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Asymptotic behavior of solutions to a quadratic nonlinear Schrödinger system with mass resonance

By

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Abstract

We consider the asymptotic behavior of a solution to the quadratic nonlinear Schrödinger system with gauge invariant form. The system arises in the model of the nonlinear optics describes a nonlinear interaction between the laser beam and plasma waves. We show the various aspects of this particular system and show the finite time blow up for $4 \le n \le 6$ and the scattering behavior of the solution in two dimensions.

§ 1. Introduction

We consider the following system of the quadratic nonlinear Schrödinger equation:

$$\begin{cases} i\partial_t u_1 + \dfrac{1}{2m_1}\Delta u_1 = \overline{u}_1 u_2, & t \in \mathbb{R},\ x \in \mathbb{R}^n, \\ i\partial_t u_2 + \dfrac{1}{2m_2}\Delta u_2 = u_1^2, & t \in \mathbb{R},\ x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $u_j = u_j(t,x) : \mathbb{R}\times\mathbb{R}^n \to \mathbb{C}$ $(j = 1, 2)$ are the unknown functions, $\overline{u}_1$ stands for the complex conjugate of u_1 and m_1 and m_2 are positive constants. The quadratic nonlinear Schrödinger system is a simplified equation derived from the interaction model between the laser and plasma and it models the nonlinear interaction known as the Raman amplification phenomenon. The original model is described by the following modified Zakharov system (cf. Colin-Colin-Ohta [4]):

$$\begin{cases} \big(i\partial_t + ik\partial_z + k\Delta_\perp + \partial_z^2\big)A = -(\nabla\cdot E)\overline{A}e^{-i\theta}, \\ \big(i\partial_t + k\Delta_\perp + \partial_z^2\big)E = \nabla(A^2 e^{i\theta}), \end{cases}$$

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where $A=(A_1,A_2,A_3)$ is the complex amplitude for incident laser field, $E=(E_1,E_2,E_3)$ denotes the complex electronic field, k is a large parameter which stands for the leading wave number, $\theta = kz-\omega t$ and the two dimensional Laplacian $\Delta_\perp = \partial_x^2+\partial_y^2$. We invoke the dispersion relation $\omega = k^2$ and introduce a new unknown function $F=Ee^{-i\theta}$ and a parameter vector $K=(0,0,k)$ to reduce the system into the following:

$$\begin{cases} \big(i\partial_t + ik\partial_z + k\Delta_\perp + \partial_z^2\big)A = -ikF_3\overline{A} - (\nabla\cdot F)\overline{A}, \\ \big(i\partial_t + \omega + 2ik\partial_z + k\Delta_\perp + \partial_z^2 - k^2\big)F = iKA^2 + \nabla(A^2). \end{cases}$$

To see the envelope for the highly oscillating part, we divide the both side of the system by k and we formally take the limit $k\to\infty$ to have

$$\begin{cases} (i\partial_z + \Delta_\perp)A = -iF_3\overline{A}, \\ (2i\partial_z + \Delta_\perp)F = i\boldsymbol{e}_3A^2, \end{cases}$$

where e_3 is the third component of the three dimensional vector. By a trivial change of the coefficient of the unknown function, we obtain the simplified version of the system.

In this paper, we consider the asymptotic behavior of a solution to the quadratic nonlinear Schrödinger system (1.1), in particular, the finite time blow-up and the scattering problem. It is now worth to compare the problem to the existing known results with the single nonlinear Schrödinger equation of gauge invariant nonlinearity:

$$\begin{cases} i\partial_t u + \dfrac{1}{2}\Delta u = \lambda|u|^{\alpha-1}u, \quad t\in(-T,T),\ x\in\mathbb{R}^n, \\ \qquad u(0,x) = u_0(x), \qquad\qquad x\in\mathbb{R}^n, \end{cases} \tag{1.2}$$

Ginibre-Velo [7] showed the local well-posedness for (1.2) for $u_0\in H^1(\mathbb{R}^n)$ when $1<\alpha<1+4/(n-2)$. Besides the solution $u(t)$ satisfies the following conservation laws:

$$\begin{aligned} \|u(t)\|_{L^2} &= \|u_0\|_{L^2}, \\ E\big(u(t)\big) \equiv &\|\nabla u(t)\|_{L^2}^2 - \frac{2}{\alpha+1}\|u(t)\|_{L^{\alpha+1}}^{\alpha+1} = E(u_0). \end{aligned} \tag{1.3}$$

Y. Tsutsumi [30] showed the local well-posedness for (1.2) for $u_0\in L^2(\mathbb{R}^n)$ when $1<\alpha<1+4/n$ and the L^2 conservation laws. The L^2 critical case $\alpha=1+4/n$ and the H^1 critical case $\alpha = 1+4/(n-2)$ was shown by Cazenave-Weissler [3]. Existence of the global solution and scattering theory as well as the finite time blow up results are heavily depends on the conservation laws and the invariance under some transform: It is well understood that the simpler model (1.2) has the following invariance:

(1) (the Galilei invariance) If $u(t,x)$ solves (1.2), then so does

$$v(t,x) = u(t,x-pt)e^{i(x\cdot p - |p|^2\frac{t}{2})} \tag{1.4}$$

where $p\in\mathbb{R}^n$.

(2) (the Pseudo conformal invariance): If $u(t,x)$ solves (1.2), then so does

$$v(t,x) = e^{\frac{i|x|^2}{2t}} t^{-\frac{n}{2}} u\left(-\frac{1}{t}, \frac{x}{t}\right) \tag{1.5}$$

when $\alpha = 1 + 4/n$.

The corresponding invariant transformation to the system (1.1) holds when the mass parameters m_1 and m_2 have a special (resonance) relation $2m_1 = m_2$:

(1) (Galilei transform) If $(u_1(t,x), u_2(t,x))$ solves (1.1), then so does

$$\begin{cases} u_{1g}(t,x) = u_1(t, x - tp_1) e^{i(x\cdot p_1 - |p_1|^2 \frac{t}{2m_1})}, \\ u_{2g}(t,x) = u_2(t, x - tp_2) e^{i(x\cdot p_2 - |p_2|^2 \frac{t}{2m_2})}, \end{cases} \tag{1.6}$$

where $p_1, p_2 \in \mathbb{R}^n$ denotes the moment parameter satisfying $2p_1 = p_2$.

