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Hardy spaces with variable exponent

By

Mitsuo Izuki*, Eiichi Nakai** and Yoshihiro Sawano***

Abstract

In this paper we first make a view of Lebesgue spaces with variable exponent. After reviewing fundamental properties such as completeness, duality and associate spaces, we reconsider Hardy spaces with variable exponent. We supplement what we obtained in our earlier paper. In Part I we collect some known basic properties toghther with their proofs. In Part II we summarize and reinforce what we obtained in [30, 36].

Contents

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*Department of Mathematics, Tokyo Denki University, Adachi-ku, Tokyo 120-8551, Japan
e-mail: izuki@mail.dendai.ac.jp
**Department of Mathematics, Ibaraki University, Mito, Ibaraki 310-8512, Japan
e-mail: enakai@mx.ibaraki.ac.jp
***Department of Mathematics and Information Science, Tokyo Metropolitan University, 1-1 Minami-Ohsawa, Hachioji, Tokyo 192-0397, Japan.
e-mail: ysawano@tmu.ac.jp

Notation

In the whole paper we will use the following notation:

(1) Given a measurable set $S \subset \mathbb{R}^n$, we denote the Lebesgue measure by $|S|$ and the characteristic function by χ_S.

(2) Given a measurable set $S \subset \mathbb{R}^n$ and a function f on $\mathbb{R}^n$, we denote the mean value of f on S by f_S or $\fint_S f(x)\,dx$, namely, $f_S = \fint_S f(x)\,dx := \frac{1}{|S|}\int_S f(x)\,dx$.

(3) The set $\mathbb{N}_0$ consits of all non-negative integers.

(4) Given a malti-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in {\mathbb{N}_0}^n$, we write

$$|\alpha| := \sum_{\nu=1}^{n} \alpha_\nu.$$

In addition the derivative of f is denoted by

$$D^\alpha f := \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}.$$

(5) A symbol C always stands for a positive constant independent of the main parameters.

(6) An open cube $Q \subset \mathbb{R}^n$ is always asssumed to have sides parallel to the coordinate axes. Namely we can write $Q = Q(x,r) := \prod_{\nu=1}^n (x_\nu - r/2\,,\, x_\nu + r/2)$ using $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $r > 0$.

(7) We define an open ball by

$$B(x,r) := \{y \in \mathbb{R}^n \,:\, |x-y| < r\},$$

where $x \in \mathbb{R}^n$ and $r > 0$.

(8) Given a positive number s, a cube $Q = Q(x,r)$ and an open ball $B = B(x,r)$, we define $sQ := Q(x,sr)$ and $sB := B(x,sr)$.

(9) The set $\Omega \subset \mathbb{R}^n$ is measurable and satisfies $|\Omega| > 0$.

(10) The set $C^\infty_{\mathrm{comp}}(\Omega)$ consists of all compactly supported and infinitely differentiable functions f defined on Ω.

(11) The uncentered Hardy–Littlewood maximal operator M is given by

$$Mf(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)|\, dy,$$

where the supremum is taken over all open ballls B containing x. We can replace the open balls $\{B\}$ by the open cubes $\{Q\}$.

(12) By "a variable exponent", we mean a measurable function $p(\,\cdot\,) : \Omega \to (0,\infty)$. The symbol "$(\,\cdot\,)$" emphasizes that the function p does not always mean a constant exponent $p \in (0,\infty)$. Given a variable exponent $p(\,\cdot\,)$ we define the following:

   (a) $p_- := \mathrm{ess.inf}_{x\in\Omega} p(x) = \sup\{a \,:\, p(x) \ge a \text{ a.e. } x \in \Omega\}$.

   (b) $p_+ := \mathrm{ess.sup}_{x\in\Omega} p(x) = \inf\{a \,:\, p(x) \le a \text{ a.e. } x \in \Omega\}$.

   (c) $\Omega_0 := \{x \in \Omega \,:\, 1 < p(x) < \infty\} = p^{-1}((1,\infty))$.

   (d) $\Omega_1 := \{x \in \Omega \,:\, p(x) = 1\} = p^{-1}(1)$.

   (e) $\Omega_\infty := \{x \in \Omega \,:\, p(x) = \infty\} = p^{-1}(\infty)$.

(*f*) the conjugate exponent $p'(\cdot)$:

$$p'(x) := \begin{cases} \infty & (x \in \Omega_1), \\ \frac{p(x)}{p(x)-1} & (x \in \Omega_0), \\ 1 & (x \in \Omega_\infty), \end{cases}$$

namely, $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ always holds for a.e. $x \in \Omega$. In particular, if $p(\cdot)$ equals to a constant p, then of course $p'(\cdot) = p'$ is the usual conjugate exponent.

(13) We adopt the following definition of the Fourier transform and its inverse:

$$\mathcal{F}f(\xi) := \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}\,dx,\ \mathcal{F}^{-1}f(x) := \int_{\mathbb{R}^n} f(\xi)e^{2\pi i x\cdot\xi}\,d\xi$$

for $f \in L^1(\mathbb{R}^n)$.

(14) Using this definition of Fourier transform and its inverse, we also define

$$\varphi(D)f(x) := \mathcal{F}^{-1}[\varphi\cdot\mathcal{F}f](x) = \langle f, \mathcal{F}^{-1}\varphi(x - \cdot)\rangle \tag{0.1}$$

for $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

Part I
Basic theory on function spaces with variable exponents

§ 1. Introduction

Recently, in harmonic analysis, partial differential equations, potential theory and applied mathematics, many authors investigate function spaces with variable exponents. In particular, spaces with variable exponent are necessary in the field of electronic fluid mechanics and the applications to the recovery of graphics.

The theory of Lebesgue spaces with variable exponent dates back to Orlicz's paper [33] and Nakano's books in 1950 and 1951 [31, 32]. In particular, the definition of Musielak-Orlicz spaces is clearly written in [31]. Later, Kováčik–Rákosník [19] clarified fundamental properties of Lebesgue spaces with variable exponents and Sobolev spaces with variable exponents. This important achievement leads to the present hot discussion of function spaces with variable exponents.

Here is a table of brief history of function spaces with variable exponents:

- Orlicz [33] (1931) $\cdots$ $L^{p(\cdot)}(\Omega)$ with $1 \le p_- \le p_+ < \infty$.
- Nakano [32] (1951) $\cdots$ $L^{p(\cdot)}(\Omega)$ with $1 \le p_- \le p_+ < \infty$.
- Sharapudinov [37] (1979) $\cdots$ $L^{p(\cdot)}([0,1])$ with $1 \le p_- \le p_+ \le \infty$.
- Kováčik–Rákosník [19] (1991) $\cdots$ $L^{p(\cdot)}(\Omega)$ with $1 \le p_- \le p_+ \le \infty$, basic theory.

One of the important problems is to prove the boundedness of the Hardy-Littlewood maximal operator M. Once this is established, we can expect that this boundedness can be applied to many parts of analysis. Actually, many authors tackled this hard problem. The paper [10] by Diening is a pioneering one. Based upon the paper [10], Cruz-Uribe, Fiorenza and Neugebauer [5, 6] have given sufficient conditions for M to be bounded on Lebesgue spaces with variable exponents and the condition is referred to as the log-Hölder condition.

Due to the extrapolation theorem by Cruz-Uribe–Fiorenza–Martell–Pérez [4] about Lebesgue spaces with variable exponent, we can prove the boundedness of singular integral operators of Calderón–Zygmund type, the boundedness of commutators generated by BMO functions and singular integral operators and the Fourier multiplier results.

§ 2. The usual Lebesgue spaces-Elementary properties

In this section, we review classical Lebesgue spaces.

Definition 2.1. Let $1 \le p < \infty$. The Lebesgue space $L^p(\Omega)$ is the set of all complex-valued measurable functions f defined on Ω satisfying $\|f\|_{L^p(\Omega)} < \infty$, where

$$\|f\|_{L^p(\Omega)} := \begin{cases} \left(\int_\Omega |f(x)|^p \, dx\right)^{1/p} & (1 \le p < \infty), \\ \operatorname{ess.sup}_{x \in \Omega} |f(x)| & (p = \infty). \end{cases}$$

Theorem 2.2 (Hölder's inequality). *Let $1 \le p \le \infty$. We have that for all $f \in L^p(\Omega)$ and all $g \in L^{p'}(\Omega)$,*

$$\int_\Omega |f(x)g(x)| \, dx \le \|f\|_{L^p(\Omega)} \|g\|_{L^{p'}(\Omega)}.$$

Applying Hölder's inequality, we obtain the following.

Theorem 2.3 (Minkowski's inequality). *Let $1 \le p \le \infty$. We have that for all $f, g \in L^p(\Omega)$,*

$$\|f+g\|_{L^p(\Omega)} \le \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}.$$

Corollary 2.4. *If $1 \le p \le \infty$, then $\| \cdot \|_{L^p(\Omega)}$ is a norm.*

§ 3. Lebesgue spaces with variable exponents

Lebesgue spaces with variable exponent have been studied intenstively for these two decades right after some basic properties was established by Kováčik–Rákosník [19]. We refer to the surveys [16, 17, 34] and a new book [7] for recent developments. In this section we state and recall some known basic properties.

