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The Hardy-Littlewood maximal operator on Lebesgue spaces with variable exponent

By

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Abstract

The aim of the present paper is to consider the boundedness of the Hardy-Littlewood maximal operator on the generalized Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ with variable exponent. On the generalized Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ with variable exponent, the boundedness of the Hardy-Littlewood maximal operator was proved by Diening and Cruz-Uribe, Fiorenza and Neugebauer at the beginning of this century. In this paper we rearrange their proof. After giving a simple proof of the boundedness of the Hardy-Littlewood maximal operator, we provide some examples showing the necessity of some regularity conditions on $p(\cdot)$ for the boundedness. As an application of the auxiliary pointwise estimate for the Hardy-Littlewood maximal operator we prove some density results for generalized Lebesgue and Sobolev spaces with variable exponent.

Contents

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§ 1. Introduction

The aim of this paper is to develop a theory of variable Lebesgue spaces. Mainly we consider the Hardy-Littlewood maximal operator. This paper contains some of well-known results and their proofs for convenience.

On the generalized Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ with variable exponent, the boundedness of the Hardy-Littlewood maximal operator (see (1.1) for its definition) was proved by Diening [6] (2004) and Cruz-Uribe, Fiorenza and Neugebauer [4, 5] (2003, 2004). In the present paper we rearrange their proof. Our proof may be simpler than the original. This idea was given in [28, 29] by Mizuta and Shimomura.

For a variable exponent $p(\cdot) : \mathbb{R}^n \to (0, \infty)$, let

$$p_- := \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x), \quad p_+ := \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x).$$

Let $L^{p(\cdot)}(\mathbb{R}^n)$ be the set of all measurable functions f on $\mathbb{R}^n$ such that $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} < \infty$, where

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} := \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \le 1\right\}.$$

If $1 \le p_- \le p_+ < \infty$, then $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ is a norm and thereby $L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach space. For a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the Hardy-Littlewood maximal function $Mf(x)$ is defined by

$$Mf(x) := \sup_{B\ni x} ⨍_B |f(y)|\,dy, \quad ⨍_B |f(y)|\,dy := \frac{1}{|B|}\int_B |f(y)|\,dy, \tag{1.1}$$

where the supremum is taken over all balls B containing x. It is well known that the operator M is bounded on $L^p(\mathbb{R}^n)$ if $1 < p \le \infty$. Here, to discuss the difference between the case for variable Lebesgue spaces and the classical case, we recall its proof in Section 8. See Theorem 5.1 as well, where a plausible analogy is not available. Indeed, in Theorem 5.1, for a measurable function $p(\cdot) : \mathbb{R}^n \to [1, \infty)$, we shall show that $p(\cdot)$ is a constant function, if there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^n} \{Mf(x)\}^{p(x)}\, dx \le C \int_{\mathbb{R}^n} |f(x)|^{p(x)}\, dx$$

for all measurable functions $f : \mathbb{R}^n \to \mathbb{C}$. This carries over to the non-doubling setting. Diening [6] and Cruz-Uribe, Fiorenza and Neugebauer [4, 5] proved the following:

Theorem 1.1 ([4, 5, 6]). *If $p(\cdot)$ satisfies*

$$1 < p_- \le p_+ < \infty, \tag{1.2}$$

$$|p(x) - p(y)| \le \frac{c_*}{\log(1/|x-y|)} \quad \text{for} \quad |x-y| \le \frac{1}{2}, \tag{1.3}$$

$$|p(x) - p(y)| \le \frac{c^*}{\log(e+|x|)} \quad \text{for} \quad |y| \ge |x|, \tag{1.4}$$

for some positive constants c_* and c^*, then the operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

The boundedness follows from the next pointwise estimate and the boundedness of M on $L^{p_-}(\mathbb{R}^n)$ for $p_- > 1$.

Theorem 1.2 ([4, 5, 6]). *If $p(\cdot)$ satisfies* (1.3), (1.4) *and* $1 \le p_- \le p_+ < \infty$, *then there exists a positive constant C, dependent only on n and $p(\cdot)$, such that, for all functions f with $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le 1$,*

$$Mf(x)^{p(x)} \le C(M(|f(\cdot)|^{p(\cdot)/p_-})(x)^{p_-} + (e+|x|)^{-np_-}) \quad \text{for all } x \in \mathbb{R}^n.$$

We prove Theorem 1.2 in Section 4 by using the idea of Mizuta and Shimomura in [28, 29].

Admitting Theorem 1.2, let us prove Theorem 1.1. In the present paper we use C to denote various positive constants, which may differ from line to line. Also, here and below it will be understood that

$$Mf(x)^{p(x)} = \{Mf(x)\}^{p(x)} \quad (x \in \mathbb{R}^n).$$

By no means $Mf(x)^{p(x)}$ is equal to $M[|f|^{p(\cdot)}](x)$. Remark that both appear in the proof and they should be dealt as different things.

Proof of Theorem 1.1. By homogeneity, it is enough to prove that, there exists a positive constant C such that $\|Mf\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C$ for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ with $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le 1$. Note that $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le 1$ is equivalent to

$$\int_{\mathbb{R}^n} |f(x)|^{p(x)}\, dx \le 1.$$

In this case, letting $g(x) := |f(x)|^{p(x)/p_-}$, we have $\|g\|_{L^{p_-}(\mathbb{R}^n)} \le 1$. By Theorem 1.2 and the boundedness of M on $L^{p_-}(\mathbb{R}^n)$ for $p_- > 1$ (see Theorem 8.2), we have

$$\begin{aligned}\int_{\mathbb{R}^n} Mf(x)^{p(x)}\,dx &\le C\int_{\mathbb{R}^n} Mg(x)^{p_-}\,dx + C\int_{\mathbb{R}^n} \frac{dx}{(e+|x|)^{np_-}}\\ &\le C\int_{\mathbb{R}^n} g(x)^{p_-}\,dx + C\\ &\le C.\end{aligned}$$

This shows that

$$\int_{\mathbb{R}^n} \left(\frac{Mf(x)}{C}\right)^{p(x)} dx \le 1,$$

since $1 < p_- \le p_+ < \infty$. That is $\|Mf\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C$. □

Before we go further, we state properties of $p(\cdot)$. From $p_+ < \infty$ and (1.3) it follows that

$$|p(x) - p(y)| \le \frac{C}{\log(e + 1/|x-y|)} \quad \text{for all } x, y \in \mathbb{R}^n. \tag{1.5}$$

From (1.4) it follows that there exists a constant p_∞ such that

$$|p(x) - p_\infty| \le \frac{C}{\log(e+|x|)} \quad \text{for all } x \in \mathbb{R}^n. \tag{1.6}$$

The theory of Lebesgue spaces with variable exponent dates back to Nakano's books in 1950 and 1951 [33, 34] and Orlicz [36]. In particular, Nakano defined Musielak-Orlicz spaces explicitly in [33]. However, it remained intact for a long time until the advent of the papers Sharapudinov [39] and Kováčik–Rákosník [20]. Finally, the pioneering works [6, 8] by Diening paved the way with which to connect harmonic analysis and variable exponent Lebesgue spaces.