(2) (Pseudo conformal transform) When $n = 4$ and $2m_1 = m_2$, let $(u_1(t,x), u_2(t,x))$ solves (1.1), then so does

$$\begin{cases} u_{1p}(t,x) = e^{\frac{im_1|x|^2}{2t}} t^{-\frac{n}{2}} u_1\left(-\frac{1}{t}, \frac{x}{t}\right), \\ u_{2p}(t,x) = e^{\frac{im_2|x|^2}{2t}} t^{-\frac{n}{2}} u_2\left(-\frac{1}{t}, \frac{x}{t}\right). \end{cases} \tag{1.7}$$

Regarding those invariance, the system (1.1) has an analogous feature to the single gauge invariant nonlinear Schrödinger equation (1.2). Along with this idea, Hayashi-Ozawa-Tanaka [16] showed the local well-posedness result for (1.1), if $4 \le n \le 6$, then for any $(u_{10}, u_{20}) \in H^1(\mathbb{R}^n)^2$ and the solution (u_1, u_2) satisfies the following conservation laws:

$$\begin{aligned} &\|u_1(t)\|_{L^2}^2 + \|u_2(t)\|_{L^2}^2 = \|u_{10}\|_{L^2}^2 + \|u_{20}\|_{L^2}^2, \\ E(u,v)(t) \equiv &\frac{1}{m_1}\|\nabla u_1(t)\|_{L^2}^2 + \frac{1}{2m_2}\|\nabla u_2(t)\|_{L^2}^2 + 2\mathrm{Re}\int_{\mathbb{R}^n} u_1^2 \overline{u}_2 dx = E(u_{10}, u_{20}). \end{aligned} \tag{1.8}$$

They showed the similar well-posedness result in L^2 of (1.1) in the same paper [16].

Finally, we close this section by giving some notations used in this paper. We denote by $\mathcal{F}\phi$ or $\hat{\phi}$ the Fourier transform of ϕ. We introduce the free Schrödinger evolution group

$$U_j(t) = e^{\frac{it}{2m_j}\Delta}$$

which generates the L^2 isometry group and it is known the factorization into the dilation operator $D_j(t)$ and the multiplication operator $M_j(t) = e^{\frac{im_j|x|^2}{2t}}$ to have

$$U_j(t) = M_j(t) D_j(t) \mathcal{F} M_j(t),$$

where

$$D_j(t)\phi(x) = \frac{m_j}{it}\phi\left(\frac{m_j x}{t}\right).$$

§ 2. Finite time blow-up

In this section, we consider the system (1.1) imposing the initial data.

$$\begin{cases} i\partial_t u_1 + \dfrac{1}{2m_1}\Delta u_1 = \overline{u}_1 u_2, & t \in [-T,T],\ x \in \mathbb{R}^n, \\ i\partial_t u_2 + \dfrac{1}{2m_2}\Delta u_2 = u_1^2, & t \in [-T,T],\ x \in \mathbb{R}^n, \\ u_1(0,x) = u_{10}(x), \quad u_2(0,x) = u_{20}(x), & x \in \mathbb{R}^n \end{cases} \tag{2.1}$$

It is well known that when $\lambda = -1$ and $\alpha \geq 1+4/n$, there exist solutions of (1.2) blowing up in finite time for certain initial data (Glassey [8], Weinstein [32]). Above results are proved based on the lack of the positivity of the energy $E(u_0)$. Their results are proved in the framework of $H^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n; |x|^2dx)$ to use the previous results are based on the pseudo conformal conservation law that satisfied by the solution in $H^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n; |x|^2dx)$. Ogawa-Tsutsumi [22] removed the assumption that the solution in $L^2(\mathbb{R}^n; |x|^2dx)$ for the radially symmetric solution when $n \geq 2$, and they extended the result for blow-up of solutions of (1.2) without radially symmetry when $n = 1$ (see also [23]). Hayashi-Ozawa-Tanaka [16] showed that there exist solutions of (1.1) blowing up in finite time for certain initial data under the condition $2m_1 = m_2$ in the framework of $H^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n; |x|^2dx)$. We are able to prove the blow-up result for (1.1) based on the idea of Ogawa-Tsutsumi [22].

Theorem 2.1. *Let $4 \leq n \leq 6$, $2m_1 = m_2$. Suppose that $(u_{10}, u_{20}) \in H^1(\mathbb{R}^n)^2$: radially symmetric, $E(u_{10}, u_{20}) < 0$. Then the maximal existence time for solution (u_1, u_2) of (2.1) is finite.*

Remark. In the case of non-radially symmetric solution, we are able to prove grow-up of the solution to (1.1) based on a variation of virial identity (cf. Nawa [21]).

The proof of the Theorem 2.1 is based on the following lemma and modified virial identity.

Lemma 2.2. *Let $n \geq 2$ and u be a radially symmetric function in $H^1(\mathbb{R}^n)$. Then for any $R > 0$, u satisfies*

$$\|u\|_{L^\infty(R<r)} \leq CR^{-(n-1)/2}\|u\|_{L^2(R<r)}^{1/2}\|\nabla u\|_{L^2(R<r)}^{1/2}$$

where $r = |x|$ and C is a constant independent of u and R.

Proposition 2.3 (modified virial identity). *Let $4 \le n \le 6$. Suppose that Ψ is a vector valued function in $(W^{3,\infty}(\mathbb{R}^n))^n$. Then the H^1 solution $(u_1(t), u_2(t))$ of* (2.1) *satisfies*

$$
\begin{aligned}
&\mathrm{Im}\int_{\mathbb{R}^n} u_{10}\Psi\cdot\nabla\overline{u}_{10}dx + \frac{1}{2}\mathrm{Im}\int_{\mathbb{R}^n} u_{20}\Psi\cdot\nabla\overline{u}_{20}dx\\
&- \mathrm{Im}\int_{\mathbb{R}^n} u_1(t)\Psi\cdot\nabla\overline{u}_1(t)dx - \frac{1}{2}\mathrm{Im}\int_{\mathbb{R}^n} u_2(t)\Psi\cdot\nabla\overline{u}_2(t)dx\\
&= \int_0^t \Big\{ -\frac{1}{4m_1}\mathrm{Re}\int_{\mathbb{R}^n}\Delta(\nabla\cdot\Psi)|u_1|^2dx - \frac{1}{8m_2}\mathrm{Re}\int_{\mathbb{R}^n}\Delta(\nabla\cdot\Psi)|u_2|^2dx\\
&\qquad + \frac{1}{m_1}\mathrm{Re}\int_{\mathbb{R}^n}\partial_j\Psi_i\partial_j u_1\partial_i\overline{u}_1dx + \frac{1}{2m_2}\mathrm{Re}\int_{\mathbb{R}^n}\partial_j\Psi_i\partial_j u_2\partial_i\overline{u}_2dx\\
&\qquad\quad + \frac{1}{2}\mathrm{Re}\int_{\mathbb{R}^n}\nabla\cdot\Psi u_1^2\overline{u}_2dx\Big\}d\tau.
\end{aligned}
$$

Here, we give a short proof of Theorem 2.1 based on the idea of Ogawa-Tsutsumi [22].