§ 3.1. Elementary properties

Definition 3.1. Given a measurable function $p(\,\cdot\,) : \Omega \to [1, \infty]$, we define the Lebesgue space with variable exponent

$$L^{p(\,\cdot\,)}(\Omega) := \{f \,:\, \rho_p(f/\lambda) < \infty \text{ for some } \lambda > 0\},$$

where

$$\rho_p(f) := \int_{\{x\in\Omega\,:\,p(x)<\infty\}} |f(x)|^{p(x)} dx + \|f\|_{L^\infty(\{x\in\Omega\,:\,p(x)=\infty\})}.$$

Moreover, define

$$\|f\|_{L^{p(\,\cdot\,)}(\Omega)} := \inf\left\{\lambda > 0 : \rho_p(f/\lambda) \le 1\right\}.$$

Remark 1. We easily see that, if $p(\,\cdot\,)$ equals to a constant p_0, then

$$L^{p(\,\cdot\,)}(\Omega) = L^{p_0}(\Omega) \quad \text{and} \quad \|f\|_{L^{p(\,\cdot\,)}(\Omega)} = \|f\|_{L^{p_0}(\Omega)}$$

are true.

Now we review the definition of modular.

Definition 3.2. Let $\mathcal{M}(\Omega)$ be the set of all complex-valued measurable functions defined on Ω and $X \subset \mathcal{M}(\Omega)$. A functional $\rho : X \to [0, \infty]$ is said to be a modular if the following conditions are fulfilled:

(*a*) $\rho(0) = 0$.

(*b*) For all $f \in X$ and $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, we have $\rho(\lambda f) = \rho(f)$.

(*c*) ρ is convex, namely, we have that for all f, $g \in X$ and all $0 \le t \le 1$,

$$\rho(tf + (1-t)g) \le t\rho(f) + (1-t)\rho(g).$$

(*d*) For every $f \in X$ such that $0 < \rho(f) < \infty$, the function

$$(0, \infty) \ni \lambda \mapsto \rho(\lambda f) \tag{3.1}$$

is left-continuous, namely, $\lim_{\lambda\to 1-0} \rho(\lambda f) = \rho(f)$ holds.

(*e*) If $\rho(f) = 0$, then $f = 0$.

A modular ρ is said to be a continuous modular if (d)' is satisfied:

$(d)'$ For every $f \in X$ such that $0 < \rho(f) < \infty$, the function defined by (3.1) is continuous.

Theorem 3.3. *Let $p(\,\cdot\,) : \Omega \to [1, \infty]$ be a variable exponent. Then $\rho_p(\,\cdot\,)$ is a modular. If $p(\,\cdot\,)$ satisfies $\operatorname{ess.sup}_{x\in\Omega\,\setminus\,\Omega_\infty} p(x) < \infty$, then $\rho_p(\,\cdot\,)$ is a continuous modular.*

Lemma 3.4. *Assume $0 < \|f\|_{L^{p(\,\cdot\,)}(\Omega)} < \infty$.*

(1) $\rho_p\left(\frac{f}{\|f\|_{L^{p(\,\cdot\,)}(\Omega)}}\right) \leq 1$.

(2) *If $\operatorname{ess.sup}_{x\in\Omega\,\setminus\,\Omega_\infty} p(x) < \infty$, then $\rho_p\left(\frac{f}{\|f\|_{L^{p(\,\cdot\,)}(\Omega)}}\right) = 1$ holds.*

Theorem 3.5. *Let $p(\,\cdot\,) : \Omega \to [1, \infty]$ be a variable exponent. Then $\|\cdot\|_{L^{p(\,\cdot\,)}(\Omega)}$ is a norm (often referred to as the Luxemberg–Nakano norm).*

Lemma 3.6. *Let $p(\,\cdot\,) : \Omega \to [1, \infty]$ be a variable exponent.*

(1) *If $\|f\|_{L^{p(\,\cdot\,)}(\Omega)} \leq 1$, then we have $\rho_p(f) \leq \|f\|_{L^{p(\,\cdot\,)}(\Omega)} \leq 1$.*

(2) *Conversely if $\rho_p(f) \leq 1$, then $\|f\|_{L^{p(\,\cdot\,)}(\Omega)} \leq 1$ holds.*

(3) *Assume that $1 \leq \tilde{p}_+ = \sup_{x\in\Omega\setminus\Omega_\infty} p(x) < \infty$. If $\rho_p(f) \leq 1$, then $\|f\|_{L^{p(\,\cdot\,)}} \leq \rho_p(f)^{1/\tilde{p}_+} \leq 1$.*

Finally, we remark that $L^{p(\,\cdot\,)}(\mathbb{R}^n)$ is a complete space.

Theorem 3.7. *The norm $\|\cdot\|_{L^{p(\,\cdot\,)}(\Omega)}$ is complete, that is, $L^{p(\,\cdot\,)}(\Omega)$ is a Banach space.*

§ 3.2. The associate space

Given a measurable function $p(\,\cdot\,) : \Omega \to [1, \infty]$, we defined the Lebesgue space with variable exponent by Definition 3.1.

For $p(\,\cdot\,) : \Omega \to [1, \infty]$, we define $p'(\,\cdot\,) : \Omega \to [1, \infty]$ as

$$1 = \frac{1}{p(x)} + \frac{1}{p'(x)}.$$

By no means the function $p'(\,\cdot\,)$ stands for the derivative of $p(\,\cdot\,)$.

The aim of this section is to state results related to duality.

Theorem 3.8 (Generalized Hölder's inequality). *Let $p(\cdot) : \Omega \to [1,\infty]$ be a variable exponent. Then, for all $f \in L^{p(\cdot)}(\Omega)$ and all $g \in L^{p'(\cdot)}(\Omega)$,*

$$\int_\Omega |f(x)g(x)|\,dx \le r_p \|f\|_{L^{p(\cdot)}(\Omega)} \|g\|_{L^{p'(\cdot)}(\Omega)},$$

where

$$r_p = 1 + \frac{1}{p_-} - \frac{1}{p_+}.$$

It is well known that $L^p(\Omega)\,(1 \le p < \infty)$ has $L^{p'}(\Omega)$ as its dual. This is not the case when $p = \infty$. The notion of associated spaces is close to dual spaces, which is used in the theory of function spaces. It is sometimes referred to as the Köthe dual. In the case of variable Lebesgue spaces the definition is given as follows:

Definition 3.9. Let $p(\cdot) : \Omega \to (1,\infty)$ be a variable exponent. The associate space of $L^{p'(\cdot)}(\Omega)$ and its norm are defined as follows:

$$L^{p'(\cdot)}(\Omega)' = \left\{ f \text{ is measurable } : \|f\|_{L^{p'(\cdot)}(\Omega)'} < \infty \right\},$$

$$\|f\|_{L^{p'(\cdot)}(\Omega)'} := \sup \left\{ \left| \int_\Omega f(x)g(x)\,dx \right| : \|g\|_{L^{p'(\cdot)}(\Omega)} \le 1 \right\}.$$

Remark 2. The condition $\|g\|_{L^{p'(\cdot)}(\Omega)} \le 1$ is equivalent to $\rho_{p'}(g) \le 1$ by virtue of Lemma 3.6.

Theorem 3.10. *Given a variable exponent $p(\cdot) : \Omega \to [1,\infty]$, write*

$$r_p := 1 + \frac{1}{p_-} - \frac{1}{p_+}.$$

Then we have that for all $f \in L^{p(\cdot)}(\Omega)$,

$$\|f\|_{L^{p(\cdot)}(\Omega)} \le \|f\|_{L^{p'(\cdot)}(\Omega)'} \le r_p \|f\|_{L^{p(\cdot)}(\Omega)}, \tag{3.2}$$

in particular, $L^{p(\cdot)}(\Omega) = L^{p'(\cdot)}(\Omega)'$ holds with norm equivalence.

In order to prove Theorem 3.10 we use the next lemma.

Lemma 3.11. *Let $p(\cdot) : \Omega \to [1,\infty]$ be a variable exponent. If $\|f\|_{L^{p'(\cdot)}(\Omega)'} \le 1$, then $\rho_p(f) \le \|f\|_{L^{p'(\cdot)}(\Omega)'}$ holds.*

In order to prove Lemma 3.11, we use the following Lemmas 3.12 and 3.13.