Compared with the proof on classical Lebesgue spaces in Section 8, a barrier of the proof of the boundedness on Lebesgue spaces with variable exponent is the disability of using Lemma 8.4. As is described in (8.4) appearing in the proof of Lemma 8.4, we have

$$\int_{\mathbb{R}^n} |g(x)|\,dx = \int_0^\infty |\{x \in \mathbb{R}^n \,:\, |g(x)| > t\}|\,dt \tag{1.7}$$

for all measurable functions $f : \mathbb{R}^n \to \mathbb{C}$. As is remarked above and in Theorem 5.1, it is not possible to prove

$$\int_{\mathbb{R}^n} \{Mf(x)\}^{p(x)}\,dx \le C\int_{\mathbb{R}^n} |f(x)|^{p(x)}\,dx \tag{1.8}$$

unless the function $p(\cdot)$ is a constant.

If one mimics the argument in Lemma 8.4, then we are faced with the difficulty: Indeed, we cannot apply the change of variables in (8.4). From (1.7), we readily have

$$\int_{\mathbb{R}^n} |f(x)|^{p(x)}\,dx = \int_0^\infty |\{x\in\mathbb{R}^n\,:\,|f(x)|^{p(x)}>t\}|\,dt$$

but it is not possible to change variables $t \mapsto s^{p(x)}$ because we can not deal with $s^{p(x)-1}$ in a satisfactory manner.

Faced to such difficulties, we seek a method of proving a weak-type inequalities:

$$\sup_{\lambda>0} \lambda \|\chi_{\{Mf>\lambda\}}\|_{L^{p(\cdot)}} \le C\|f\|_{L^{p(\cdot)}}. \tag{1.9}$$

Note that (1.9) is a consequence of Theorem 1.2. The thrust of considering (1.9) is the weak inequality

$$\|\chi_{\{x\in\mathbb{R}^n\,:\,Mf(x)>\lambda\}}\|_{L^p} = |\{x\in\mathbb{R}^n\,:\,Mf(x)>\lambda\}|^{1/p} \le C\,\lambda^{-1}\|f\|_{L^p(\mathbb{R}^n)},$$

which is discussed in (8.3). The proof seems similar to the proof of Theorem 8.2(1), which asserts that the Hardy-Littlewood maximal operator M is of weak type $(1,1)$, namely,

$$|\{x\in\mathbb{R}^n\,:\,Mf(x)>\lambda\}| \le C\,\lambda^{-1}\|f\|_{L^1(\mathbb{R}^n)}$$

holds for all $\lambda>0$ and all $f\in L^1(\mathbb{R}^n)$. However, since there is no way to control

$$\|\chi_{\{Mf>\lambda\}}\|_{L^{p(\cdot)}}$$

even after we prove Theorem 8.2(1), this method does not seem to work. Inequality (1.9) is proved by Cruz-Uribe, Diening and Fiorenza [1]. See Proposition 9.1 below.

§ 2. Weight class A_p

Recently it turns out that the theory of maximal operators on variable Lebesgue spaces has a lot to do with the theory of weights.

By "a weight" w, we mean that it is a non-negative a.e. $\mathbb{R}^n$ and locally integrable function. Below we write

$$w(S) := \int_S w(x)\,dx$$

for a weight w and a measurable set S.

Definition 2.1. Let $w>0$ be a weight and $1\le p<\infty$ a constant. The weighted Lebesgue space $L^p_w(\mathbb{R}^n)$ is defined by

$$L^p_w(\mathbb{R}^n) := \left\{f \text{ is measurable and complex-valued} \,:\, \|f\|_{L^p_w(\mathbb{R}^n)}<\infty\right\},$$

where

$$\|f\|_{L^p_w(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx\right)^{1/p}.$$

Of course if $w(x) \equiv 1$, then $L^p_w(\mathbb{R}^n)$ means the usual Lebesgue space $L^p(\mathbb{R}^n)$.

Proposition 2.2 (Muckenhoupt [30]). *Let $w > 0$ a.e. $\mathbb{R}^n$ be a weight.*

(1) *If $1 < p < \infty$, then the following three conditions are equivalent:*

(a) *The weight w is the Muckenhoupt A_p weight, i.e.,*

$$[w]_{A_p} := [w]_{A_p(\mathbb{R}^n)} = \sup_B w_B \left([w^{-1/(p-1)}]_B \right)^{p-1} < \infty, \tag{2.1}$$

where the supremum is taken over all open balls B.

(b) *The Hardy-Littlewood maximal operator M is bounded on $L^p_w(\mathbb{R}^n)$.*

(c) *M is of weak type (p,p) on $L^p_w(\mathbb{R}^n)$, namely, we have that for all $f \in L^p_w(\mathbb{R}^n)$ and all $\lambda > 0$,*

$$w(\{x \in \mathbb{R}^n : Mf(x) > \lambda\})^{1/p} \le C\,\lambda^{-1}\,\|f\|_{L^p_w(\mathbb{R}^n)}.$$

(2) *The following two conditions are equivalent:*

(a) *The weight w is the Muckenhoupt A_1 weight, i.e.,*

$$Mw(x) \le C\,w(x),$$

or equivalently

$$[w]_{A_1} = [w]_{A_1(\mathbb{R}^n)} := \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} \frac{Mw(x)}{w(x)} < \infty \tag{2.2}$$

holds.

(b) *M is of weak type $(1,1)$ on $L^1_w(\mathbb{R}^n)$, namely, we have that for all $f \in L^1_w(\mathbb{R}^n)$ and all $\lambda > 0$,*

$$w(\{x \in \mathbb{R}^n : Mf(x) > \lambda\}) \le C\,\lambda^{-1}\,\|f\|_{L^1_w(\mathbb{R}^n)}.$$

Example 2.3 ([11]). Let $a \in \mathbb{R}$. We consider the power weight $|x|^a$ defined on $\mathbb{R}^n$.

(1) Let $1 < p < \infty$. Then the weight $|x|^a$ is the Muckenhoupt A_p weight if and only if $-n < a < n(p-1)$.

(2) The weight $|x|^a$ is the Muckenhoupt A_1 weight if and only if $-n < a \le 0$.

The theory carries over to the spaces on open sets. Let Ω be an open set in $\mathbb{R}^n$ and, for measurable functions f on Ω, define

$$Mf(x) := \sup_B \fint_{B\cap\Omega} |f(y)|\, dy, \quad \fint_{B\cap\Omega} |f(y)|\, dy := \frac{1}{|B|}\int_{B\cap\Omega} |f(y)|\, dy. \tag{2.3}$$

where the supremum is taken over all balls B containing x.

In analogy with (2.1) and (2.2) for an open set Ω, we write

$$[w]_{A_p(\Omega)} := \sup_B w_{B\cap\Omega}\left([w^{-1/(p-1)}]_{B\cap\Omega}\right)^{p-1} < \infty, \tag{2.4}$$

where B runs over all balls and

$$[w]_{A_1(\Omega)} := \operatorname*{ess\,sup}_{x\in\Omega} \frac{Mw(x)}{w(x)} < \infty.$$

In the definition above, we are tempted to use cubes instead of balls because we need a geometric property of cubes.