Proof of Theorem 2.1. We assume that the solution $(u_1(t), u_2(t))$ exists globally in time and derive a contradiction. Suppose that $\phi \in C^3([0,\infty))$, we take a cut-off function ϕ as follows:

$$
\phi(r) \equiv \begin{cases} r, & 0 \le r < 1,\\ r-(r-1)^3, & 1 \le r < 1+\dfrac{\sqrt{3}}{3},\\ \text{smooth},\quad \phi' < 0, & 1+\dfrac{\sqrt{3}}{3} \le r < 2,\\ 0, & 2 \le r. \end{cases} \tag{2.2}
$$

Let $R > 0$ is a large constant to be determined later. We put for $R > 0$,

$$
\Psi(x) = \Psi_R(x) = \frac{x}{r}\phi_R(x) = R\frac{x}{r}\phi\left(\frac{r}{R}\right).
$$

We see that

$$
\begin{aligned}
&\mathrm{Im}\int_{\mathbb{R}^n} u_{10}\Psi_R\cdot\nabla\overline{u}_{10}dx+\frac{1}{2}\mathrm{Im}\int u_{20}\Psi_R\cdot\nabla\overline{u}_{20}dx\\
&\quad-\mathrm{Im}\int u_1(t)\Psi_R\cdot\nabla\overline{u}_1(t)dx-\frac{1}{2}\mathrm{Im}\int_{\mathbb{R}^n} u_2(t)\Psi_R\cdot\nabla\overline{u}_2(t)dx\\
&=\int_0^t\bigg\{-\int_{R<r}\sigma(r)\left(\frac{1}{4m_1}|u_1|^2+\frac{1}{8m_2}|u_2|^2\right)dx\\
&\qquad+\frac{1}{m_1}\int_{R<r}|\nabla u_1|^2dx+\frac{1}{m_1}\int_{R<r<2R}\phi_R'(r)|\nabla u_1|^2dx\\
&\qquad+\frac{1}{2m_2}\int_{R<r}|\nabla u_2|^2dx+\frac{1}{2m_2}\int_{R<r<2R}\phi_R'|\nabla u_2|^2dx\\
&\qquad+\frac{n}{2}\mathrm{Re}\int_{r<R}u_1^2\overline{u}_2dx+\frac{1}{2}\mathrm{Re}\int_{R<r<2R}\left(\phi_R'(r)+\frac{n-1}{r}\phi_R(r)\right)u_1^2\overline{u}_2dx\bigg\}d\tau\\
&=\int_0^t\bigg\{2E_0-\frac{1}{m_1}\int_{R<r}a(r)|\nabla u_1|^2dx-\frac{1}{2m_2}\int_{R<r}a(r)|\nabla u_2|^2dx\\
&\qquad+\frac{1}{2}\mathrm{Re}\int_{R<r}b(r)u_1^2\overline{u}_2dx-\int_{R<r}\sigma(r)\left(\frac{1}{4m_1}|u_1|^2+\frac{1}{8m_2}|u_2|^2\right)dx\bigg\}d\tau,
\end{aligned}
$$

where we set

$$
a(r)\equiv\begin{cases}1-\phi_R'(r), & R\le r<2R,\\ 1, & 2R<r,\end{cases}
$$

and for $4\le n\le 6$

$$
b(r)\equiv\begin{cases}n-\phi_R'(r)-\dfrac{n-1}{r}\phi_R(r), & R\le r<2R,\\ n, & 2R<r,\end{cases}
$$

and

$$
\sigma(r)\equiv\phi_R'''(r)+(n-1)\left\{\frac{2}{r}\phi_R''(r)+\frac{n-1}{r^2}\phi_R'(r)-\frac{n-3}{r^3}\phi_R(r)\right\},\quad R<r<2R.
$$

Using Lemma 2.2, we have for $\varepsilon > 0$,

$$\begin{aligned}
&\mathrm{Re}\int_{R<r} b(r)u_1^2\overline{u}_2dx \\
&\le \|u_1\|^2_{L^2(R<r)}\|bu_2\|_{L^\infty(R<r)} \\
&\le CR^{-\frac{n-1}{2}}\|u_1\|^2_{L^2}\|bu_2\|^{1/2}_{L^2}\left(\int_{R<r}|\nabla(bu_2)|^2dx\right)^{1/4} \\
&\le \varepsilon\int_{R<r}|\nabla(bu_2)|^2dx + C(\varepsilon)\left(R^{-\frac{n-1}{2}}\|u\|^2_{L^2}\|bu_2\|^{1/2}_{L^2}\right)^{\frac{4}{3}} \\
&\le \varepsilon\int_{R<r} b^2|\nabla u_1|^2dx + \varepsilon\int_{R<r}|u_2|^2|\nabla b|^2dx \\
&\quad + C(\varepsilon)R^{-\frac{2(n-1)}{3}}\|u_1\|^{\frac{8}{3}}_{L^2}\|bu_2\|^{\frac{2}{3}}_{L^2}.
\end{aligned}$$

Since

$$|\nabla b(r)| \le \frac{C_1}{R} \text{ and } |\sigma(r)| \le \frac{C_2}{R^2}, \quad R < r,$$

we obtain

$$\begin{aligned}
&\mathrm{Im}\int_{\mathbb{R}^n} u_{10}\Psi_R\cdot\nabla\overline{u}_{10}dx + \frac{1}{2}\mathrm{Im}\int_{\mathbb{R}^n} u_{20}\Psi_R\cdot\nabla\overline{u}_{20}dx \\
&- \mathrm{Im}\int_{\mathbb{R}^n} u_1(t)\Psi_R\cdot\nabla\overline{u}_1(t)dx - \frac{1}{2}\mathrm{Im}\int_{\mathbb{R}^n} u_1(t)\Psi_R\cdot\nabla\overline{u}_2(t)dx \\
\le& \int_0^t\Bigg\{2E_0 - \frac{1}{m_1}\int_{R<r} a(r)|\nabla u_1|^2dx + \int_{R<r}\left(\varepsilon b^2(r) - \frac{a(r)}{2m_1}\right)|\nabla u_2|^2dx \\
&\quad + \left(C_1R^{-2}\varepsilon\|u_2\|^2_{L^2} + C(\varepsilon)R^{-\frac{2(n-1)}{3}}\|u_1\|^{\frac{8}{3}}_{L^2}\|bu_2\|^{\frac{2}{3}}_{L^2} + C_2R^{-2}\left(\|u_1\|^2_{L^2} + \|u_2\|^2_{L^2}\right)\right\}d\tau.
\end{aligned}$$

We show that for sufficiently small $\varepsilon > 0$;

$$\varepsilon b^2(r) - \frac{a(r)}{2m_2} \le 0, \quad R < r. \tag{2.3}$$

For $\left(1 + \frac{\sqrt{3}}{3}\right)R < r$,

$$b(r) < C, \quad \text{and} \quad a(r) > 1,$$

where $c > 0$ is independent of R. For $R < r < \left(1 + \frac{\sqrt{3}}{3}\right)R$,

$$\varepsilon b^2(r) - a(r) = \varepsilon\left(\frac{3(r-R)^2}{R^2}\left(1 + \frac{n-1}{3\sqrt{3}}\right)\right)^2 - \frac{3(r-R)^2}{R^2}$$

Since

$$\frac{3(r-R)^2}{R^2} < 1, \quad R < r < \left(1 + \frac{\sqrt{3}}{3}\right)R,$$

choosing

$$\varepsilon < \left(1 + \frac{n-1}{3\sqrt{3}}\right)^{-2},$$

we obtain (2.3).