Lemma 3.12. *Let $p(\cdot) : \Omega \to [1,\infty]$ be a variable exponent. If $\|f\|_{L^{p'(\cdot)}(\Omega)'} < \infty$ and $\rho_{p'}(g) < \infty$, then we have*

$$\left| \int_\Omega f(x)g(x)\,dx \right| \le \|f\|_{L^{p'(\cdot)}(\Omega)'} \max\{1,\, \rho_{p'}(g)\}.$$

Lemma 3.13. *If* $1 < p(x) < \infty$ *a.e.* $x \in \Omega$, $\rho_p(f) < \infty$ *and* $\|f\|_{L^{p'(\cdot)}(\Omega)'} \le 1$, *then* $\rho_p(f) \le 1$ *holds.*

Lemma 3.11 is a direct consequence of Lemma 3.13. Indeed, if $\|f\|_{L^{p'(\cdot)}(\Omega)'} \le 1$, then we have

$$\rho_p\left(\frac{f}{\|f\|_{L^{p'(\cdot)}(\Omega)'}}\right) = \int_\Omega \left(\frac{|f(x)|}{\|f\|_{L^{p'(\cdot)}(\Omega)'}}\right)^{p(x)} dx \le 1$$

from Lemma 3.13. Note that

$$\left(\frac{1}{\|f\|_{L^{p'(\cdot)}(\Omega)'}}\right)^{p(x)} \ge \frac{1}{\|f\|_{L^{p'(\cdot)}(\Omega)'}}$$

because $\|f\|_{L^{p'(\cdot)}(\Omega)'} \le 1$. Hence,

$$\frac{\rho_p(f)}{\|f\|_{L^{p'(\cdot)}(\Omega)'}} = \frac{1}{\|f\|_{L^{p'(\cdot)}(\Omega)'}} \int_\Omega (|f(x)|)^{p(x)}\, dx \le \int_\Omega \left(\frac{|f(x)|}{\|f\|_{L^{p'(\cdot)}(\Omega)'}}\right)^{p(x)} dx \le 1.$$

Hence, we have $\rho_p(f) \le \|f\|_{L^{p'(\cdot)}(\Omega)'}$.

§ 3.3. Norm convergence, modular convergence and convergence in measure

Here we investigate the relations between several types of convergences.

Theorem 3.14. *Let* $p(\cdot) : \Omega \to [1, \infty]$ *be a variable exponent and* $f_j \in L^{p(\cdot)}(\Omega)$ $(j = 1, 2, 3, \ldots)$.

(1) *If* $\lim_{j\to\infty} \|f_j\|_{L^{p(\cdot)}(\Omega)} = 0$, *then* $\lim_{j\to\infty} \rho_p(f_j) = 0$ *holds.*

(2) *Assume that* $|\Omega\backslash\Omega_\infty| > 0$. *The following two conditions* (A) *and* (B) *are equivalent:*

(A) $\operatorname{ess.sup}_{x\in\Omega\backslash\Omega_\infty} p(x) < \infty$.

(B) *If* $\lim_{j\to\infty} \rho_p(f_j) = 0$, *then* $\lim_{j\to\infty} \|f_j\|_{L^{p(\cdot)}(\Omega)} = 0$ *holds.*

Theorem 3.15. *If a sequence* $\{f_j\}_{j=1}^\infty \in L^{p(\cdot)}(\Omega)$ *converges in* $L^{p(\cdot)}(\Omega)$, *then* f_j *converges to* 0 *in the sense of the Lebesgue measure, namely,*

$$\lim_{j\to\infty} |\{x \in \Omega \,:\, |f_j(x)| > \varepsilon\}| = 0 \tag{3.3}$$

for all $\varepsilon > 0$.

As an example of $p(\,\cdot\,)$ satisfying the requirement of Theorem 3.15, we can list

$$p(x) = 2 + \infty \cdot \chi_{B(0,1)}(x) = \begin{cases} 2 & (x \notin B(0,1)), \\ \infty & (x \in B(0,1)). \end{cases}$$

Here we assumed $B(0,1) \subset \Omega$.

§ 3.4. Duality (The generalized F. Riesz representation theorem)

Here we show that a counterpart of the $L^p(\Omega)$-$L^{p'}(\Omega)$ duality is available in the variable setting.

Definition 3.16. Let $p(\,\cdot\,) : \Omega \to [1, \infty]$ be a variable exponent. The dual space of $L^{p(\,\cdot\,)}(\Omega)$ and its norm are defined by

$$L^{p(\,\cdot\,)}(\Omega)^* := \left\{ T : L^{p(\,\cdot\,)}(\Omega) \to \mathbb{C} \,:\, T \text{ is linear and bounded} \right\},$$
$$\|T\|_{L^{p(\,\cdot\,)}(\Omega)^*} := \sup \left\{ |T(u)| \,:\, \|u\|_{L^{p(\,\cdot\,)}(\Omega)} \le 1 \right\}.$$

It is natural to ask ourselves whether $L^{p'(\cdot)}(\Omega)$ is naturally identified with the dual of $L^{p(\cdot)}(\Omega)$. Half of the answer is given by the next theorem.

Theorem 3.17. *Let $p(\,\cdot\,) : \Omega \to [1, \infty]$ be a variable exponent. Given a function $f \in L^{p'(\,\cdot\,)}(\Omega)$ we define the functional*

$$T_f(u) := \int_\Omega f(x)u(x)\, dx \quad (u \in L^{p(\,\cdot\,)}(\Omega)).$$

Then, the integral defining $T_f u$ converges absolutely. Also, the functional T_f belongs to $L^{p(\,\cdot\,)}(\Omega)^*$ and the estimate

$$\|f\|_{L^{p'(\,\cdot\,)}(\Omega)} \le \|T_f\|_{L^{p(\,\cdot\,)}(\Omega)^*} \le (1 + 1/p_- - 1/p_+)\|f\|_{L^{p'(\,\cdot\,)}(\Omega)}.$$

In particular $L^{p'(\,\cdot\,)}(\Omega) \subset L^{p(\,\cdot\,)}(\Omega)^*$ is true.

When $p_+ < \infty$, then we can give an affirmative answer to the above question.

Theorem 3.18. *Let $p(\,\cdot\,) : \Omega \to [1, \infty)$ be a variable exponent such that*

$$p_+ < \infty.$$

For all linear functionals $F \in L^{p(\,\cdot\,)}(\Omega)^*$ there uniquely exists a function $f \in L^{p'(\,\cdot\,)}(\Omega)$ such that

$$F(u) = \int_\Omega f(x)u(x)\, dx \quad (u \in L^{p(\,\cdot\,)}(\Omega)).$$

Moreover, we have the norm estimate

$$\|f\|_{L^{p'(\,\cdot\,)}(\Omega)} \le \|F\|_{L^{p(\,\cdot\,)}(\Omega)^*} \le (1 + 1/p_- - 1/p_+)\|f\|_{L^{p'(\,\cdot\,)}(\Omega)}. \tag{3.4}$$

In particular $L^{p(\,\cdot\,)}(\Omega)^* \subset L^{p'(\,\cdot\,)}(\Omega)$ is true.

§ 3.5. Some estimates of the norms

The following is a crucial inequality and very useful, because it is by no means easy to measure the $L^{p(\cdot)}(\mathbb{R}^n)$-norm of the characteristic functions.

Lemma 3.19 ([30]). *Suppose that* $p(\cdot)$ *is a function satisfying* (5.1), (5.2) *and* $0 < p_- \le p_+ < \infty$.

1. *For all cubes* $Q = Q(z, r)$ *with* $z \in \mathbb{R}^n$ *and* $r \le 1$, *we have* $|Q|^{1/p_-(Q)} \lesssim |Q|^{1/p_+(Q)}$. *In particular, we have*

$$|Q|^{1/p_-(Q)} \sim |Q|^{1/p_+(Q)} \sim |Q|^{1/p(z)} \sim \|\chi_Q\|_{L^{p(\cdot)}}, \tag{3.5}$$

   where $p_+(Q) = \mathrm{ess.sup}_{x\in Q} p(x)$ *and* $p_-(Q) = \mathrm{ess.inf}_{x\in Q} p(x)$.

2. *For all cubes* $Q = Q(z, r)$ *with* $z \in \mathbb{R}^n$ *and* $r \ge 1$, *we have*

$$\|\chi_Q\|_{L^{p(\cdot)}} \sim |Q|^{1/p_\infty}.$$

Here the implicit constants in $\sim$ *do not depend on* z *and* $r > 0$.

Remark 3. The equivalence (3.5) can be implicitly found in [12, Lemma 2.5].

§ 4. Banach function spaces

In this subsection we outline the definition of Banach function spaces and the Fatou lemma. For further information we refer to Bennet–Sharpley [3].

Definition 4.1. Let $\mathcal{M}(\Omega)$ be the set of all measurable and complex-valued functions on Ω. A linear space $X \subset \mathcal{M}(\Omega)$ is said to be a Banach function space if there exists a functional $\|\cdot\|_X : \mathcal{M}(\Omega) \to [0, \infty]$ with the following conditions:
Let f, g, $f_j \in \mathcal{M}(\Omega)$ $(j = 1, 2, \ldots)$.

(1) $f \in X$ holds if and only if $\|f\|_X < \infty$.