The next result is an analogy of the one due to Lerner, Ombrosi and Pérez [25]. Let Q be a cube and $x \in \mathbb{R}^n$. Define $\mathcal{D}(Q)$ the set of all dyadic cubes with respect to Q. More precisely, let $Q := x + [-r/2, r/2]^n$. Then a dyadic cube with respect to Q is a cube that can be expressed as

$$Q \cap (x + (r/2^{\nu+1})m + [0, r/2^{\nu+1}]^n), \quad m \in \mathbb{Z}^n,\ \nu \in \mathbb{Z}.$$

Denote by $\mathcal{D}(Q)_x$ the subset of all cubes in $\mathcal{D}(Q)$ that contain x.

Theorem 2.4. *Let $w \in A_1(\Omega)$. Define*

$$M_{Q,\mathrm{dyadic};\Omega} w(x) := \sup_{R\in\mathcal{D}(Q)_x} \frac{1}{|R|}\int_{R\cap\Omega} w(y)\, dy \quad (x \in \mathbb{R}^n).$$

If we set $\delta := \frac{1}{2^{n+1}[w]_{A_1(\Omega)}}$, then we have

$$\left(\fint_{Q\cap\Omega} M_{Q,\mathrm{dyadic};\Omega} w(x)^\delta w(x)\, dx\right)^{\frac{1}{1+\delta}} \le 2\fint_{Q\cap\Omega} w(x)\, dx$$

for all cubes Q.

Observe that $M_{Q,\mathrm{dyadic};\Omega}$ is controlled by M; $M_{Q,\mathrm{dyadic};\Omega} w \le CMw$.

Proof. First we note that, for any positive constant r, we have

$$[\min(w, r)]_{A_1} \le [w]_{A_1},$$

from the definition of the Muckenhoupt A_1 weight. Then, by replacing w with $\min(w, r)$ with $r > 0$, we can and do assume that $w \in L^\infty(\mathbb{R}^n)$. Abbreviate $⨍_{Q\cap\Omega} w(x)\,dx$ to μ. Then we have

$$\begin{aligned}
&⨍_{Q\cap\Omega} M_{Q,\mathrm{dyadic};\Omega}w(x)^\delta w(x)\,dx \\
&:= \frac{1}{|Q|}\int_0^\infty \delta\lambda^{\delta-1} w\left\{x \in Q\cap\Omega \,:\, M_{Q,\mathrm{dyadic};\Omega}w(x) > \lambda\right\} d\lambda \\
&= \frac{1}{|Q|}\left(\int_0^\mu + \int_\mu^\infty\right) \delta\lambda^{\delta-1} w\left\{x \in Q\cap\Omega \,:\, M_{Q,\mathrm{dyadic};\Omega}w(x) > \lambda\right\} d\lambda \\
&\le \mu^{\delta+1} + \frac{1}{|Q|}\int_\mu^\infty \delta\lambda^{\delta-1} w\left\{x \in Q\cap\Omega \,:\, M_{Q,\mathrm{dyadic};\Omega}w(x) > \lambda\right\} d\lambda.
\end{aligned}$$

Let $\lambda > \mu$. Then we can decompose

$$\{x \in Q\cap\Omega \,:\, M_{Q,\mathrm{dyadic};\Omega}w(x) > \lambda\} = \bigcup_j Q_j \cap \Omega$$

into a union of dyadic cubes $\{Q_j\}_j$ such that

$$\frac{1}{|Q_j|}\int_{Q_j\cap\Omega} w(x)\,dx > \lambda \ge \frac{1}{2^n|Q_j|}\int_{Q_j\cap\Omega} w(x)\,dx = \frac{1}{2^n|Q_j|} w(Q_j\cap\Omega)$$

and that

$$|Q_j \cap Q_{j'}| = 0 \quad (j \ne j').$$

Hence

$$\begin{aligned}
&w\left\{x \in Q\cap\Omega \,:\, M_{Q,\mathrm{dyadic};\Omega}w(x) > \lambda\right\} \\
&= \sum_j w(Q_j\cap\Omega) \\
&\le 2^n \sum_j |Q_j|\lambda \\
&= 2^n\lambda\,|\{x \in Q\cap\Omega \,:\, M_{Q,\mathrm{dyadic};\Omega}w(x) > \lambda\}|.
\end{aligned}$$

Inserting this estimate, we obtain

$$\begin{aligned}
&\frac{1}{|Q|}\int_\mu^\infty \delta\lambda^{\delta-1} w\left\{x \in Q\cap\Omega \,:\, M_{Q,\mathrm{dyadic};\Omega}w(x) > \lambda\right\} d\lambda \\
&\le \frac{2^n}{|Q|}\int_\mu^\infty \delta\lambda^{\delta}\,|\{x \in Q\cap\Omega \,:\, M_{Q,\mathrm{dyadic};\Omega}w(x) > \lambda\}|\,d\lambda \\
&\le \frac{2^n}{|Q|}\int_0^\infty \delta\lambda^{\delta}\,|\{x \in Q\cap\Omega \,:\, M_{Q,\mathrm{dyadic};\Omega}w(x) > \lambda\}|\,d\lambda \\
&= \frac{2^n\delta}{1+\delta}⨍_{Q\cap\Omega} M_{Q,\mathrm{dyadic};\Omega}w(x)^{1+\delta}\,dx.
\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}
&\fint_{Q\cap\Omega} M_{Q,\mathrm{dyadic};\Omega}w(x)^{\delta}w(x)\,dx \\
&\le \mu^{\delta+1} + \frac{2^n\delta}{\delta+1}\fint_{Q\cap\Omega} M_{Q,\mathrm{dyadic};\Omega}w(x)^{1+\delta}\,dx \\
&\le \mu^{\delta+1} + \frac{2^n\delta}{\delta+1}\fint_{Q\cap\Omega} M_{Q,\mathrm{dyadic};\Omega}w(x)^{\delta}Mw(x)\,dx \\
&\le \mu^{\delta+1} + \frac{2^n\delta[w]_{A_1(\Omega)}}{\delta+1}\fint_{Q\cap\Omega} M_{Q,\mathrm{dyadic};\Omega}w(x)^{\delta}w(x)\,dx \\
&\le \mu^{\delta+1} + \frac{1}{2}\fint_{Q\cap\Omega} M_{Q,\mathrm{dyadic};\Omega}w(x)^{\delta}w(x)\,dx.
\end{aligned}$$

Now that we are assuming that $w \in L^{\infty}(\mathbb{R}^n)$, it follows from the absorbing argument that

$$\fint_{Q\cap\Omega} M_{Q,\mathrm{dyadic};\Omega}w(x)^{\delta}w(x)\,dx \le 2\mu^{\delta+1} = 2\left(\fint_{Q\cap\Omega} w(x)\,dx\right)^{1+\delta}.$$

The proof is therefore complete. □

§ 3. Boundedness of the Hardy-Littlewood maximal operator on domains

In this section we recall some known results. To formulate results let us use the following notations, which are standard in the setting of variable exponents.

