{aligned} \tag{2.3}$$

It is easy to see that $f$ admits the $\gamma_j$-part and $\tau_j$-part, $j = 2, 3$, near the origin and they are written as $f_{\gamma_2}(x) = f_{\gamma_3}(x) = x_1^2 x_2^2$ and $f_{\tau_2}(x) = f_{\tau_3}(x) \equiv 0$. Consider the $\gamma_1$-part and $\tau_1$-part of $f$. The situation depends on the parameter $k$ as follows.

- $(k = 0)$ $f_{\gamma_1}$ and $f_{\tau_1}$ cannot be defined.
- $(k = 1)$ $f_{\gamma_1}$ cannot be defined but $f_{\tau_1}(x) = x_1 e^{-1/x_2^2}$.
- $(k = 2)$ $f_{\gamma_1}(x) = f(x)$ and $f_{\tau_1}(x) \equiv 0$.
- $(k \geq 3)$ $f_{\gamma_1}(x) = x_1^2 x_2^2$ and $f_{\tau_1}(x) \equiv 0$.

From the above, we see that $f \in \hat{\mathcal{E}}(U)$ if and only if $k \geq 2$; $f \in \hat{\mathcal{E}}[P](U)$ if and only if $k \geq 1$. Notice the case of $k = 1$; $\tau_1 \cap \Gamma_+(f) = \emptyset$ but $f_{\tau_1}(x) = x_1 e^{-1/x_2^2} \not\equiv 0$ (see Remark 1 (v)).

## § 3. Main results

Let us explain our results relating to the behavior of the oscillatory integral $I(\tau)$ in (1.1) as $\tau \to +\infty$.

Throughout this section, the functions $f$, $\varphi$ satisfy the following conditions:

(A) $f : U \to \mathbb{R}$ is a $C^\infty$ function satisfying that $f(0) = 0$, $\nabla f(0) = (0, \ldots, 0)$ and $\Gamma_+(f) \neq \emptyset$, where $U$ is an open neighborhood of the origin in $\mathbb{R}^n$;

(B) $\varphi : \mathbb{R}^n \to \mathbb{C}$ is a $C^\infty$ function whose support is contained in $U$.

A $C^\infty$ function $f$ is said to be *nondegenerate* over $\mathbb{R}$ with respect to the Newton polyhedron $\Gamma_+(f)$ if for every compact face $\gamma$ of $\Gamma_+(f)$, the polynomial $f_\gamma$ satisfies

$$\nabla f_\gamma = \left( \frac{\partial f_\gamma}{\partial x_1}, \ldots, \frac{\partial f_\gamma}{\partial x_n} \right) \neq (0, \ldots, 0) \quad \text{on the set } \{x \in \mathbb{R}^n; x_1 \cdots x_n \neq 0\}.$$

As mentioned in the Introduction, it is known that, if $f$ is real analytic near the origin, then the oscillatory integral (1.1) has an asymptotic expansion of the form (1.2). More precisely, Varchenko in [17] and Arnold, Gusein-Zade and Varchenko [1] obtained the following results.

**Theorem 3.1** ([17],[1]). *If $f$ is real analytic on $U$ and is nondegenerate over $\mathbb{R}$ with respect to its Newton polyhedron, then we have the following.*

*(i) The progression $\{\alpha\}$ in (1.2) belongs to finitely many arithmetic progressions, which are obtained by using the theory of toric varieties based on the geometry of the Newton polyhedron $\Gamma_+(f)$;*

*(ii) $\beta(f) \leq -1/d(f)$;*

*(iii) If at least one of the following conditions is satisfied:*

(*a*) $d(f) > 1$;

(*b*) $f$ *is nonnegative or nonpositive on* $U$;

(*c*) $1/d(f)$ *is not an odd integer and* $f_{\tau_0}$ *does not vanish on* $U \cap (\mathbb{R} \setminus \{0\})^n$,

*then* $\beta(f) = -1/d(f)$ *and* $\eta(f) = m(f)$.

Now, let us explain our results. They need the following condition:

(C) $f$ belongs to the class $\hat{\mathcal{E}}(U)$ and is nondegenerate over $\mathbb{R}$ with respect to its Newton polyhedron.

**Theorem 3.2.** *If* $f$ *satisfies the condition* (*C*) *and the support of* $\varphi$ *is contained in a sufficiently small neighborhood of the origin, then we have*

$$I(\tau) \sim \sum_{\alpha} \sum_{k} C_{\alpha k}(\varphi) \tau^{\alpha} (\log \tau)^{k-1} \quad \text{as } \tau \to +\infty, \tag{3.1}$$

*where* $\{\alpha\}$ *belongs to the same progressions as in the case of* $f_0$, *which is the principal part of* $f$. (*Since* $f_0$ *is a polynomial, the precise progressions in the case of* $f_0$ *can be constructed by using toric resolution as in [17].*)

From the above theorem, for a given $f$ satisfying the condition (C), the oscillation index $\beta(f)$ and its multiplicity $\eta(f)$ can be defined in a similar fashion to the real analytic case as in the Introduction.

The following theorem shows that the oscillation index $\beta(f)$ can be accurately estimated by using the Newton distance $d(f)$.

**Theorem 3.3.** *If* $f$ *satisfies the condition* (*C*) *and the support of* $\varphi$ *is contained in a sufficiently small neighborhood of the origin, then there exists a positive constant* $C(\varphi)$ *depending on* $\varphi$ *but being independent of* $\tau$ *such that*

$$|I(\tau)| \leq C(\varphi) \tau^{-1/d(f)} (\log \tau)^{m(f)-1} \quad \text{for } \tau \geq 1.$$

*This implies* $\beta(f) \leq -1/d(f)$.

*Remark* 5. The above theorem is not only a generalization to (ii) in Theorem 3.1 but also is a slightly stronger result even if $f$ is real analytic. Indeed, from the argument in [17],[1], we obtain the estimate $|I(\tau)| \leq C(\varphi)\tau^{-1/d(f)}(\log \tau)^{m(f)}$ for $\tau \geq 1$, but more delicate computation of coefficients in the asymptotic expansion (3.1) can improve this.

Next, let us consider the case that the equations $\beta(f) = -1/d(f)$ and $\eta(f) = m(f)$ hold.

**Theorem 3.4.** *If $f$ satisfies the condition $(C)$ and at least one of the following three conditions is satisfied:*

*(a)* $d(f) > 1$*;*

*(b)* $f$ *is nonnegative or nonpositive on* $U$*;*

*(c)* $1/d(f)$ *is not an odd integer and* $f_{\tau_0}$ *does not vanish on* $U \cap (\mathbb{R} \setminus \{0\})^n$*,*

*then the equations* $\beta(f) = -1/d(f)$ *and* $\eta(f) = m(f)$ *hold.*

*Outline of proofs of the above theorems.* By careful investigation of the properties of functions satisfying the condition (C), we can construct a toric resolution of singularities for these functions. Once this resolution is obtained, the assertions in the theorems can be shown through the arguments in Varchenko [17].

In [10], we give not only complete proofs but also more detailed results about the behavior of oscillatory integrals under the same hypotheses.

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