Hence by (2.3) and choosing $R > 0$ sufficiently large, we obtain for some $\eta > 0$,

$$\begin{aligned} &2E_0 - \frac{1}{m_1}\int_{R<r} a(r)|\nabla u_1|^2 dx + \int_{R<r}\left(\varepsilon b^2(r) - \frac{a(r)}{2m_1}\right)|\nabla u_2|^2 dx \\ &\quad + \left(C_1 R^{-2}\varepsilon\|u_2\|_{L^2}^2 + C(\varepsilon)R^{-\frac{2(n-1)}{3}}\|u_1\|_{L^2}^{\frac{8}{3}}\|bu_2\|_{L^2}^{\frac{2}{3}} + C_2R^{-2}\left(\|u_1\|_{L^2}^2 + \|u_2\|_{L^2}^2\right)\right\} \\ &\geq -\eta. \end{aligned}$$

Therefore we have

$$\begin{aligned} &\mathrm{Im}\int_{\mathbb{R}^n} u_{10}\Psi_R\cdot\nabla\overline{u}_{10}dx + \frac{1}{2}\mathrm{Im}\int_{\mathbb{R}^n} u_{20}\Psi_R\cdot\nabla\overline{u}_{20}dx + \eta t \\ &\leq \mathrm{Im}\int_{\mathbb{R}^n} u_1(t)\Psi_R\cdot\nabla\overline{u}_1(t)dx + \frac{1}{2}\mathrm{Im}\int_{\mathbb{R}^n} u_2(t)\Psi_R\cdot\nabla\overline{u}_2(t)dx. \end{aligned} \tag{2.4}$$

Put

$$\Phi(r) = \int_0^r \phi_R(s)ds.$$

Since $\Phi \in L^\infty(\mathbb{R}^n)$ and

$$\nabla\Phi(r) = \frac{x}{r}\cdot\phi_R(r) = \Psi_R(r).$$

From (1.1), we see that

$$i\int_{\mathbb{R}^n}\Phi u_1\partial_t\overline{u}_1 dx = \frac{1}{2m_1}\int_{\mathbb{R}^n}\nabla\Phi u_1\cdot\nabla\overline{u}_1dx - \frac{1}{2m_1}\int_{\mathbb{R}^n}\Phi|\nabla u_1|^2dx - \int_{\mathbb{R}^n}\Phi u_1^2\overline{u}_2dx$$

and

$$i\int_{\mathbb{R}^n}\Phi u_2\partial_t\overline{u}_2 dx = \frac{1}{2m_2}\int_{\mathbb{R}^n}\nabla\Phi u_2\cdot\nabla\overline{u}_1dx - \frac{1}{2m_2}\int_{\mathbb{R}^n}\Phi|\nabla u_2|^2dx - \int_{\mathbb{R}^n}\Phi\overline{u_1}^2u_2dx.$$

By taking the imaginary part,

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^n}\Phi|u_1|dx + \frac{d}{dt}\int_{\mathbb{R}^n}\Phi|u_2|dx \\ &\quad = 2\mathrm{Re}\int_{\mathbb{R}^n}\Phi u_1\overline{u}_{1t}dx + 2\mathrm{Re}\int_{\mathbb{R}^n}\Phi u_2\overline{u}_{2t}dx \\ &\quad = -\frac{1}{m_1}\mathrm{Im}\int_{\mathbb{R}^n}\nabla\Phi u_1\cdot\nabla\overline{u}_1dx - \frac{1}{m_2}\mathrm{Im}\int_{\mathbb{R}^n}\nabla\Phi u_2\cdot\nabla\overline{u}_2dx. \end{aligned}$$

Using the relation $2m_1 = m_2$, we obtain

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^n}\Phi|u_1|dx + \frac{d}{dt}\int_{\mathbb{R}^n}\Phi|u_2|dx \\ &\quad = -\frac{1}{m_1}\left(\mathrm{Im}\int_{\mathbb{R}^n}\Psi_R u_1\cdot\nabla\overline{u}_1dx + \frac{1}{2}\mathrm{Im}\int_{\mathbb{R}^n}\Psi_R u_2\cdot\nabla\overline{u}_2dx\right). \end{aligned}$$

Noting

$$\mathrm{Im}\int_{\mathbb{R}^n}\Phi u_1^2\overline{u}_2dx=-\mathrm{Im}\int_{\mathbb{R}^n}\Phi\overline{u}_1^2u_2dx,$$

we obtain by (2.4)

$$\begin{aligned}&\int_{\mathbb{R}^n}\Phi|u_1|dx+\int_{\mathbb{R}^n}\Phi|u_2|dx\\ \le&-\frac{\eta}{2m_1}t^2-\frac{t}{m_1}\left(\mathrm{Im}\int_{\mathbb{R}^n}\Psi_R u_{10}\cdot\nabla\overline{u}_{10}dx+\frac{1}{2}\mathrm{Im}\int_{\mathbb{R}^n}\Psi_R u_{20}\cdot\nabla\overline{u}_{20}dx\right)\\ &+\int_{\mathbb{R}^n}\Phi|u_{10}|^2dx+\int_{\mathbb{R}^n}\Phi|u_{10}|^2dx.\end{aligned}\tag{2.5}$$

Therefore the left hand side of (2.5) becomes negative in finite time, that implies a contradiction, since $\Phi(r)>0$ except when $r=0$. Hence the solution of (2.1) must blow up in finite time. □

§ 3. Scattering result

In this section, we consider the scattering problem for the nonlinear Schrödinger system (1.1) imposing the final state condition.

$$\begin{cases} i\partial_t u_1+\dfrac{1}{2m_1}\Delta u_1=\overline{u}_1u_2, & t\in\mathbb{R},\ x\in\mathbb{R}^2,\\ i\partial_t u_2+\dfrac{1}{2m_2}\Delta u_2=u_1^2, & t\in\mathbb{R},\ x\in\mathbb{R}^2,\\ (u_1(t),u_2(t))\to(u_{1a}(t),u_{2a}(t)) & \text{as } t\to\infty \text{ in } L^2.\end{cases}\tag{3.1}$$

In the case of $1<\alpha\le 1+\frac{2}{n}$, Barab [1] showed that a smooth global solution u for (1.2) does not converge of a solution to a free Schrödinger equation as $t\to\pm\infty$. In the case of $1+\frac{2}{n}<\alpha<1+\frac{4}{n}$, Tsutsumi-Yajima [31] showed that the solution for (1.2) behaves like a solution to the free Schrödinger equation. From these results, we see that the exponent $\alpha=1+\frac{2}{n}$ is a border line between the existence of the scattering state or not. Therefore, in two dimensional case the system (1.1) is critical situation to consider the scattering problem. Hayashi-Li-Naumkin [10] showed that either of the following three cases occurs depending on the relation between m_1 and m_2, there exists an asymptotic free solution ($2m_1\ne m_2$ and $m_1\ne m_2$), there exists a modified wave operator under some condition ($2m_1=m_2$) and there exists an asymptotic free solution under some restricted condition ($m_1=m_2$). We call the condition $2m_1=m_2$ the mass resonance condition. When the mass resonance condition is satisfied, Hayashi-Li-Naumkin [10] constructed the modified wave operators under the special assumptions on both amplitude and argument of two scattering states. They assume the ratio of the argument of scattering state to the argument of another scattering state is one to two. This assumption is removable with a minor modification of the correction term.