Given a measurable set Ω of $\mathbb{R}^n$, we recall that we wrote

$$Mf(x) = \sup_{B\ni x}\fint_B |f(y)|\,dy, \quad \fint_B |f(y)|\,dy := \frac{1}{|B|}\int_{B\cap\Omega}|f(y)|\,dy.$$

We write

$$p_- := \operatorname*{ess\,inf}_{x\in\Omega} p(x), \quad p_+ := \operatorname*{ess\,sup}_{x\in\Omega} p(x).$$

Definition 3.1.

(1) The set $\mathcal{P}(\Omega)$ consists of all variable exponents $p(\cdot) : \Omega \to [1,\infty]$ such that $1 < p_- \le p_+ < \infty$.

(2) The set $\mathcal{B}(\Omega)$ consists of all variable exponents $p(\,\cdot\,) \in \mathcal{P}(\Omega)$ such that the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\Omega)$.

(3) A measurable function $r(\,\cdot\,) : \Omega \to (0,\infty)$ is said to be locally log-Hölder continuous if

$$|r(x) - r(y)| \le \frac{C}{-\log(|x-y|)} \quad (|x-y| \le 1/2)$$

is satisfied. The set $LH_0(\Omega)$ consists of all locally log-Hölder continuous functions.

(4) A measurable function $r(\,\cdot\,) : \Omega \to (0, \infty)$ is said to be log-Hölder continuous at ∞ if

$$|r(x) - r(y)| \le \frac{C}{\log(e + |x|)} \quad (|y| \ge |x|)$$

is satisfied. The set $LH_\infty(\Omega)$ consists of all measurable functions being log-Hölder continuous at ∞.

(5) The set $LH(\Omega)$ consists of all measurable functions satisfying the two log-Hölder continuous properties above, namely, $LH(\Omega) := LH_0(\Omega) \cap LH_\infty(\Omega)$.

It could not be better if everything were settled in the framework of $\mathcal{P}(\Omega)$. However, we have a counterexample. See Section 6.

Before we proceed further, a helpful remark may be in order.

Remark 1. We can easily check the following facts.

(1) As we have seen, given a measurable function $r(\,\cdot\,) : \Omega \to (0, \infty)$, we see that the following two conditions are equivalent:

(*a*) $r(\,\cdot\,) \in LH_\infty(\Omega)$.

(*b*) There exists a constant r_∞ such that

$$|r(x) - r_\infty| \le \frac{C}{\log(e + |x|)} \quad (x \in \Omega).$$

(2) Let a variable exponent $p(\,\cdot\,) : \Omega \to [1, \infty)$ satisfy $p_+ < \infty$. Then $p(\,\cdot\,) \in LH(\Omega)$ if and only if $1/p(\,\cdot\,) \in LH(\Omega)$.

(3) Let $p(\cdot) \in \mathcal{P}(\Omega)$. Then $p(\cdot) \in LH(\Omega)$ holds if and only if $1/p(\cdot) \in LH(\Omega)$ holds.

There are some famous results on sufficient conditions of variable exponents for the boundedness of the Hardy-Littlewood maximal operator. If a variable exponent $p(\cdot) : \mathbb{R}^n \to [1, \infty]$ satisfies $1 < p_- \le p_+ \le \infty$, we define

$$\|f\|_{L^{p(\cdot)}} := \|\chi_{\{p<\infty\}} f\|_{L^{p(\cdot)}(\{p<\infty\})} + \|\chi_{\{p=\infty\}} f\|_{L^\infty}.$$

Proposition 3.2.

(1) Diening [6] (2004): *If Ω is bounded, then $\mathcal{P}(\Omega) \cap LH_0(\Omega) \subset \mathcal{B}(\Omega)$ holds.*

(2) Cruz-Uribe–Fiorenza–Neugebauer [4] (2004): *Let Ω be an open set of $\mathbb{R}^n$. Then $\mathcal{P}(\Omega) \cap LH(\Omega) \subset \mathcal{B}(\Omega)$.*

(3) Cruz-Uribe–Diening–Fiorenza [1] (2009) and Diening–Harjulehto–Hästö–Mizuta–Shimomura [9] (2009): *If a variable exponent $p(\cdot) : \mathbb{R}^n \to [1, \infty]$ satisfies $1 < p_- \le p_+ \le \infty$ and $1/p(\,\cdot\,) \in LH(\mathbb{R}^n)$, then the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.*

Next we state a necessary condition for the boundedness of the Hardy-Littlewood maximal operator.

Proposition 3.3 ([9]). *Let $p(\cdot):\Omega\to[1,\infty]$ be a variable exponent. If M is bounded on $L^{p(\cdot)}(\Omega)$, then $p_->1$ holds.*

The proof is originally by Diening, Harjulehto, Hästö, Mizuta and Shimomura [9]. However, Lerner extended this result to Banach function spaces when $\Omega=\mathbb{R}^n$ (see [24, Theorem 1.2] and [22, Corollary 1.3]). Here we transform Lerner's proof to our setting.

Proof of Proposition 3.3. First we show that, if M is bounded on $L^{p(\cdot)}(\Omega)$, then M is also bounded on $L^{p(\cdot)/(1+\delta)}(\Omega)$ for some $\delta>0$: Since M is assumed bounded on $L^{p(\cdot)}(\Omega)$, there exists a constant $C_0>0$ such that

$$\|Mf\|_{L^{p(\cdot)}}\le C_0\|f\|_{L^{p(\cdot)}}.$$

Define

$$g(x):=\sum_{j=0}^{\infty}\frac{1}{(2C_0)^j}M^jf(x),$$

where it will be understood that $M^0f(x)=|f(x)|$. Observe also that $\|g\|_{L^{p(\cdot)}}\sim\|f\|_{L^{p(\cdot)}}$. Since M is sublinear, we have

$$\begin{aligned}Mg(x)&=M\left[\sum_{j=0}^{\infty}\frac{1}{(2C_0)^j}M^jf\right](x)\\&=\lim_{J\to\infty}M\left[\sum_{j=0}^{J}\frac{1}{(2C_0)^j}M^jf\right](x)\\&\le\sum_{j=0}^{\infty}\frac{1}{(2C_0)^j}M^{j+1}f(x)\\&\le 2C_0g(x).\end{aligned}$$

This means that g is an A_1-weight and the A_1-constant is less than $2C_0$. Thus, we are in the position of using the reverse Hölder inequality (Theorem 2.4) and we obtain

$$M[g^{1+\delta}](x)\le C|g(x)|^{1+\delta}\quad(x\in\Omega).$$

Here the constants C and δ depend only upon n and C_0. Thus, we obtain

$$\|M[|f|^{1+\delta}]\|_{L^{p(\cdot)/(1+\delta)}}\le\|M[g^{1+\delta}]\|_{L^{p(\cdot)/(1+\delta)}}\le C(\|g\|_{L^{p(\cdot)}})^{1+\delta}\le C(\|f\|_{L^{p(\cdot)}})^{1+\delta}.$$

The function $f\in L^{p(\cdot)}(\Omega)$ being arbitrary, it follows that the operator M is bounded on $L^{p(\cdot)/(1+\delta)}(\Omega)$.