(2) (Norm property):

- $(A1)$ (Positivity): $\|f\|_X \ge 0$.
- $(A2)$ (strict Positivity) $\|f\|_X = 0$ if and only if $f = 0$ a.e..
- (B) (Homogeneity): $\|\lambda f\|_X = |\lambda| \cdot \|f\|_X$.
- (C) (The triangle inequality): $\|f + g\|_X \le \|f\|_X + \|g\|_X$.

(3) (Symmetry): $\|f\|_X = \|\,|f|\,\|_X$.

(4) (Lattice property): If $0 \le g \le f$ a.e., then $\|g\|_X \le \|f\|_X$.

(5) (Fatou property): If $0 \le f_1 \le f_2 \le \dots$ and $\lim_{j\to\infty} f_j = f$, then $\lim_{j\to\infty} \|f_j\|_X = \|f\|_X$.

(6) For all measurable sets F with $|F| < \infty$, we have $\|\chi_F\|_X < \infty$.

(7) For all measurable sets F with $|F| < \infty$, there exists a constant $C_F > 0$ such that $\int_F |f(x)|\,dx \le C_F \|f\|_X$.

Example 4.2. Both the usual Lebesgue spaces $L^p(\Omega)$ with constant exponent $1 \le p \le \infty$ and the Lebesgue spaces $L^{p(\,\cdot\,)}(\Omega)$ with variable exponent $p(\,\cdot\,) : \Omega \to [1, \infty]$ are Banach function spaces.

Lemma 4.3 (The Fatou lemma)**.** *Let X be a Banach function space and $f_j \in X$ $(j = 1, 2, \dots)$. If f_j converges to a function f a.e. Ω and $\liminf_{j\to\infty} \|f_j\|_X < \infty$, then we have $f \in X$ and $\|f\|_X \le \liminf_{j\to\infty} \|f_j\|_X$.*

Remark 4. In the proof of Lemma 3.4 we use the Fatou lemma with $X = L^1(\{p(x) < \infty\}),\ L^\infty(\Omega_\infty)$.

Part II
Hardy spaces with variable exponent

The role of this part is to survey Hardy spaces with variable exponent. In this part we summarize what we obtained in [30, 36].

§ 5. Fundamental properties

Let $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ be an exponent such that $0 < p_- = \inf_{x\in\mathbb{R}^n} p(x) \le p_+ = \sup_{x\in\mathbb{R}^n} p(x) < \infty$. Here and below, for the sake of simplicity, we shall postulate the following conditions on $p(\,\cdot\,)$.

$$\text{(5.1)} \qquad \text{(log-Hölder continuity)} \quad |p(x) - p(y)| \lesssim \frac{1}{\log(1/|x-y|)} \quad \text{for} \quad |x-y| \le \frac{1}{2},$$

$$\text{(5.2)} \qquad \text{(decay condition)} \quad |p(x) - p(y)| \lesssim \frac{1}{\log(e+|x|)} \quad \text{for} \quad |y| \ge |x|.$$

Remark that (5.1) and (5.2) are necessary when we consider the property of maximal operators.

§ 5.1. Definition of Hardy spaces

Recall that the space $L^{p(\,\cdot\,)}(\mathbb{R}^n)$, the Lebesgue space with variable exponent $p(\,\cdot\,)$, is defined as the set of all measurable functions f for which the quantity $\int_{\mathbb{R}^n} |\varepsilon f(x)|^{p(x)}\,dx$ is finite for some $\varepsilon > 0$. The quasi-norm is given by

$$\|f\|_{L^{p(\,\cdot\,)}} := \inf\left\{\lambda > 0 \,:\, \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \le 1\right\}$$

for such a function f.

In the celebrated paper [13], by using a suitable family $\mathcal{F}_N$, C. Fefferman and E. Stein defined the Hardy space $H^p(\mathbb{R}^n)$ with the norm given by

$$\|f\|_{H^p} := \left\|\sup_{t>0}\sup_{\varphi\in\mathcal{F}_N} |t^{-n}\varphi(t^{-1}\cdot) * f|\right\|_{L^p}, \quad f \in \mathcal{S}'(\mathbb{R}^n)$$

for $0 < p < \infty$. Here, in this part, we aim to replace $L^p(\mathbb{R}^n)$ with $L^{p(\,\cdot\,)}(\mathbb{R}^n)$ and investigate the function space obtained in this way.

The aim of the present paper is to review the definition of Hardy spaces with variable exponents and then to consider and apply the atomic decomposition. As is the case with the classical theory, we choose a suitable subset $\mathcal{F}_N \subset \mathcal{S}(\mathbb{R}^n)$, which we describe.

Definition 5.1.

1. Topologize $\mathcal{S}(\mathbb{R}^n)$ by the collection of semi-norms $\{p_N\}_{N\in\mathbb{N}}$ given by

$$p_N(\varphi) := \sum_{|\alpha|\le N} \sup_{x\in\mathbb{R}^n} (1+|x|)^N |\partial^\alpha\varphi(x)|$$

for each $N \in \mathbb{N}$. Define

$$\mathcal{F}_N := \{\varphi \in \mathcal{S}(\mathbb{R}^n) : p_N(\varphi) \le 1\}. \tag{5.3}$$

2. Let $f \in \mathcal{S}'(\mathbb{R}^n)$. Denote by $\mathcal{M}f$ the grand maximal operator given by

$$\mathcal{M}f(x) := \sup\{|t^{-n}\psi(t^{-1}\cdot) * f(x)| \,:\, t > 0, \quad \psi \in \mathcal{F}_N\},$$

where we choose and fix a large integer N.

3. The Hardy space $H^{p(\,\cdot\,)}(\mathbb{R}^n)$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ for which the quantity

$$\|f\|_{H^{p(\,\cdot\,)}} := \|\mathcal{M}f\|_{L^{p(\,\cdot\,)}}$$

is finite.

The definition of $\mathcal{F}_N$ dates back to the original work [38].

The following theorem about the definition of $H^{p(\cdot)}(\mathbb{R}^n)$ is obtained in [30].

Theorem 5.2 ([30, Theorem 1.2 and 3.3]). *Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be a function such that $\int_{\mathbb{R}^n} \varphi(x)\,dx \neq 0$. We define*

$$\|f\|_{H^{p(\cdot)}_{\varphi,*}} := \left\| \sup_{t>0} |t^{-n}\varphi(t^{-1}\cdot) * f| \right\|_{L^{p(\cdot)}}, \quad f \in \mathcal{S}'(\mathbb{R}^n). \tag{5.4}$$

Then the norms $\|\cdot\|_{H^{p(\cdot)}_{\varphi,*}}$ and $\|f\|_{H^{p(\cdot)}}$ are equivalent.

Note that it can happen that $0 < p_- < 1 < p_+ < \infty$ in our setting.

§ 5.2. Poisson integral characterization

Now we consider the Poisson integral characterization. Recall that $f \in \mathcal{S}'(\mathbb{R}^n)$ is a bounded distribution in terms of Stein, if $f * \varphi \in L^\infty(\mathbb{R}^n)$ for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$, and that $e^{-t\sqrt{-\Delta}}f = \mathcal{F}^{-1}(e^{-t|\xi|}\mathcal{F}f)$ $\quad (f \in \mathcal{S}'(\mathbb{R}^n))$ denotes the Poisson semi-group for bounded distributions f. We refer to [38, p.89] for more details. Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be chosen to satisfy

$$\chi_{Q(0,1)} \le \mathcal{F}\psi \le \chi_{Q(0,2)}. \tag{5.5}$$

With this preparation in mind, we can define

$$e^{-t\sqrt{-\Delta}}f := [e^{-t\sqrt{-\Delta}}(1-\psi)] * f + e^{-t\sqrt{-\Delta}}[\psi * f],$$

if f is a bounded distribution.

We have the following characterization.

Theorem 5.3 ([30, Theorem 3.4]). *Suppose that $p(\cdot)$ satisfies* (5.1), (5.2) *and $0 < p_- \le p_+ < \infty$. Let $f \in \mathcal{S}'(\mathbb{R}^n)$. Then the following are equivalent.*

1. *$f \in H^{p(\cdot)}(\mathbb{R}^n)$,*
2. *f is a bounded distribution and $\sup_{t>0} |e^{-t\sqrt{-\Delta}}f| \in L^{p(\cdot)}(\mathbb{R}^n)$.*

§ 5.3. Atomic decomposition

Here is another key result which we shall highlight. To formulate we adopt the following definition of the atomic decomposition.

Definition 5.4 ($(p(\cdot), q)$-atom). Let $p(\cdot) : \mathbb{R}^n \to (0,\infty)$, $0 < p_- \le p_+ < q \le \infty$ and $q \ge 1$. Fix an integer $d \ge d_{p(\cdot)} := \min\{d \in \mathbb{N} \cup \{0\} : p_-(n+d+1) > n\}$. A function a on $\mathbb{R}^n$ is called a $(p(\cdot), q)$-atom if there exists a cube Q such that (a1) $\mathrm{supp}(a) \subset Q$,

(a2) $\|a\|_{L^q} \le \dfrac{|Q|^{1/q}}{\|\chi_Q\|_{L^{p(\cdot)}}}$,

(a3) $\displaystyle\int_{\mathbb{R}^n} a(x)x^\alpha\,dx = 0$ for $|\alpha| \le d$.