We introduce a weighted Sobolev spaces $H^{s,m}$ defined for $s, m \in \mathbb{R}$, the generator of Galilei transformation and dilation operator $D(t)$ for $t \in \mathbb{R}$ by

$$H^{s,m} = H^{s,m}(\mathbb{R}^n) \equiv \Big\{ \phi \in \mathcal{S}' ; \|(1+|x|^2)^{\frac{m}{2}} (1-\Delta)^{\frac{s}{2}} \phi\|_{L^2} < \infty \Big\},$$

$$|J_j|^\beta(t) = U_j(t)|x|^\beta U_j(-t) = t^\beta M_j(t)(-\Delta)^{\frac{\beta}{2}} M_j(-t)$$

and

$$D(t)\phi = \frac{1}{it}\phi\left(\frac{x}{t}\right)$$

respectively. We also denote $H^{s,0}(\mathbb{R}^n) = H^s(\mathbb{R}^n)$.

Theorem 3.1. *Let $2m_1 = m_2$, $1 < \gamma < 2$. Then there exists $\varepsilon > 0$ with the following property: For any $u_{1+}, u_{2+} \in H^{0,\gamma}$ with $\arg \hat{u}_{1+}, \arg \hat{u}_{2+} \in H^\gamma$, $|\hat{u}_{1+}(\xi)| = \sqrt{2}|\hat{u}_{2+}(\xi)|$ (a.e. $\xi \in \mathbb{R}^2$) and $\|\hat{u}_{1+}\|_{H^\gamma} + \|\hat{u}_{2+}\|_{H^\gamma} \le \varepsilon$, (3.1) has a unique global solution $(u_1(t), u_2(t)) \in (C(\mathbb{R}; L^2))^2$ which satisfies the estimates*

$$\left\| u_1(t) + U_1(t)\widetilde{S_1}(t)u_{1+} \right\|_{L^2} + \left\| u_2(t) + U_2(t)\widetilde{S_2}(t)u_{2+} \right\|_{L^2} \le Ct^{-b} \tag{3.2}$$

for all $t \ge 0$, where $\frac{1}{2} < b < 1$,

$$\widetilde{S_1}(t) = \mathcal{F}^{-1} D(m_1) e^{\frac{i}{2}(\theta_0(\xi) + \sqrt{2}|\hat{u}_{1+}(\xi)|\log t)} \mathcal{F}, \quad \widetilde{S_2}(t) = \mathcal{F}^{-1} D(m_2) e^{i(\pi - \theta_0(\xi) + 2|\hat{u}_{2+}(\xi)|\log t)} \mathcal{F},$$

and $\theta_0(\xi) = -\arg \hat{u}_{1+}(\xi) + \frac{1}{2}\arg \hat{u}_{2+}(\xi)$.

Theorem 3.1 shows that it is possible to construct the modified wave operators without assuming the ratio of the argument of two scattering states. We here introduce a new angular modification to cancel out the difference between given two scattering states and it is the key to prove the Theorem 3.1. Roughly speaking, the asymptotic behavior of the system (1.1) may be determined by solutions of the following nonlinear ordinary differential equations:

$$\begin{cases} i\partial_t \varphi_1(t) = \dfrac{1}{t}\overline{\varphi}_1 \varphi_2, \\ i\partial_t \varphi_2(t) = \dfrac{1}{t}\varphi_1^2, \end{cases} \tag{3.3}$$

where φ_1 and φ_2 are complex-valued functions. As far as we know, the solutions of the system (3.3) have not been completely classified. If we had more informations about the profile equation, we would obtain detailed results about the asymptotic behavior of the nonlinear Schrödinger system (1.1). Hayashi-Li-Naumkin [10] constructed a special solution of the system (3.3) to show the existence of modified wave operators for the system (1.1) when $|\hat{u}_{1+}(\xi)| = \sqrt{2}|\hat{u}_{2+}(\xi)|$. Therefore we consider this amplitude condition is almost necessary and sufficient condition to show the existence of modified wave operators for the system (1.1) based on some formal calculations. We prove the Theorem 3.1 based on the method of the paper written by Hayashi-Naumkin [14] used for obtaining the asymptotic behavior of the nonlinear Schrödinger equation in critical case. Here we introduce some lemmas to prove Theorem 3.1.

Lemma 3.2. *Let $\beta > 1$. Then for any $\phi \in H^{\beta,0}$, we have*

$$\|\phi\|_{L^\infty} \le C\|\phi\|_{L^2}^{1-\frac{1}{\beta}}\|(-\Delta)^\beta\phi\|_{L^2}^{\frac{1}{\beta}}.$$

Lemma 3.3. *(i) Let $1 \le p, p_1, p_2, q_1, q_2, \sigma \le \infty$ with $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{p}$, and let $s > 0$. Then*

$$\|\phi\psi\|_{\dot{B}^s_{p,\sigma}} \le C\|\phi\|_{\dot{B}^s_{p_1,\sigma}}\|\psi\|_{L^{p_2}} + C\|\phi\|_{L^{q_1}}\|\psi\|_{\dot{B}^s_{q_2,\sigma}}. \tag{3.4}$$

(ii) Let $1 \le p, \sigma \le \infty$, $s > 0$, and let ψ be a real-valued function. Then

$$\|e^{i\psi}\|_{\dot{B}^s_{p,\sigma}} \le C(1+\|\psi\|_{L^\infty})^{[s]}\|\psi\|_{\dot{B}^s_{p,\sigma}}, \tag{3.5}$$

where $[s]$ means the largest integer not greater than s.

Lemma 3.4. *Let $1 < \beta < 2$, and let ψ be a real-valued function. Then the estimates are true.*

$$\|\phi e^{i\psi}\|_{\dot{H}^\beta} \le C\|\phi\|_{\dot{H}^\beta} + C\|\phi\|_{L^\infty}(1+\|\psi\|_{L^\infty})^{[\beta]}\|\psi\|_{\dot{H}^\beta}. \tag{3.6}$$

Lemma 3.4 follows from Lemma 3.3 immediately.

Proof of Theorem 3.1. We take two scattering data with $|\hat{u}_{1+}(\xi)| = \sqrt{2}|\hat{u}_{2+}(\xi)|$;

$$\hat{u}_{1+}(\xi) = |\hat{u}_{1+}(\xi)|e^{i\arg\hat{u}_{1+}(\xi)}, \quad \hat{u}_{2+}(\xi) = |\hat{u}_{2+}(\xi)|e^{i\arg\hat{u}_{2+}(\xi)}.$$

For each scattering data, we introduce the phase modification as follows:

$$e^{iS_1(t)} = e^{\frac{i}{2}(\theta_0(\xi)+\sqrt{2}|\hat{u}_{1+}(\xi)|\log t)}, \quad e^{iS_2(t)} = -e^{i(-\theta_0(\xi)+2|\hat{u}_{2+}(\xi)|\log t)},$$

where $\theta_0(\xi) \equiv -\arg\hat{u}_{1+}(\xi) + \frac{1}{2}\arg\hat{u}_{2+}(\xi)$.
Then we have

$$\begin{aligned} e^{iS_1(t)}\hat{u}_{1+} &= |\hat{u}_{1+}(\xi)|e^{i\arg\hat{u}_{1+}(\xi)}e^{\frac{i}{2}(\theta_0(\xi)+\sqrt{2}|\hat{u}_{1+}(\xi)|\log t)}, \\ e^{iS_2(t)}\hat{u}_{2+} &= -|\hat{u}_{2+}(\xi)|e^{i\arg\hat{u}_{2+}(\xi)}e^{i(-\theta_0(\xi)+2|\hat{u}_{2+}(\xi)|\log t)}. \end{aligned}$$