Next, with this in mind, assume that M is bounded on $L^{p(\cdot)}(\Omega)$ with $p_- = 1$. Then M is also bounded on $L^{p(\cdot)/(1+\delta)}(\Omega)$ for some $\delta > 0$. In this case, the set

$$U = \left\{ x \in \Omega \cap B(0,R) : \frac{p(x)}{1+\delta} \le \frac{1}{1+\delta/2} \right\}$$

has positive measure for large $R > 0$. Hence there exists $f \in L^{p(\cdot)/(1+\delta)}(\Omega)$ such that $\int_{\Omega \cap B(0,R)} |f(x)|\, dx = \infty$. For example, we partition U into a collection $\{U_j\}_{j=1}^{\infty}$ of measurable sets such that

$$|U_j| = 2^{-j}|U| \quad (j = 1, 2, \cdots).$$

We let

$$f(x) = \sum_{j=1}^{\infty} \frac{1}{|U_j|} \chi_{U_j} \in L^{p(\,\cdot\,)/(1+\delta)}(\mathbb{R}^n).$$

Then

$$Mf(x) \ge \frac{1}{B(x, |x| + 2R)} \int_{\Omega \cap B(x,|x|+2R)} f(y)\, dy = \infty$$

on Ω and by virtue of the generalized Hölder inequality (see, for example, [16, Theorem 3.8]), we obtain

$$\begin{aligned} \|f\|_{L^{p(\cdot)/(1+\delta)}} &\le C_R \|f\|_{L^{1/(1+\delta/2)}} \\ &= C_R \left(\sum_{j=1}^{\infty} |U_j|^{\delta/2} \right)^{1+\delta/2} \\ &< \infty \end{aligned}$$

and hence the inequality $\|Mf\|_{L^{p(\cdot)/(1+\delta)}} \le C\|f\|_{L^{p(\cdot)/(1+\delta)}}$ fails. This is a contradiction. Therefore, we have the conclusion. □

§ 4. Pointwise estimate

We aim here to prove Theorem 1.2. We supply a simpler proof by using an idea by Mizuta and Shimomura in [28, 29].

For a nonnegative function f on $\mathbb{R}^n$ and a ball $B(x, r)$, we write

$$I := I(x,r) = \fint_{B(x,r)} f(y)\, dy, \quad J := J(x,r) = \fint_{B(x,r)} f(y)^{p(y)}\, dy. \tag{4.1}$$

Then, observe that

$$Mf(x) \sim \sup_{r>0} I \quad \text{and} \quad M(|f(\cdot)|^{p(\cdot)})(x) \sim \sup_{r>0} J.$$

Let

$$\mathcal{F}_{p(\cdot)} := \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : f(y) \geq 1 \text{ or } f(y) = 0 \text{ for each } y \in \mathbb{R}^n, \text{ and } \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq 1 \right\},$$
$$\mathcal{G} := \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : 0 \leq f(y) < 1 \text{ for each } y \in \mathbb{R}^n \right\}.$$

It counts that we do not postulate any condition of $\|f\|_{L^{p(\cdot)}}$ on $\mathcal{G}$. To prove Theorem 1.2 we state two lemmas concerning pointwise estimates.

Lemma 4.1. *Let $p(\cdot)$ satisfy* (1.3) *and $1 \leq p_- \leq p_+ < \infty$. Then there exists a positive constant C, dependent only on n and $p(\cdot)$, such that, for all functions $f \in \mathcal{F}_{p(\cdot)}$ and for all balls $B(x,r)$,*

$$I \leq C J^{1/p(x)}.$$

Lemma 4.2. *Let $p(\cdot)$ satisfy* (1.4) *and $1 \leq p_- \leq p_+ < \infty$. Then there exists a positive constant C, dependent only on n and $p(\cdot)$, such that, for all functions $f \in \mathcal{G}$ and for all balls $B(x,r)$,*

$$I \leq C(J^{1/p(x)} + (e + |x|)^{-n}).$$

Proof of Lemma 4.1. Let $B = B(x,r)$ and let $f \in \mathcal{F}_{p(\cdot)}$.

Case 1: $J > 1$. In this case $1 < J \leq 1/|B| = 1/(v_n r^n)$, since $\int f(y)^{p(y)}\,dy \leq 1$, where v_n denotes the volume of the unit ball in $\mathbb{R}^n$. Let $m = [1 + \log 1/v_n]$, where $[a]$ denotes the integer part of $a \in \mathbb{R}$. Then

$$1 < J \leq \frac{1}{v_n r^n} \leq \frac{e^m}{r^n} \leq \left(e + \frac{1}{r}\right)^{m+n}. \tag{4.2}$$

Let $K = J^{1/p(x)}$. Then, for $y \in B(x,r)$, using (1.5) and (4.2), we have

$$\begin{aligned} |(p(x) - p(y)) \log K| &= \frac{|p(x) - p(y)|}{p(x)} \log J \\ &\leq \frac{|p(x) - p(y)|}{p_-}(m+n) \log\left(e + \frac{1}{r}\right) \\ &\leq \frac{C(m+n)}{p_- \log(e + 1/|x-y|)} \log\left(e + \frac{1}{r}\right) \leq C, \end{aligned}$$

that is, $K^{p(x)} \sim K^{p(y)}$. Hence

$$\begin{aligned} I &= \fint_B f(y)\chi_{\{f\le K\}}(y)\,dy + \fint_B f(y)\chi_{\{f>K\}}(y)\,dy \\ &\le \fint_B K\,dy + \fint_B f(y)\left(\frac{f(y)}{K}\right)^{p(y)-1} dy \\ &= K + \fint_B \frac{K}{K^{p(y)}} f(y)^{p(y)}\,dy \\ &\le C\left(K + \frac{K}{K^{p(x)}}J\right) = CK = CJ^{1/p(x)}. \end{aligned}$$

Case 2: $J \le 1$. Recall that $f \in \mathcal{F}_{p(\cdot)}$. In this case, using $f(y) \le f(y)^{p(y)}$, we have

$$I \le J \le J^{1/p(x)}.$$

Therefore we have the conclusion. □

Proof of Lemma 4.2. Let $B = B(x,r)$ and $f \in \mathcal{G}$. Let

$$\begin{aligned} E_1 &= \{y \in B : |y| < |x|,\ (e+|y|)^{-n-1} \le f(y) < 1\}, \\ E_2 &= \{y \in B : |y| < |x|,\ 0 \le f(y) < (e+|y|)^{-n-1}\}. \\ E_3 &= \{y \in B : |y| \ge |x|,\ (e+|x|)^{-n-1} \le f(y) < 1\}, \\ E_4 &= \{y \in B : |y| \ge |x|,\ 0 \le f(y) < (e+|x|)^{-n-1}\}. \end{aligned}$$

Observe that $\{E_i\}_{i=1}^4$ partitions B. So we have

$$\frac{1}{|B|}\int_B f(y)\,dy = \sum_{i=1}^4 \frac{1}{|B|}\int_{E_i} f(y)\,dy.$$