The set of all such pairs (a, Q) will be denoted by $A(p(\cdot), q)$.

Under this definition, we define the atomic Hardy spaces with variable exponents. Here and below we denote

$$\underline{p} := \min(p_-, 1). \tag{5.6}$$

Definition 5.5 (Sequence norm $\mathcal{A}(\{\kappa_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty)$ and $H^{p(\cdot),q}_{\mathrm{atom}}(\mathbb{R}^n)$)**.** Given sequences of nonnegative numbers $\{\kappa_j\}_{j=1}^\infty$ and cubes $\{Q_j\}_{j=1}^\infty$, define

(5.7)

$$\mathcal{A}(\{\kappa_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty) := \inf\left\{\lambda > 0 \,:\, \int_{\mathbb{R}^n}\left\{\sum_{j=1}^\infty\left(\frac{\kappa_j\chi_{Q_j}(x)}{\lambda\|\chi_{Q_j}\|_{L^{p(\cdot)}}}\right)^{\underline{p}}\right\}^{\frac{p(x)}{\underline{p}}} dx \le 1\right\}.$$

The atomic Hardy space $H^{p(\cdot),q}_{\mathrm{atom}}(\mathbb{R}^n)$ is the set of all functions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that it can be written as

$$f = \sum_{j=1}^\infty \kappa_j a_j \text{ in } \mathcal{S}'(\mathbb{R}^n), \tag{5.8}$$

where $\{\kappa_j\}_{j=1}^\infty$ is a sequence of nonnegative numbers, $\{(a_j, Q_j)\}_{j=1}^\infty \subset \mathcal{A}(p(\cdot), q)$ and $\mathcal{A}(\{\kappa_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty)$ is finite. One defines

$$\|f\|_{H^{p(\cdot),q}_{\mathrm{atom}}} := \inf \mathcal{A}(\{\kappa_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty),$$

where the infimum is taken over all admissible expressions as in (5.8).

Suppose that $0 < p_- \le p_+ < \infty$. Under these definitions, in Section 6 we formulate the following.

Theorem 5.6. *The variable Hardy norms given in Theorem 5.2 and the ones given by means of atoms are isomorphic as long as*

$$q > p_+ \ge 1, \text{ or } q = 1 > p_+.$$

Remark that we could not specify the condition of q precisely in [30] but as the calculation in [36] shows $q > p_+ \geq 1$ or $q = 1 > p_+$ suffices.

§ 6. Atomic decompositions

In this section we consider decomposition.

Here, we define an index $d_{p(\cdot)} \in \mathbb{N} \cup \{0\}$ by

$$d_{p(\cdot)} := \min \{d \in \mathbb{N} \cup \{0\} : p_-(n + d + 1) > n\} . \tag{6.1}$$

For a nonnegative integer d, let $\mathcal{P}_d(\mathbb{R}^n)$ denote the set of all polynomials having degree at most d.

Let $p(\cdot) : \mathbb{R}^n \to (0, \infty)$, $0 < p_- \leq p_+ < q \leq \infty$ and $q \geq 1$. Recall that we have defined $(p(\cdot), q)$-atoms in Definition 5.4.

In the variable setting as well, we have that atoms have $L^{p(\cdot)}$-norm less than 1. We denote by $A(p(\cdot), q)$ the set of all pairs (a, Q) such that a is a $(p(\cdot), q)$-atom and that Q is the corresponding cube.

Remark 5.

1. Define another variable exponent $\tilde{q}(\cdot)$ by

$$\frac{1}{p(x)} = \frac{1}{q} + \frac{1}{\tilde{q}(x)} \quad (x \in \mathbb{R}^n). \tag{6.2}$$

Then we have

$$\|f \cdot g\|_{L^{p(\cdot)}} \lesssim \|g\|_{L^q} \|f\|_{L^{\tilde{q}(\cdot)}} \tag{6.3}$$

for all measurable functions f and g [21].

2. A direct consequence of Lemma 3.19 and (6.3) is that $\|a\|_{L^{p(\cdot)}} \lesssim 1$ whenever $(a, Q) \in A(p(\cdot), q)$.

Of course, as is the case when $p(\cdot)$ is a constant, Remark 5 can be extended as follows:

Proposition 6.1 (cf. [30, Proposition 4.2])**.**

1. *Let* $q > \max(1, p_+)$*. If* $p(\cdot)$ *satisfies* $0 < p_- \leq p_+ < \infty$ *as well as* (5.1) *and* (5.2)*, then we have*

$$\|a\|_{H^{p(\cdot)}} \lesssim 1$$

for any $(a, Q) \in A(p(\cdot), q)$.

2. If $p(\cdot)$ *satisfies* $0 < p_- \le p_+ < 1$ *as well as* (5.1) *and* (5.2), *then we have*

$$\|a\|_{H^{p(\cdot)}} \lesssim 1$$

for any $(a, Q) \in A(p(\cdot), 1)$.

The function space $H^{p(\cdot),q}_{\mathrm{atom}}(\mathbb{R}^n)$ was defined to be the set of all functions f such that it can be written in the form $f = \sum_{j=1}^{\infty} \kappa_j a_j$ in $\mathcal{S}'(\mathbb{R}^n)$, where $\mathcal{A}(\{\kappa_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}) < \infty$ and $\{(a_j, Q_j)\}_{j\in\mathbb{N}} \subset A(p(\cdot), q)$. One defines

$$\|f\|_{H^{p(\cdot),q}_{\mathrm{atom}}} := \inf \mathcal{A}(\{\kappa_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}),$$

where the infimum is taken over all expressions as above.

Observe that if $p(\cdot) \equiv p_+ = p_-$, that is, $p(\cdot)$ is a constant function, then we can recover classical Hardy spaces. Unlike the classical case, $(p(\cdot), \infty)$-atoms are not dealt separately. Consequently we have two types of results for $(p(\cdot), \infty)$-atoms.

Definition 6.2 ($H^{p(\cdot),\infty}_{\mathrm{atom},*}(\mathbb{R}^n)$, [30, Definition 4.3])**.** Let $p(\cdot) : \mathbb{R}^n \to (0, \infty)$, $0 < p_- \le p_+ < q \le \infty$ and $q \ge 1$. Then $f \in \mathcal{S}'(\mathbb{R}^n)$ is in $H^{p(\cdot),\infty}_{\mathrm{atom},*}(\mathbb{R}^n)$ if and only if there exist sequences of nonnegative numbers $\{\kappa_j\}_{j=1}^{\infty}$ and $\{(a_j, Q_j)\}_{j=1}^{\infty} \subset A(p(\cdot), \infty)$ such that

$$f = \sum_{j=1}^{\infty} \kappa_j a_j \text{ in } \mathcal{S}'(\mathbb{R}^n), \text{ and that } \sum_j \int_{Q_j} \left(\frac{\kappa_j}{\|\chi_{Q_j}\|_{L^{p(\cdot)}}} \right)^{p(x)} dx < \infty. \tag{6.4}$$

For sequences of nonnegative numbers $\{\kappa_j\}_{j=1}^{\infty}$ and cubes $\{Q_j\}_{j=1}^{\infty}$, define

$$\mathcal{A}^*(\{\kappa_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}) := \inf \left\{ \lambda > 0 : \int_{Q_j} \sum_j \left(\frac{\kappa_j}{\lambda \|\chi_{Q_j}\|_{L^{p(\cdot)}}} \right)^{p(x)} dx \le 1 \right\}.$$

Now we formulate our atomic decomposition theorem. Let us begin with the space $H^{p(\cdot),q}_{\mathrm{atom},*}(\mathbb{R}^n)$ with $q = \infty$.

Theorem 6.3 ([30, Theorem 4.5])**.** *If* $p(\cdot)$ *satisfies* $0 < p_- \le p_+ < \infty$, (5.1) *and* (5.2), *then, for all* $f \in \mathcal{S}'(\mathbb{R}^n)$,

$$\|f\|_{H^{p(\cdot)}} \sim \|f\|_{H^{p(\cdot),\infty}_{\mathrm{atom}}} \sim \|f\|_{H^{p(\cdot),\infty}_{\mathrm{atom},*}}.$$

The atomic decomposition for $\mathcal{A}(p(\cdot), q)$ can be also obtained.

Theorem 6.4 (cf. [30, Theorem 4.6])**.** *Suppose either* (*i*) *or* (*ii*) *holds;*

(*i*) $0 < p_- \le p_+ < q \le \infty$ *and* $p_+ \ge 1$*;*

(ii) $0 < p_- \le p_+ < 1 \le q \le \infty$.