Multiplying both sides of (1.1) by $\mathcal{F}U_j(-t)$, we obtain

$$\begin{aligned} i\partial_t(\mathcal{F}U_1(-t)u_1) &= \mathcal{F}U_1(-t)(\overline{u}_1u_2), \\ i\partial_t(\mathcal{F}U_2(-t)u_2) &= \mathcal{F}U_2(-t)(u_1^2). \end{aligned}$$

Note that $e^{iS_1(t)}\hat{u}_{1+}$ and $e^{iS_2(t)}\hat{u}_{2+}$ satisfy the equations

$$i\partial_t(e^{iS_1(t)}\hat{u}_{1+}) = \frac{1}{t}\overline{e^{iS_1(t)}\hat{u}_{1+}}e^{iS_2(t)}\hat{u}_{2+},$$
$$i\partial_t(e^{iS_2(t)}\hat{u}_{2+}) = \frac{1}{t}(e^{iS_1(t)}\hat{u}_{1+})^2.$$

Therefore we can rewrite the first equation of (1.1) as follows;

$$\begin{aligned}
&i\partial_t\left(D\left(\frac{1}{m_1}\right)\mathcal{F}U_1(-t)u_1 - e^{iS_1(t)}\hat{u}_{1+}\right)\\
&= D\left(\frac{1}{m_1}\right)\mathcal{F}U_1(-t)\overline{u}_1u_2 - \frac{1}{t}\overline{e^{iS_1(t)}\hat{u}_{1+}}e^{iS_2(t)}\hat{u}_{2+}\\
&= D\left(\frac{1}{m_1}\right)\mathcal{F}U_1(-t)\left(\overline{u}_1u_2 + \frac{1}{t}U_1(t)\mathcal{F}^{-1}D(m_1)\overline{e^{iS_1(t)}\hat{u}_{1+}}e^{iS_2(t)}\hat{u}_{2+}\right)\\
&= D\left(\frac{1}{m_1}\right)\mathcal{F}U_1(-t)\Bigg(\overline{u}_1u_2 + \frac{1}{t}M_1(t)D_1(t)D(m_1)\overline{e^{iS_1(t)}\hat{u}_{1+}}e^{iS_2(t)}\hat{u}_{2+}\\
&\qquad + \frac{1}{t}R_1(t)D(m_1)\overline{e^{iS_1(t)}\hat{u}_{1+}}e^{iS_2(t)}\hat{u}_{2+}\Bigg),
\end{aligned} \tag{3.7}$$

where $R_1(t) = M_1(t)D_1(t)\mathcal{F}(M_1(t)-1)\mathcal{F}^{-1}$.
Similarly, the second equation of (1.1) can be rewritten as follows;

$$\begin{aligned}
&i\partial_t\left(D\left(\frac{1}{m_2}\right)\mathcal{F}U_2(-t)u_2 - e^{iS_2(t)}\hat{u}_{2+}\right)\\
&= D\left(\frac{1}{m_2}\right)\mathcal{F}U_2(-t)u_1^2 - \frac{1}{t}(e^{iS_1(t)}\hat{u}_{1+})^2\\
&= D\left(\frac{1}{m_2}\right)\mathcal{F}U_2(-t)\left(u_1^2 + \frac{1}{t}U_2(t)\mathcal{F}^{-1}D(m_2)(e^{iS_1(t)}\hat{u}_{1+})^2\right)\\
&= D\left(\frac{1}{m_2}\right)\mathcal{F}U_2(t)\Bigg(u_1^2 + \frac{1}{t}M_2(t)D_2(t)D(m_2)(e^{iS_1(t)}\hat{u}_{1+})^2\\
&\qquad + \frac{1}{t}R_2(t)D(m_2)(e^{iS_1(t)}\hat{u}_{1+})^2\Bigg),
\end{aligned} \tag{3.8}$$

where $R_2(t) = M_2(t)D_2(t)\mathcal{F}(M_2(t)-1)\mathcal{F}^{-1}$. For simplicity, we put

$$w_1 = e^{iS_1(t)}\hat{u}_{1+}, \quad w_2 = e^{iS_2(t)}\hat{u}_{2+}.$$

Integrating each side of (3.7) and (3.8), we obtain

$$\begin{aligned}
&u_1(t) + M_1(t)D_1(t)D(m_1)w_1\\
&\quad = -i\int_t^\infty U_1(t-\tau)\left(\overline{u}_1u_2 - \overline{(-M_1(\tau)D_1(\tau)D(m_1)w_1)}(-M_2(\tau)D_2(\tau)D(m_2)w_2)\right)d\tau\\
&\qquad - R_1(t)D(m_1)w_1 + i\int_t^\infty U_1(t-\tau)R_1(\tau)D(m_1)\overline{w}_1w_2d\tau,
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}&u_2(t) + M_2(t)D_2(t)D(m_2)w_2\\&\quad = -i\int_t^\infty U_2(t-\tau)\left(u_1^2 - (-M_1(\tau)D_1(\tau)D(m_1)w_1)^2\right)d\tau\\&\qquad - R_2(t)D(m_2)w_2 + i\int_t^\infty U_2(t-\tau)R_2(\tau)D(m_2)\left(w_1\right)^2 d\tau.\end{aligned}\tag{3.10}$$

To show the existence of $u = (u_1, u_2)$ satisfying (3.9) and (3.10), we shall prove that the map defined by the right-hand side of (3.9) and (3.10) are a contraction mapping on

$$X \equiv \left\{\phi = (\phi_1, \phi_2) \in \left(C([T,\infty); L^2)\right)^2; \|\phi - \tilde{w}\|_X < \infty\right\},$$

with

$$\tilde{w} = (\tilde{w}_1, \tilde{w}_2) \equiv (-M_1(t)D_1(t)D(m_1)w_1, -M_2(t)D_2(t)D(m_2)w_2),$$

$$\|\phi\|_X \equiv \sum_{j=1}^{2}\sup_{t\in[T,\infty)}\left(t^{\frac{\beta}{2}+\mu}\|\phi_j(t)\|_{L^2} + t^\mu\||J_j|^\beta\phi_j(t)\|_{L^2}\right),$$

where $1 < \beta < \gamma < 2$, $\gamma - \beta > 2\mu > 0$.
Let us consider the linearized version of (3.9) and (3.10),

$$\begin{aligned}&u_1(t) + M_1(t)D_1(t)D(m_1)w_1\\&\quad = -i\int_t^\infty U_1(t-\tau)\left(\overline{v}_1v_2 - \overline{(-M_1(\tau)D_1(\tau)D(m_1)w_1)}(-M_2(\tau)D_2(\tau)D(m_2)w_2)\right)d\tau\\&\qquad - R_1(t)D(m_1)w_1 + i\int_t^\infty U_1(t-\tau)R_1(\tau)D(m_1)\overline{w}_1w_2 d\tau,\end{aligned}\tag{3.11}$$

$$\begin{aligned}&u_2(t) + M_2(t)D_2(t)D(m_2)w_2\\&\quad = -i\int_t^\infty U_2(t-\tau)\left(v_1^2 - (-M_1(\tau)D_1(\tau)D(m_1)w_1)^2\right)d\tau\\&\qquad - R_2(t)D(m_2)w_2 + i\int_t^\infty U_2(t-\tau)R_2(\tau)D(m_2)\left(w_1\right)^2 d\tau,\end{aligned}\tag{3.12}$$

where $v = (v_1, v_2) \in X_\rho \equiv \{\phi \in X; \|\phi - \tilde{w}\|_X \le \rho\}$