Case 1: Integration over E_1. By (1.4), we have

$$\begin{aligned} |(p(x)-p(y))\log f(y)| &= |p(x)-p(y)|\log(1/f(y)) \\ &\le \frac{C}{\log(e+|y|)}\log((e+|y|)^{n+1}) \\ &= C, \end{aligned}$$

that is, $f(y)^{p(x)} \sim f(y)^{p(y)}$. Let $K = J^{1/p(x)}$. Then

$$\begin{aligned} \frac{1}{|B|}\int_{E_1} f(y)\,dy &\le \frac{1}{|B|}\int_{E_1} K\,dy + \frac{1}{|B|}\int_{E_1} f(y)\left(\frac{f(y)}{K}\right)^{p(x)-1} dy \\ &\le C\left(K + \frac{K}{K^{p(x)}}\cdot\frac{1}{|B|}\int_{E_1} f(y)^{p(y)}\,dy\right) \\ &\le C\left(K + \frac{K}{K^{p(x)}}J\right) = CK = CJ^{1/p(x)}. \end{aligned}$$

Case 2: Integration over E_2.

(a) If $r \le |x|/2$, then $|x| \sim |y|$. Hence

$$\begin{aligned}\frac{1}{|B|}\int_{E_2} f(y)\,dy &\le \frac{1}{|B|}\int_{E_2}\frac{dy}{(e+|y|)^{n+1}}\\ &\le C\frac{1}{|B|}\int_{E_2}\frac{dy}{(e+|x|)^{n+1}}\\ &= C\frac{|E_2|}{|B|(e+|x|)^{n+1}}\\ &\le C\frac{1}{(e+|x|)^{n+1}}.\end{aligned}$$

(b) If $r > |x|/2$ and $|x| > 1$, then, using the fact that $|B| = v_n r^n$, we obtain

$$\begin{aligned}\frac{1}{|B|}\int_{E_2} f(y)\,dy &\le \frac{1}{|B|}\int_{E_2}\frac{dy}{(e+|y|)^{n+1}}\\ &\le \frac{1}{|B|}\int_{\mathbb{R}^n}\frac{dy}{(e+|y|)^{n+1}}\\ &\le \frac{C}{r^n}\\ &\le \frac{C}{(e+|x|)^n}.\end{aligned}$$

(c) If $|x| \le 1$, then the estimate is simple:

$$\frac{1}{|B|}\int_{E_2} f(y)\,dy \le 1 \le \frac{4^n}{(e+|x|)^n}.$$

Case 3: Integration over E_3. By (1.4), we have

$$|(p(x)-p(y))\cdot\log f(y)| = |p(x)-p(y)|\cdot\log\frac{1}{f(y)} \le C\frac{\log((e+|x|)^{n+1})}{\log(e+|x|)} = C,$$

that is, $f(y)^{p(x)} \sim f(y)^{p(y)}$. Then, by the same calculation as Case 1, we conclude

$$\frac{1}{|B|}\int_{E_3} f(y)\,dy \le 2J^{1/p(x)}.$$

Case 4: Integration over E_4. A crude estimate using $0 \le f(z) \le (1+|z|)^{-n-1}$ for all $z \in E_4$ suffices

$$\frac{1}{|B|}\int_{E_4} f(y)\,dy \le \frac{1}{|B|}\int_{E_4}\frac{dy}{(e+|x|)^{n+1}} = \frac{|E_4|}{|B|(e+|x|)^{n+1}} \le \frac{1}{(e+|x|)^{n+1}}.$$

Therefore, we have the conclusion. □

Proof of Theorem 1.2. Let $\|f\|_{L^{p(\cdot)}} \le 1$. We may assume that f is nonnegative. Split f by $f = f_1 + f_2$, where

$$f_1 := f\chi_{\{f\ge 1 \text{ or } f=0\}}, \quad f_2 := f\chi_{\{0<f<1\}}.$$

Write $\overline{p}(x) := p(x)/p_-$. Then $\overline{p}$ still satisfies (1.3), (1.4) and $1 \le \overline{p}_- \le \overline{p}_+ < \infty$. In this case $\|f_1\|_{L^{\overline{p}(\cdot)}} \le 1$, since $f_1(y)^{\overline{p}(y)} \le f_1(y)^{p(y)} \le f(y)^{p(y)}$, that is, $f_1 \in \mathcal{F}_{\overline{p}(\cdot)}$ and $f_2 \in \mathcal{G}$. Let

$$I := I(x,r) = ⨍_{B(x,r)} f(y)\,dy, \quad \bar{J} := \bar{J}(x,r) = ⨍_{B(x,r)} f(y)^{\overline{p}(y)}\,dy,$$
$$I_i := I_i(x,r) = ⨍_{B(x,r)} f_i(y)\,dy, \quad \bar{J}_i := \bar{J}_i(x,r) = ⨍_{B(x,r)} f_i(y)^{\overline{p}(y)}\,dy, \quad i = 1,2.$$

By Lemmas 4.1 and 4.2 we have

$$I = I_1 + I_2 \le C\bar{J}_1^{1/\overline{p}(x)} + C(\bar{J}_2^{1/\overline{p}(x)} + (e+|x|)^{-n}) \le C(\bar{J}^{1/\overline{p}(x)} + (e+|x|)^{-n}).$$

Then

$$I^{p(x)} \le C(\bar{J}^{p_-} + (e+|x|)^{-np_-}),$$

that is,

$$\left(⨍_{B(x,r)} f(y)\,dy\right)^{p(x)} \le C\left(\left(⨍_{B(x,r)} f(y)^{p(y)/p_-}\,dy\right)^{p_-} + (e+|x|)^{-np_-}\right),$$

for all balls $B(x,r)$. Then we have the conclusion. □

With Lemmas 4.1 and 4.2 proven, the proofs of Theorems 1.1 and 1.2 are complete.

§ 5. Modular inequalities

An interesting result is proved by Lerner [21, Theorem 1.1], where Lerner considered the size of A_∞ constants. In this section, we give an alternative proof. Our proof can be extended to the setting of the non-doubling measures readily. Recall that

$$\mathcal{P}(\mathbb{R}^n) := \{p(\cdot) \in L^\infty(\mathbb{R}^n) \,:\, p(\cdot) \text{ is positive and satisfies } 1 < p_- \le p_+ < \infty\}.$$

Theorem 5.1 ([21]). *If $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, then the following two conditions are equivalent:*

(*a*) *We have that for all* $f \in L^{p(\,\cdot\,)}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \{Mf(x)\}^{p(x)}\,dx \le C \int_{\mathbb{R}^n} |f(x)|^{p(x)}\,dx. \tag{5.1}$$

(*b*) *The variable exponent* $p(\,\cdot\,)$ *equals to a constant.*

The implication $(b) \Rightarrow (a)$ is well known, see Section 8 for its proof. It counts that $(a) \Rightarrow (b)$ is true. This implies a difference between the norm inequality and the modular inequality (5.1). In particular we see that the inequality (5.1) shows a stronger condition than the norm one. Izuki [14] has considered the similar problems for some operators arising from multiresolution analyses and wavelets.