Assume $p(\cdot)$ *satisfies* (5.1) *and* (5.2). *Then, for all* $f \in \mathcal{S}'(\mathbb{R}^n)$, $\|f\|_{H^{p(\cdot)}} \sim \|f\|_{H^{p(\cdot),q}_{\rm atom}}$.

§ 7. Applications of atomic decomposition

This section is a small modification of [30, Section 5]. We first state Theorem 7.2 based on Theorem 6.4, which refines [30, Theorem 5.2]. And then we recall what we obtained in [30].

§ 7.1. Molecular decomposition

Now we investigate molecular decomposition as an application of Theorems 6.3 and 6.4. Here we present a definition of molecules.

Definition 7.1 (Molecules [30, Definition 5.1]). Let $0 < p_- \le p_+ < q \le \infty$, $q \ge 1$ and $d \in [d_{p(\cdot)}, \infty) \cap \mathbb{Z}$ be fixed. One says that $\mathfrak{M}$ is a $(p(\cdot), q)$-molecule centered at a cube Q if it satisfies the following conditions.

1. On $2\sqrt{n}Q$, $\mathfrak{M}$ satisfies the estimate $\|\mathfrak{M}\|_{L^q(2\sqrt{n}Q)} \le \dfrac{|Q|^{\frac{1}{q}}}{\|\chi_Q\|_{L^{p(\cdot)}}}$.
2. Outside $2\sqrt{n}Q$, we have $|\mathfrak{M}(x)| \le \dfrac{1}{\|\chi_Q\|_{L^{p(\cdot)}}}\left(1 + \dfrac{|x-z|}{\ell(Q)}\right)^{-2n-2d-3}$. This condition is called the decay condition.
3. If α is a multiindex with length less than d, then we have
$$\int_{\mathbb{R}^n} x^\alpha \mathfrak{M}(x)\, dx = 0.$$
This condition is called the moment condition.

By definition $(p(\cdot), q)$-atoms are $(p(\cdot), q)$-molecules modulo a multiplicative constant.

As we did in [30], we are able to prove the following result.

Theorem 7.2 (cf. [30, Theorem 5.2]). *Let* $0 < p_- \le p_+ < q \le \infty$ *and* $d \in \mathbb{Z} \cup [d_{p(\cdot)}, \infty)$. *Assume either*

$$p_+ < 1 = q \text{ or } q > p_+ = 1.$$

Assume in addition that $p(\cdot)$ *satisfies* (5.1) *and* (5.2).

Suppose that $\{Q_j\}_{j=1}^\infty = \{Q(z_j, \ell_j)\}_{j=1}^\infty$ *is a sequence of cubes and, for each* $j \in \mathbb{N}$, *that we are given a* $(p(\,\cdot\,), q)$*-molecule* $\mathfrak{M}_j$ *centered at* Q_j. *If a sequence of positive numbers* $\{\kappa_j\}_{j=1}^\infty$ *satisfies*

$$\mathcal{A}\left(\{\kappa_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty\right) = 1, \text{ that is, } \int_{\mathbb{R}^n} \left(\sum_{j=1}^\infty \left|\frac{\kappa_j \chi_{Q_j}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}}}\right|^{\underline{p}}\right)^{\frac{p(x)}{\underline{p}}} dx \leq 1,$$

then we have

$$\left\|\sum_{j=1}^\infty \kappa_j \mathfrak{M}_j\right\|_{H^{p(\cdot)}} \lesssim 1. \tag{7.1}$$

§ 7.2. Boundedness of singular integral operators

If we combine Theorems 6.4 and 7.2, then we obtain the following theorem.

Theorem 7.3 ([30, Theorem 5.5])**.** *Assume that* $p(\,\cdot\,)$ *satisfies* $0 < p_- \leq p_+ < \infty$, (5.1) *and* (5.2). *Let* $k \in \mathcal{S}(\mathbb{R}^n)$ *and write*

$$A_m := \sup_{x \in \mathbb{R}^n} |x|^{n+m} |\nabla^m k(x)| \quad (m \in \mathbb{N} \cup \{0\}).$$

Define a convolution operator T *by*

$$Tf(x) := k * f(x) \quad (f \in L^2(\mathbb{R}^n)).$$

Then, T *can be extended also to an* $H^{p(\cdot)}(\mathbb{R}^n)$*-*$H^{p(\cdot)}(\mathbb{R}^n)$ *operator and the norm depends only on* $\|\mathcal{F}k\|_{L^\infty}$ *and a finite number of collections* $A_1, A_2, \ldots, A_N$ *with* N *depending only on* $p(\,\cdot\,)$.

§ 7.3. Littlewood-Paley characterization

Now we consider the Littlewood-Paley characterization of the function spaces.

We are going to characterize $H^{p(\cdot)}(\mathbb{R}^n)$ by means of the Littlewood-Paley decomposition.

The following lemma is a natural extension with $|\cdot|$ in the definition of $\mathcal{M}f$ replaced by $\ell^2(\mathbb{Z})$.

We introduce the $\ell^2(\mathbb{Z})$-valued function space $H^{p(\cdot)}(\mathbb{R}^n; \ell^2(\mathbb{Z}))$. Suppose that we are given a sequence $\{f_j\}_{j=-\infty}^\infty \subset \mathcal{S}'(\mathbb{R}^n)$.

Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\chi_{Q(0,1)} \leq \psi \leq \chi_{Q(0,2)}$. We set $\psi_k(\xi) := \psi(2^{-k}\xi)$. With this in mind, we define

$$\|\{f_j\}_{j=-\infty}^\infty\|_{H^{p(\cdot)}(\ell^2)} := \left\|\sup_{k \in \mathbb{Z}} \left(\sum_{j=-\infty}^\infty |\psi_k(D) f_j|^2\right)^{\frac{1}{2}}\right\|_{L^{p(\cdot)}}.$$

Observe that this is a natural vector-valued extension of

$$\|f\|_{H^{p(\cdot)}} \sim \left\| \sup_{k\in\mathbb{Z}} |2^{kn}\mathcal{F}^{-1}\psi(2^k\cdot) * f| \right\|_{L^{p(\cdot)}}.$$

We characterize Hardy spaces with variable exponents. Let us set $\varphi_j(x) := \varphi(2^{-j}x)$, $\varphi_j(D)f := \mathcal{F}^{-1}[\varphi(2^{-j}\cdot)\mathcal{F}f]$ for $f \in \mathcal{S}'(\mathbb{R}^n)$.

Theorem 7.4 ([30, Theorem 5.7]). *Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be a function supported on $Q(0,4) \setminus Q(0,1/4)$ such that*

$$\sum_{j=-\infty}^{\infty} |\varphi_j(\xi)|^2 > 0$$

for $\xi \in \mathbb{R}^n \setminus \{0\}$. Then the following norm is an equivalent norm of $H^{p(\cdot)}(\mathbb{R}^n)$:

$$\|f\|_{\dot{F}^0_{p(\cdot)2}} := \left\| \left(\sum_{j=-\infty}^{\infty} |\varphi_j(D)f|^2 \right)^{\frac{1}{2}} \right\|_{L^{p(\cdot)}}, \quad f \in \mathcal{S}'(\mathbb{R}^n). \tag{7.2}$$

§ 8. Campanato spaces with variable growth conditions

§ 8.1. Definition of Campanato spaces with variable growth conditions

Recall that $d_{p(\cdot)}$ is defined in (6.1) to be

$$d_{p(\cdot)} := \min\{d \in \mathbb{N} \cup \{0\} : p_-(n+d+1) > n\}.$$

Let $L^q_{\mathrm{comp}}(\mathbb{R}^n)$ be the set of all $L^q(\mathbb{R}^n)$-functions having compact support. Given a nonnegative integer d, let

$$L^{q,d}_{\mathrm{comp}}(\mathbb{R}^n) := \left\{ f \in L^q_{\mathrm{comp}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} f(x)x^\alpha\,dx = 0,\ |\alpha| \le d \right\}.$$

Likewise if Q is a cube, then we write

$$L^{q,d}(Q) := \left\{ f \in L^q(Q) : \int_Q f(x)x^\alpha\,dx = 0,\ |\alpha| \le d \right\}.$$

If d is as in (6.1), then $L^{q,d}_{\mathrm{comp}}(\mathbb{R}^n)$ is dense in $H^{p(\cdot),q}_{\mathrm{atom}}(\mathbb{R}^n)$, Indeed, it contains all the finite linear combinations of $(p(\cdot),q)$-atoms from the definition of $H^{p(\cdot),q}_{\mathrm{atom}}(\mathbb{R}^n)$.

Recall that $\mathcal{P}_d(\mathbb{R}^n)$ is the set of all polynomials having degree at most d. For a locally integrable function f, a cube Q and a nonnegative integer d, there exists a unique polynomial $P \in \mathcal{P}_d(\mathbb{R}^n)$ such that, for all $q \in \mathcal{P}_d(\mathbb{R}^n)$,

$$\int_Q (f(x) - P(x))q(x)\,dx = 0.$$

Denote this unique polynomial P by $P_Q^d f$. It follows immediately from the definition that $P_Q^d g = g$ if $g \in \mathcal{P}_d(\mathbb{R}^n)$.