We first consider (3.11). Let $0 \le \delta \le \beta < \gamma$. Since $R_1 = M_1(t)D_1(M_1 - 1)\mathcal{F}^{-1}$ and $|J_1|^\beta R_1(t) = R_1(t)(-\Delta)^{\frac{\beta}{2}}$, we have

$$\begin{aligned}&\||J_1|^\delta R_1(t)D(m_1)w_1\|_{L^2}\\&\quad = \|R_1(t)(-\Delta)^{\frac{\delta}{2}}D(m_1)w_1\|_{L^2}\\&\quad = \|(M_1 - 1)\mathcal{F}^{-1}(-\Delta)^{\frac{\delta}{2}}D(m_1)w_1\|_{L^2}.\end{aligned}$$

For $0 < \gamma < 1$, we have

$$\left|e^{\frac{i|x|}{2t}} - 1\right| = 2\left|\sin\frac{|x|}{4t}\right| \leq C\frac{|x|^{2\gamma}}{|t|^{\gamma}}.$$

By the above inequality and Lemma 3.4, we have

$$\begin{aligned}
&\|(M_1 - 1)\mathcal{F}^{-1}(-\Delta)^{\frac{\delta}{2}} D(m_1)w_1\|_{L^2} \\
&\quad\leq Ct^{-\frac{\gamma-\delta}{2}}\|w_1\|_{\dot{H}^\gamma} \\
&\quad\leq Ct^{-\frac{\gamma-\delta}{2}}\{\|\hat{u}_{1+}\|_{\dot{H}^\gamma} + \|\hat{u}_{1+}\|_{L^\infty}(1 + \|S_1(t)\|_{L^\infty})\|S_1(t)\|_{\dot{H}^\gamma}\} \\
&\quad\leq Ct^{-\frac{\gamma-\delta}{2}}(\log t)^2\|\hat{u}_+\|_{H^\gamma}.
\end{aligned}$$

Thus we see

$$\||J_1|^\delta R_1(t)D(m_1)w_1\|_{L^2} \leq Ct^{-\frac{\gamma-\delta}{2}}(\log t)^2\|\hat{u}_+\|_{H^\gamma}. \tag{3.13}$$

Similarly, we have

$$\begin{aligned}
&\int_t^\infty \||J_1|^\delta R_1(t)D(m_1)\overline{w}_1 w_2\|_{L^2}\frac{d\tau}{\tau} \\
&= \int_t^\infty \|R_1(t)(-\Delta)^{\frac{\delta}{2}}D(m_1)\overline{w}_1 w_2\|_{L^2}\frac{d\tau}{\tau} \\
&\leq C\int_t^\infty \tau^{-1-\frac{\gamma-\delta}{2}}\|\overline{w}_1 w_2\|_{\dot{H}^\gamma}d\tau \\
&\leq C\int_t^\infty \tau^{-1-\frac{\gamma-\delta}{2}}(\|w_1\|_{\dot{H}^\gamma}\|w_2\|_{L^\infty} + \|w_1\|_{L^\infty}\|w_2\|_{\dot{H}^\gamma})d\tau \\
&\leq Ct^{-\frac{\gamma-\delta}{2}}(\log t)^2\|\hat{u}_+\|_{H^\gamma}.
\end{aligned} \tag{3.14}$$

Also we obtain

$$\begin{aligned}
\|v_j\|_{L^\infty} &\leq \|v_j - \tilde{w}_j\|_{L^\infty} + \|\tilde{w}_j\|_{L^\infty} \\
&\leq Ct^{-1}\||J_j|^\beta(v_j - \tilde{w}_j)\|_{L^2}^{\frac{1}{\beta}}\|v_j - \tilde{w}_j\|_{L^2}^{1-\frac{1}{\beta}} + Ct^{-1}\|\hat{u}_{j+}\|_{L^\infty} \\
&\leq Ct^{-1}(\rho t^{-\mu} + \varepsilon)
\end{aligned} \tag{3.15}$$

for $j = 1, 2$. Combining the above estimates (3.13), (3.14) and (3.15), we see

$$\begin{aligned}
\|u_1(t) - \tilde{w}_1(t)\|_{L^2} &\le C \int_t^\infty \|\overline{v}_1 v_2 - \overline{\tilde{w}}_1 \tilde{w}_2\|_{L^2} d\tau + \|R_1(t) D(m_1) w_1\|_{L^2} \\
&+ \int_t^\infty \|U_1(t-\tau) R_1(\tau) D(m_1) \overline{w}_1 w_2\|_{L^2} d\tau \\
&\le C \int_t^\infty \{\|v_2\|_{L^\infty} \|v_1 - \tilde{w}_1\|_{L^2} + \|\tilde{w}_1\|_{L^\infty} \|v_2 - \tilde{w}_2\|_{L^2}\} \, d\tau \\
&+ C t^{-\frac{\gamma}{2}} (\log t)^2 \|\hat{u}_+\|_{H^\gamma} \\
&\le C \int_t^\infty \tau^{-1} (\rho \tau^{-\mu} + \varepsilon) \rho \tau^{-\beta/2 - \mu} d\tau + C \int_t^\infty \varepsilon \tau^{-1} \rho \tau^{-\beta/2-\mu} d\tau \\
&+ C t^{-\frac{\gamma}{2}} (\log t)^2 \|\hat{u}_+\|_{H^\gamma} \\
&\le C \varepsilon \rho t^{-\frac{\beta}{2} - \mu} + C \rho^2 t^{-\beta/2 - 2\mu} + C \varepsilon t^{-\gamma/2} (\log t)^2
\end{aligned} \tag{3.16}$$

for all $t \ge T$ if $T > 0$ is sufficient large.