Here we shall supply a new proof without using the notion of A_∞-weights, which was obtained by carefully reexamining the original proof of Lerner [21].

Proof of Theorem 5.1. As is remarked above, the heart of the matters is to prove that (a) implies (b). The indicator function testing (5.1) essentially suffices. Assume that (a) holds and that $p(\cdot)$ is not a.e. equal to a constant function on a ball B. Let

$$p_-(B) = \operatorname*{ess\,inf}_{x\in B} p(x), \quad p_+(B) = \operatorname*{ess\,sup}_{x\in B} p(x).$$

For $\varepsilon > 0$, we write

$$E_\varepsilon := \{x \in B \,:\, p(x) > p_+(B) - \varepsilon\}.$$

Since $p_-(B) < p_+(B)$, there exists $\varepsilon > 0$ such that $p_+(B) - 2\varepsilon > p_-(B) + \varepsilon$. In this case we have $0 < |B \setminus E_{2\varepsilon}| < |B|$ and $|E_\varepsilon| > 0$.

Let $t > 1$. Then, from (5.1) by letting $f := t\chi_{B\setminus E_{2\varepsilon}}$, we obtain

$$\begin{aligned}\int_B M[t\chi_{B\setminus E_{2\varepsilon}}](x)^{p(x)}\,dx &\le \int_{\mathbb{R}^n} M[t\chi_{B\setminus E_{2\varepsilon}}](x)^{p(x)}\,dx\\ &\le C\int_{\mathbb{R}^n} (t\chi_{B\setminus E_{2\varepsilon}}(x))^{p(x)}\,dx\\ &= C\int_{B\setminus E_{2\varepsilon}} t^{p(x)}\,dx\\ &\le Ct^{p_+(B)-2\varepsilon}|B\setminus E_{2\varepsilon}|.\end{aligned}$$

Since $M[t\chi_{B\setminus E_{2\varepsilon}}](x) \ge \frac{|B\setminus E_{2\varepsilon}|}{|B|}\chi_{E_\varepsilon}(x)t$, it follows that

$$\begin{aligned}\int_B M[t\chi_{B\setminus E_{2\varepsilon}}](x)^{p(x)}\,dx &\ge \int_B \left(\frac{|B\setminus E_{2\varepsilon}|}{|B|}\right)^{p(x)} \chi_{E_\varepsilon}(x)t^{p(x)}\,dx\\ &\ge \left(\frac{|B\setminus E_{2\varepsilon}|}{|B|}\right)^{p_+(B)} |E_\varepsilon|t^{p_+(B)-\varepsilon}.\end{aligned}$$

From both inequalities we have

$$t^{\varepsilon} \leq C \left(\frac{|B|}{|B \setminus E_{2\varepsilon}|} \right)^{p_+(B)} \frac{|B \setminus E_{2\varepsilon}|}{|E_{\varepsilon}|},$$

for any $t > 1$. This is a contradiction. □

The proof carries over the setting of the (non-doubling) metric measure spaces, where the notion of A_∞-weights is immature. Recall that in the metric measure space (X, d, μ), the uncentered maximal operator

$$M'_k f(x) := \sup \left\{ \frac{1}{\mu(B(y, kr))} \int_{B(y,r)} |f(z)| \, d\mu(z) \, : \, B(y, r) \ni x \right\}$$

and the centered maximal operator

$$M_k f(x) := \sup \left\{ \frac{1}{\mu(B(x, kr))} \int_{B(x,r)} |f(y)| \, d\mu(y) \, : \, r > 0 \right\}$$

satisfy

$$\|M'_3 f\|_{L^p(\mu)} \leq \frac{p \, 2^p}{p-1} \|f\|_{L^p(\mu)}, \quad \|M_2 f\|_{L^p(\mu)} \leq \frac{p \, 2^p}{p-1} \|f\|_{L^p(\mu)}, \tag{5.2}$$

respectively. Here

$$\|f\|_{L^p(\mu)} = \left(\int_X |f(x)|^p \, d\mu(x) \right)^{1/p}.$$

For estimates (5.2) for M'_3 and M_2 we refer to [35] and [37, 41] respectively.

Mimicking the above proof, we can prove the following for a measurable function.

Theorem 5.2. *Let $p(\cdot) : X \to [1, \infty)$ be a measurable function.*

1. *Let $k \geq 3$. If there exists a constant $C > 0$ such that*

$$\int_X M'_k f(x)^{p(x)} \, d\mu(x) \leq C \int_X |f(x)|^{p(x)} \, d\mu(x),$$

if and only if $p(\cdot)$ is equal to an a.e. constant function.

2. *Let $k \geq 2$. If there exists a constant $C > 0$ such that*

$$\int_X M_k f(x)^{p(x)} \, d\mu(x) \leq C \int_X |f(x)|^{p(x)} \, d\mu(x),$$

if and only if $p(\cdot)$ is equal to an a.e. constant function.

§ 6. Counterexamples

In this section, to show the necessity of some regularity on $p(\cdot)$, we give several examples of $p(\cdot)$ for which the Hardy-Littlewood maximal operator M is not bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ with $n = 1$.

We will use the following fundamental facts in Propositions 6.1 and 6.3, respectively: For $a > 0$, we shall show

$$M[|\cdot|^{-\theta}\chi_{(0,a]}](x) \geq C|x|^{-\theta}\chi_{[-a,0)\cup(0,a]}(x), \quad \text{if} \quad 0 < \theta < 1,$$

(see (6.2)) and

$$M[|\cdot|^{-\theta}\chi_{[a,\infty)}](x) \geq C|x|^{-\theta}\chi_{(-\infty,-a]\cup[a,\infty)}(x), \quad \text{if} \quad \theta > 0.$$

We learned these propositions from Diening's talk.

The variable exponent $p(\cdot)$ in the following proposition doesn't satisfy the local log-Hölder continuity condition (1.3):

Proposition 6.1. *Let $n = 1$ and $p(\cdot) := 4\chi_{(-\infty,0)} + 2\chi_{[0,\infty)}$. Then the operator M is not bounded on $L^{p(\cdot)}(\mathbb{R})$.*

Proof. Let $f(x) := |x|^{-1/3}\chi_{(0,1)}(x)$. Then

$$\int_{-\infty}^{\infty} \left|\frac{f(x)}{\sqrt{3}}\right|^{p(x)} dx = \int_0^1 \left|\frac{x^{-1/3}}{\sqrt{3}}\right|^2 dx = \int_0^1 \frac{x^{-2/3}}{3}\, dx = 1. \tag{6.1}$$

Hence $\|f\|_{L^{p(\cdot)}} = \sqrt{3}$. On the other hand, for $x \in (-1, 0)$,

$$Mf(x) \geq \frac{1}{2|x|}\int_x^{-x} f(y)\, dy = \frac{1}{2|x|}\int_0^{-x} |x|^{-1/3}\, dy \geq \frac{|x|^{-1/3}}{2}. \tag{6.2}$$

Then, for any $\lambda > 1$,

$$\int_{-1}^0 \left|\frac{Mf(x)}{\lambda}\right|^4 dx \geq \frac{1}{(2\lambda)^4}\int_{-1}^0 |x|^{-4/3}\, dx = \infty.$$