Definition 8.1 ([30, Definition 6.1], $\mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$)**.** Let $1 \le q \le \infty$. Let $\phi : \mathcal{Q} \to (0,\infty)$ be a function and $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$. One denotes

$$\|f\|_{\mathcal{L}_{q,\phi,d}} = \sup_{Q\in\mathcal{Q}} \frac{1}{\phi(Q)} \left(\frac{1}{|Q|} \int_Q |f(x) - P_Q^d f(x)|^q \, dx \right)^{1/q},$$

when $q < \infty$ and

$$\|f\|_{\mathcal{L}_{q,\phi,d}} = \sup_{Q\in\mathcal{Q}} \frac{1}{\phi(Q)} \|f - P_Q^d f\|_{L^\infty(Q)}.$$

when $q = \infty$. Then the Campanato space $\mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$ is defined to be the sets of all $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ such that $\|f\|_{\mathcal{L}_{q,\phi,d}} < \infty$. One considers elements in $\mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$ modulo polynomials of degree d so that $\mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$ is a Banach space. When one writes $f \in \mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$, then f stands for the representative of $\{f + P : P \text{ is a polynomial of degree } d\}$.

Here and below we abuse notation slightly. We write $\phi(x,r) := \phi(Q(x,r))$ for $x \in \mathbb{R}^n$ and $r > 0$.

§ 9. Duality $H^{p(\cdot)}(\mathbb{R}^n)$-$\mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$

In this section, we shall prove that the dual spaces of $H^{p(\cdot)}(\mathbb{R}^n)$ are generalized Campanato spaces with variable growth conditions when $0 < p_- \le p_+ \le 1$.

§ 9.1. Dual of $H^{p_0}(\mathbb{R}^n) \cap H^1(\mathbb{R}^n)$ with $0 < p_0 \le 1$

In this subsection, let p_0 be a constant with $0 < p_0 \le 1$. This subsection is an auxiliary step to investigate $H^{p(\cdot)}(\mathbb{R}^n)^*$.

If $p(\cdot)$ is a constant function, then the dual is known to exist [14].

Keeping this in mind, we now seek to investigate the structure of $\mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$.

Recall that $\mathrm{bmo}(\mathbb{R}^n)$, the local BMO, is the set of all locally integrable functions f such that

$$\|f\|_{\mathrm{bmo}} := \sup_{\substack{Q\in\mathcal{Q}\\ |Q|\le 1}} \fint_Q \left| f(x) - \fint_Q f(y)\,dy \right| dx + \sup_{\substack{Q\in\mathcal{Q}\\ |Q|=1}} \fint_Q |f(x)|\,dx < \infty.$$

Then from the definition of the norms $\|\cdot\|_{\mathrm{BMO}}$ and $\|\cdot\|_{\mathrm{bmo}}$ we have $\|f\|_{\mathrm{BMO}} \lesssim \|f\|_{\mathrm{bmo}}$. By the well-known $H^1(\mathbb{R}^n)$-$\mathrm{BMO}(\mathbb{R}^n)$ duality, $\mathrm{bmo}(\mathbb{R}^n)$ is canonically embedded into the dual space of $H^1(\mathbb{R}^n)$.

Theorem 9.1 ([30, Theorem 7.3]). *Let* $0 < p_0 \le 1$ *and* $1 \le q \le \infty$. *Set* $\phi_1(Q) := |Q|^{\frac{1}{p_0}-1}$ *and* $\phi_2(Q) := |Q|^{\frac{1}{p_0}-1} + 1$ *for* $Q \in \mathcal{Q}$. *Then we have* $\mathcal{L}_{q,\phi_2,d}(\mathbb{R}^n) \hookrightarrow \mathcal{L}_{q,\phi_1,d}(\mathbb{R}^n) + \mathrm{bmo}(\mathbb{R}^n)$ *in the sense of continuous embedding. More quantitatively, if we choose* $\psi \in \mathcal{S}(\mathbb{R}^n)$ *so that* $\chi_{Q(0,1)} \le \psi \le \chi_{Q(0,2)}$, *then we have*

$$\|\psi(D)g\|_{\mathcal{L}_{q,\phi_1,d}} \lesssim \|g\|_{\mathcal{L}_{q,\phi_2,d}}, \quad \|(1-\psi(D))g\|_{\mathrm{bmo}} \lesssim \|g\|_{\mathcal{L}_{q,\phi_2,d}}.$$

§ 9.2. Dual spaces of $H^{p(\,\cdot\,)}(\mathbb{R}^n)$

Now we specify the dual of $H^{p(\,\cdot\,)}(\mathbb{R}^n)$ with $0 < p_- \le p_+ \le 1$. It follows from the definition of the dual norm that, for all $\ell \in (H^{p(\,\cdot\,),q}_{\mathrm{atom}}(\mathbb{R}^n))^*$,

$$\|\ell\|_{(H^{p(\,\cdot\,),q}_{\mathrm{atom}}(\mathbb{R}^n))^*} = \sup\left\{|\ell(f)| : \|f\|_{H^{p(\,\cdot\,),q}_{\mathrm{atom}}} \le 1\right\}$$

is finite and $\|\ell\|_{(H^{p(\,\cdot\,),q}_{\mathrm{atom}}(\mathbb{R}^n))^*}$ is a norm on $(H^{p(\,\cdot\,),q}_{\mathrm{atom}}(\mathbb{R}^n))^*$. We prove the following theorem.

Theorem 9.2 (cf. [30, Theorem 7.5]). *Let* $p(\,\cdot\,) : \mathbb{R}^n \to (0,\infty)$, $0 < p_- \le p_+ \le 1$, $p_+ < q \le \infty$ *and* $1/q + 1/q' = 1$. *Suppose that the integer* d *is as in* (6.1). *Define*

$$\phi_3(Q) := \frac{\|\chi_Q\|_{L^{p(\,\cdot\,)}}}{|Q|} \quad (Q \in \mathcal{Q}). \tag{9.1}$$

If $p(\,\cdot\,)$ *satisfies* (5.1) *and* (5.2), *then*

$$(H^{p(\,\cdot\,),q}_{\mathrm{atom}}(\mathbb{R}^n))^* \simeq \mathcal{L}_{q',\phi_3,d}(\mathbb{R}^n)$$

with equivalent norms. More precisely, we have the following assertions.

1. *Let* $f \in \mathcal{L}_{q',\phi_3,d}(\mathbb{R}^n)$. *Then the functional*

$$\ell_f : a \in L^{q,d}_{\mathrm{comp}}(\mathbb{R}^n) \mapsto \int_{\mathbb{R}^n} a(x)f(x)\,dx \in \mathbb{C}$$

   extends to a bounded linear functional on $(H^{p(\,\cdot\,),q}_{\mathrm{atom}}(\mathbb{R}^n))^*$ *such that*

$$\|\ell_f\|_{(H^{p(\,\cdot\,),q}_{\mathrm{atom}})^*} \lesssim \|f\|_{\mathcal{L}_{q',\phi_3,d}}.$$

2. *Conversely, any linear functional* ℓ *on* $(H^{p(\,\cdot\,),q}_{\mathrm{atom}}(\mathbb{R}^n))^*$ *can be realized as above with some* $f \in \mathcal{L}_{q',\phi_3,d}(\mathbb{R}^n)$ *and we have* $\|f\|_{\mathcal{L}_{q',\phi_3,d}} \lesssim \|\ell\|_{(H^{p(\,\cdot\,),q}_{\mathrm{atom}})^*}$.

In particular, we have

$$(H^{p(\,\cdot\,)}(\mathbb{R}^n))^* \simeq \mathcal{L}_{q',\phi_3,d}(\mathbb{R}^n).$$

Namely, any $f \in \mathcal{L}_{q',\phi_3,d}(\mathbb{R}^n)$ *defines a continuous linear functional on* $(H^{p(\,\cdot\,)}(\mathbb{R}^n))^*$ *such that*

$$L_f(a) = \int_{\mathbb{R}^n} a(x)f(x)\,dx$$

for any $a \in L^{q,d}_{\rm comp}(\mathbb{R}^n)$ *and any continuous linear functional on* $(H^{p(\,\cdot\,)}(\mathbb{R}^n))^*$ *is realized with some some* $f \in \mathcal{L}_{q',\phi_3,d}(\mathbb{R}^n)$.

Note that there was no need to assume $q \gg 1$ in Theorem 9.2, since we refined Theorem 6.4. When $q \gg 1$, this theorem is recorded as [30, Theorem 7.5].