We again use $|J_1|^\beta R_1(t) = R_1(t)(-\Delta)^{\frac{\beta}{2}}$. Then multiplying (3.11) by $|J_1|^\beta$, we obtain

$$\begin{aligned}
&|J_1|^\beta (u_1(t) - \tilde{w}_1) \\
&= -i\lambda \int_t^\infty U_1(t-\tau) \left(|J_1|^\beta (\overline{v}_1 v_2 - \overline{\tilde{w}}_1 \tilde{w}_2) \right) d\tau \\
&\quad - R_1(t)(-\Delta)^{\frac{\beta}{2}} D(m_1) w_1 + i\lambda \int_t^\infty U_1(t-\tau) R_1(\tau) (-\Delta)^{\frac{\beta}{2}} D(m_1) \overline{w}_1 w_2 \frac{d\tau}{\tau}.
\end{aligned} \tag{3.17}$$

Taking the L^2-norm of the both sides of (3.17), we find

$$\begin{aligned}
&\||J_1|^\beta (u_1(t) - \tilde{w}_1(t))\|_{L^2} \\
&\le \int_t^\infty \||J_1|^\beta (\overline{v}_1 v_2 - \overline{\tilde{w}}_1 \tilde{w}_2\|_{L^2} d\tau \\
&\quad + \|R_1(t)(-\Delta)^{\frac{\beta}{2}} D(m_1) w_1\|_{L^2} + C \int_t^\infty \|R_1(\tau)(-\Delta)^{\frac{\beta}{2}} D(m_1) \overline{w}_1 w_2\|_{L^2} \frac{d\tau}{\tau}.
\end{aligned} \tag{3.18}$$

For the first term of the right hand side of (3.18), we have

$$\begin{aligned}
\||J_1|^\beta (\overline{v}_1 v_2 - \overline{\tilde{w}}_1 \tilde{w}_2)\|_{L^2} &= t^\beta \|\overline{M}_1 (\overline{v}_1 v_2 - \overline{\tilde{w}}_1 \tilde{w}_2)\|_{\dot{H}^\beta} \\
&= t^\beta \|(\overline{\overline{M}_1 v_1} \overline{M}_2 v_2 - \overline{\overline{M}_1 \tilde{w}_1} \overline{M}_2 \tilde{w}_2)\|_{\dot{H}^\beta}.
\end{aligned}$$

From direct calculation, we have

$$\begin{aligned}
&t^\beta \|(\overline{\overline{M}_1 v_1} \overline{M}_2 v_2 - \overline{\overline{M}_1 \tilde{w}_1} \overline{M}_2 \tilde{w}_2)\|_{\dot{H}^\beta} \\
&\le C t^\beta \|v_2\|_{L^\infty} \|\overline{M}_1 (v_1 - \tilde{w}_1)\|_{\dot{H}^\beta} + C t^\beta \|v_1 - \tilde{w}_1\|_{L^\infty} \|\overline{M}_2 v_2\|_{\dot{H}^\beta} \\
&\quad + C t^\beta \|v_2 - \tilde{w}_2\|_{L^\infty} \|\overline{M}_1 \tilde{w}_1\|_{\dot{H}^\beta} + C t^\beta \|\tilde{w}_1\|_{L^\infty} \|\overline{M}_2 (v_2 - \tilde{w}_2)\|_{\dot{H}^\beta} \\
&\le C \|v_2\|_{L^\infty} \||J_1|^\beta (v_1 - \tilde{w}_1)\|_{L^2} + C \|v_1 - \tilde{w}_1\|_{L^\infty} \||J_2|^\beta v_2\|_{L^2} \\
&\quad + C \|v_2 - \tilde{w}_2\|_{L^\infty} \||J_1|^\beta \tilde{w}_1\|_{L^2} + C \|\tilde{w}_1\|_{L^\infty} \||J_2|^\beta k (v_2 - \tilde{w}_2)\|_{L^2}.
\end{aligned}$$

Since $v \in X_\rho$, we see

$$\begin{aligned}\|v_j - \tilde{w}_j\|_{L^\infty} &\le Ct^{-1}\||J_j|^\beta(v_j - \tilde{w}_j)\|_{L^2}^{\frac{1}{\beta}}\|v_j - \tilde{w}_j\|_{L^2}^{1-\frac{1}{\beta}} \\ &\le C\rho t^{-1-\mu-\frac{1}{2}(\beta-1)},\end{aligned}$$

for $j = 1, 2$. We obtain by using above estimates

$$\||J_1|^\beta(\overline{v}_1 v_2 - \overline{\tilde{w}}_1\tilde{w}_2)\|_{L^2} \le C\varepsilon\rho t^{-1-\mu} + C\varepsilon\rho t^{-1-\mu-\beta+1/2}. \tag{3.19}$$

Substituting (3.19) to (3.18), we obtain

$$\begin{aligned}\||J_1|^\beta(u_1(t) - \tilde{w}_1)\|_{L^2} &\le C\rho\int_t^\infty \tau^{-1-\mu}(\varepsilon + \rho\tau^{-\mu})d\tau + C\rho t^{-\frac{\gamma-\beta}{2}}(\log t)^2 \\ &\le C\varepsilon\rho t^{-\mu} + C\rho^2 t^{-2\mu} + C\rho t^{(\gamma-\beta)/2}(\log t)^2.\end{aligned} \tag{3.20}$$

The same calculation shows the estimate for (3.12):

$$\|u_2(t) - \tilde{w}_2(t)\|_{L^2} \le C\varepsilon\rho t^{-\frac{\beta}{2}-\mu} + C\rho^2 t^{-\beta/2-2\mu} + C\varepsilon t^{-\gamma/2}(\log t)^2, \tag{3.21}$$

$$\||J_2|^\beta(u_2(t) - \tilde{w}_2)\|_{L^2} \le C\varepsilon\rho t^{-\mu} + C\rho^2 t^{-2\mu} + C\varepsilon t^{(\gamma-\beta)/2}(\log t)^2. \tag{3.22}$$

Using estimates (3.16), (3.20), (3.21) and (3.22), we find a time $T > 0$ such that $u \in X_\rho$. In the same manner we can prove the estimate

$$\|u - \tilde{u}\|_X \le \frac{1}{2}\|v - \tilde{v}\|_X,$$

where $\tilde{u} = (\tilde{u}_1, \tilde{u}_2)$ is defined by (3.11) and (3.12) with $v = (v_1, v_2)$ replaced by $\tilde{v} = (\tilde{v}_1, \tilde{v}_2)$.

Thus we conclude that the map defined by the right-hand side of (3.9) is a contraction mapping. Hence there exists a unique global solution $u = (u_1, u_2) \in (C([T,\infty); L^2))^2$ of the integral equation (3.9) and (3.10) satisfying the estimate

$$\|u_j - \tilde{w}_j\|_{L^2} \le Ct^{-\frac{1}{2}-\mu}$$

for $j = 1, 2$. u_1 and u_2 satisfies

$$u_1(t) = U_1(t-T)u_1(T) - i\int_T^t U_1(t-\tau)\overline{u}_1 u_2(\tau)d\tau \tag{3.23}$$

and

$$u_2(t) = U_2(t-T)u_2(T) - i\int_T^t U_2(t-\tau)(u_1)^2(\tau)d\tau \tag{3.24}$$

respectively. Since $u_1(T), u_2(T) \in L^2(\mathbb{R}^2)$, combining the argument due to [30] and L^2-conservation law for (1.1), we may show that (3.23) and (3.24) have a global solution in $C(\mathbb{R}; L^2(\mathbb{R}^2))$. Thus the solution $u = (u_1, u_2)$ extends to all times. This completes the proof of Theorem 3.1. □

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