§ 9.3. An open problem

Open Problem 9.3. *Do we have analogies of Theorems* 9.1 *and* 9.2 *for general cases ?*

A partial answer is;

Proposition 9.4. *When* $p_- > 1$, *then we have*

$$H^{p(\cdot)}(\mathbb{R}^n)' \sim L^{p(\cdot)}(\mathbb{R}^n)' \sim H^{p'(\cdot)}(\mathbb{R}^n).$$

How do we characterize the dual of $H^{p(\cdot)}(\mathbb{R}^n)$ for general cases, that is, without assuming $p_+ \le 1$?

Besov spaces and Triebel-Lizorkin spaces are useful tools but about the dual we have the folloing:

Proposition 9.5. *For* $0 < p < 1$,

$$h^p(\mathbb{R}^n) \sim F^0_{p2}(\mathbb{R}^n) \to B^{n/p-n}_{\infty\infty}(\mathbb{R}^n).$$

For $p = 1$,

$$h^p(\mathbb{R}^n) \sim F^0_{p2}(\mathbb{R}^n) \to \mathrm{bmo}(\mathbb{R}^n) = F^0_{\infty 2}(\mathbb{R}^n).$$

For $p > 1$,

$$h^p(\mathbb{R}^n) \sim F^0_{p2}(\mathbb{R}^n) \to F^0_{p'2}(\mathbb{R}^n).$$

So, starting from the Triebel-Lizorkin scale, the resulting duals can be both Besov spaces and Triebel-Lizorkin spaces. Once we mix the situation about p, it seems no longer possible to determine duals.

§ 10. Hölder-Zygmund spaces with variable exponents

In this section we assume that

$$0 < p_- \le p_+ < 1. \tag{10.1}$$

We consider the function spaces of Hölder-Zygmund type and we connect them in particular with the function spaces $\mathcal{L}^D_{q,\phi_3}(\mathbb{R}^n)$ that we are going to define, where again we let $\phi_3(Q) = \dfrac{\|\chi_Q\|_{L^{p(\cdot)}}}{|Q|}$ for $Q \in \mathcal{Q}$.

§ 10.1. Definition of Hölder-Zygmund spaces with variable exponents

We define Δ_h^k to be a difference operator, which is defined inductively by

$$\Delta_h^1 f = \Delta_h f := f(\cdot + h) - f, \quad \Delta_h^k := \Delta_h^1 \circ \Delta_h^{k-1}, \quad k \geq 2. \tag{10.2}$$

Definition 10.1 ([30, Definition 8.1], $\Lambda_{\phi,d}(\mathbb{R}^n)$)**.** Let $\phi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ and $d \in \mathbb{N} \cup \{0\}$. Then $\Lambda_{\phi,d}(\mathbb{R}^n)$, the Hölder space with variable exponent $p(\cdot)$, is defined to be the set of all continuous functions f such that $\|f\|_{\Lambda_{\phi,d}} < \infty$, where

$$\|f\|_{\Lambda_{\phi,d}} := \sup_{x\in\mathbb{R}^n,\, h\neq 0} \frac{1}{\phi(x,|h|)} \left|\Delta_h^{d+1} f(x)\right|.$$

One considers elements in $\Lambda_{\phi,d}(\mathbb{R}^n)$ modulo polynomials of degree d so that $\Lambda_{\phi,d}(\mathbb{R}^n)$ is a Banach space. When one writes $f \in \Lambda_{\phi,d}(\mathbb{R}^n)$, then f stands for the representative of $\{f + P \,:\, P \text{ is a polynomial of degree } d\,\}$.

Several helpful remarks may be in order.

Remark 6 ([30, Remark 8.2]).

1. Assume that there exists a constant $\mu > 0$ such that $\phi(Q) \lesssim |Q|^\mu$ for all Q with $|Q| \geq 1$. If a continuous function f satisfies $\|f\|_{\Lambda_{\phi,d}} < \infty$, then f is of polynomial order. In particular the representative of such a function f can be regarded as an element in $\mathcal{S}'(\mathbb{R}^n)$. Actually, since f is assumed continuous, f is bounded on a neighborhood $Q(0,1)$. Using $\|f\|_{\Lambda_{\phi,d}} < \infty$, inductively on $k \in \mathbb{N} \cup \{0\}$ we can show that $|f(x)| \lesssim (k+1)^{d+\mu+1}$ for all x with $k \leq |x| \leq k+1$.

2. It is absolutely necessary to assume that f is a continuous function, when $d \geq 1$. We remark that there exists a discontinuous function f such that $\Delta_h^{d+1} f(x) = 0$ for all $x, h \in \mathbb{R}^n$. See [23] for such an example.

3. The function space $\Lambda_{\phi,d}(\mathbb{R}^n)$ is used to measure the Hölder continuity uniformly, when ϕ does not depend on x. Such an attempt can be found in [22].

As for $\Lambda_{\phi,d}(\mathbb{R}^n)$, we have the following equivalence.

Theorem 10.2 ([30, Theorem 8.4])**.** *Assume that $\phi : \mathcal{Q} \to (0,\infty)$ satisfies the following conditions.*

(A1) *There exists a constant* $C > 0$ *such that*

$$C^{-1} \leq \frac{\phi(x,r)}{\phi(x,2r)} \leq C, \quad (x \in \mathbb{R}^n,\, r > 0).$$

(A2) *There exists a constant* $C > 0$ *such that*

$$C^{-1} \leq \frac{\phi(x,r)}{\phi(y,r)} \leq C, \quad (x, y \in \mathbb{R}^n,\, r > 0,\, |x-y| \leq r).$$

(A3) *There exists a constant* $C > 0$ *such that*

$$\int_0^r \frac{\phi(x,t)}{t}\, dt \leq C\phi(x,r), \quad (x \in \mathbb{R}^n,\, r > 0).$$

Then the function spaces $\Lambda_{\phi,d}(\mathbb{R}^n)$ *and* $\mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$ *are isomorphic. Speaking more precisely, we have the following* :

1. *For any* $f \in \Lambda_{\phi,d}(\mathbb{R}^n)$ *we have* $\|f\|_{\mathcal{L}_{q,\phi,d}} \lesssim \|f\|_{\Lambda_{\phi,d}}$.
2. *Any element in* $\mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$ *has a continuous representative. Moreover, whenever* $f \in \mathcal{L}_{q,\phi,d}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$, *then* $f \in \Lambda_{\phi,d}(\mathbb{R}^n)$ *and we have* $\|f\|_{\Lambda_{\phi,d}} \lesssim \|f\|_{\mathcal{L}_{q,\phi,d}}$.

§ 11. Local Hardy spaces with variable exponents

What we have been doing can be transplanted into the theory of the local Hardy spaces. For example, if $\psi \in \mathcal{S}(\mathbb{R}^n)$ is such that $\displaystyle\int_{\mathbb{R}^n} \psi(x)\, dx \neq 0$, and if we define the norm by

$$\|f\|_{h^{p(\cdot)}} = \left\| \sup_{0<t<1} \sup_{\varphi \in \mathcal{F}_N} |t^{-n}\varphi(t^{-1}\cdot) * f| \right\|_{L^{p(\cdot)}}, \tag{11.1}$$

then we see that

$$\|f\|_{h^{p(\cdot)}} \sim \left\| \sup_{j \in \mathbb{N}} |\psi_j(D) f| \right\|_{L^{p(\cdot)}}, \tag{11.2}$$

where $\psi_j(\xi) = \psi(2^{-j}\xi)$.

To conclude this paper, we establish the norms of $h^{p(\cdot)}(\mathbb{R}^n)$ and $F^0_{p(\cdot)2}(\mathbb{R}^n)$ are equivalent. Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be a bump function satisfying $\chi_{Q(0,1)} \leq \psi \leq \chi_{Q(0,2)}$ and set

$$\varphi_j := \psi(2^{-j}\cdot) - \psi(2^{-j+1}\cdot)$$

for $j \in \mathbb{N}$. In [11] Diening, Hästö and Roudenko defined the function space $F^0_{p(\,\cdot\,)2}(\mathbb{R}^n)$, the one of Triebel-Lizorkin type, with the norm

$$\|f\|_{F^0_{p(\,\cdot\,)2}} := \|\psi(D)f\|_{L^{p(\,\cdot\,)}} + \left\| \left(\sum_{j=1}^{\infty} |\varphi_j(D)f|^2 \right)^{\frac{1}{2}} \right\|_{L^{p(\,\cdot\,)}}$$

for $f \in \mathcal{S}'(\mathbb{R}^n)$.

Theorem 11.1 ([30, Theorem 9.2]). *Let $0 < p_- \le p_+ < \infty$. The function spaces $h^{p(\,\cdot\,)}(\mathbb{R}^n)$ and $F^0_{p(\,\cdot\,)2}(\mathbb{R}^n)$ are isomorphic to each other.*

Other results of the present paper have counterpart for $h^{p(\,\cdot\,)}(\mathbb{R}^n)$. For example, when we consider the local Hardy spaces, their duals will be the Besov spaces defined in [1] by virtue of the counterpart of Theorems 9.2. The proofs being analogous to the corresponding proofs for $H^{p(\,\cdot\,)}(\mathbb{R}^n)$, we omit the details.

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