That is, $\|Mf\|_{L^{p(\cdot)}} = \infty$. □

By the same argument as Proposition 6.1, we can prove the following

Corollary 6.2. *Let $n = 1$ and $p(\cdot) := 2\chi_{(-\infty,-2]} + 4\chi_{(-2,0)} + 2\chi_{[0,\infty)}$. Then the operator M is not bounded on $L^{p(\cdot)}(\mathbb{R})$.*

The variable exponent $p(\cdot)$ in the following proposition doesn't satisfy the log-Hölder type decay condition (1.4):

Proposition 6.3. *Let $n = 1$ and $p(\cdot) : \mathbb{R} \to (0, \infty)$. If $p(x) \le 2$ on $(-\infty, -k)$ and $p(x) \ge 4$ on $[k, \infty)$ for some $k \ge 0$, then the operator M is not bounded on $L^{p(\cdot)}(\mathbb{R})$.*

Proof. Let $f(x) := |x|^{-1/3}\chi_{[\max(1,k),\infty)}(x)$. Then

$$\int_{-\infty}^{\infty} \left|\frac{f(x)}{\sqrt[4]{3}}\right|^{p(x)} dx \le \int_{\max(1,k)}^{\infty} \left|\frac{|x|^{-1/3}}{\sqrt[4]{3}}\right|^{4} dx \le \int_{1}^{\infty} \frac{x^{-4/3}}{3}\, dx = 1.$$

Hence $\|f\|_{L^{p(\cdot)}} \le \sqrt[4]{3}$. On the other hand, for $x < -2\max(1,k)$,

$$(1 \ge) Mf(x) \ge \frac{1}{2|x|}\int_{x}^{-x} f(y)\, dy \ge \frac{1}{2|x|}\int_{\max(1,k)}^{-x} |x|^{-1/3}\, dy \ge \frac{|x|^{-1/3}}{4}.$$

Then, for any $\lambda > 1$,

$$\begin{aligned} \int_{-\infty}^{\infty} \left|\frac{Mf(x)}{\lambda}\right|^{p(x)} dx &\ge \int_{-\infty}^{-\max(1,k)} \left|\frac{Mf(x)}{\lambda}\right|^{2} dx \\ &\ge \frac{1}{(4\lambda)^2}\int_{-\infty}^{-\max(1,k)} |x|^{-2/3}\, dx = \infty. \end{aligned}$$

That is, $\|Mf\|_{L^{p(\cdot)}} = \infty$. □

A similar argument works and we obtain the following variant of the above proposition.

Corollary 6.4. *Let $n = 1$ and $p(\cdot) : \mathbb{R} \to (0, \infty)$ be an exponent. If*

$$\limsup_{x \to -\infty} p(x) < 2, \quad \liminf_{x \to \infty} p(x) > 4,$$

then the operator M is not bounded on $L^{p(\cdot)}(\mathbb{R})$.

The next example shows that the log-Hölder type decay condition (1.4) is necessary in a sense.

Proposition 6.5 ([4]). *Fix $p_\infty \in (1, \infty)$. Let $\phi : [0, \infty) \to [0, p_\infty - 1)$ be such that*

$$\phi(0) = \lim_{x \to \infty} \phi(x) = 0, \quad \lim_{x \to \infty} \phi(x) \log x = \infty.$$

Assume in addition that ϕ is decreasing on $[1, \infty)$. Define

$$p(x) = p_\infty - \phi(\max(x, 0)) \quad (x \in \mathbb{R}).$$

Then M is not bounded on $L^{p(\cdot)}(\mathbb{R})$.

Proof. Note first that

$$\lim_{x\to\infty}\left(1-\frac{p_\infty}{p(2x)}\right)\log x=-\infty.$$

Hence

$$\lim_{x\to\infty} x^{1-p_\infty/p(2x)}=0.$$

Thus, we can find a negative sequence $\{c_n\}_{n=1}^\infty$ such that

$$c_{n+1}<2c_n<-4,\quad |c_n|^{1-p_\infty/p(2|c_n|)}\le 2^{-n}\quad (\text{for all } n\in\mathbb{N}).$$

Define

$$f(x):=\sum_{n=1}^\infty |c_n|^{-1/p(2|c_n|)}\chi_{(2c_n,c_n)}(x).$$

Since

$$\int_{\mathbb{R}}|f(x)|^{p(x)}\,dx=\sum_{n=1}^\infty |c_n|^{-p_\infty/p(2|c_n|)}|c_n|\le 1,$$

we have $f\in L^{p(\cdot)}(\mathbb{R})$. Meanwhile, if $x\in(-c_n,-2c_n)$, then

$$Mf(x)\ge\frac{-1}{4c_n}\int_{2c_n}^{-2c_n}f(y)\,dy\ge\frac{-1}{4c_n}\int_{2c_n}^{c_n}f(y)\,dy=\frac{1}{4}|c_n|^{-1/p(-2c_n)}.$$

Hence, assuming that $|c_n|\ge 1$, we conclude

$$\begin{aligned}\int_{\mathbb{R}}\{Mf(x)\}^{p(x)}\,dx&\ge\frac{1}{4}\sum_{n=1}^\infty\int_{-c_n}^{-2c_n}|c_n|^{-p(x)/p(-2c_n)}\,dx\\&\ge\frac{1}{4}\sum_{n=1}^\infty\int_{-c_n}^{-2c_n}|c_n|^{-p(-2c_n)/p(-2c_n)}\,dx\\&=\frac{1}{4}\sum_{n=1}^\infty 1=\infty.\end{aligned}$$

This shows that $Mf\notin L^{p(\cdot)}(\mathbb{R}^n)$. □

Remark 2. Keep to the same setting as Proposition 6.5. The above proof shows that the Hardy operator

$$Hf(x)=\frac{1}{|x|}\int_{-|x|}^{|x|}f(t)\,dt\quad(x\in\mathbb{R})$$

is not bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

The next example is due to Cruz-Uribe's web page. This example shows that it does not suffice to assume the continuity only.

Proposition 6.6. *For $x \in \mathbb{R}$, let*

$$p(x) := 3 + \cos(2\pi x).$$

Then M is not bounded on $L^{p(\cdot)}(\mathbb{R})$.

The point is that M recovers the missing part of f: $Mf(x) \geq C_0|x|^{-1/3}$ for all $x > 0$. See (6.4).

Proof. Note that $p(x) \geq 3 + \cos(\pi/4)$ for $x \in [j, j+1/8]$, $j = 1, 2, \cdots$.
Let $f(x) := |x|^{-1/3} \sum_{j=1}^{\infty} \chi_{[0,1/8]}(x-j)$. Then

$$\begin{aligned}\int_{\mathbb{R}} |f(x)|^{p(x)}\,dx &= \sum_{j=1}^{\infty} \int_j^{j+1/8} |x|^{-p(x)/3}\,dx \\ &\leq \sum_{j=1}^{\infty} \int_j^{j+1/8} |x|^{-(3+\cos(\pi/4))/3}\,dx \\ &\leq \frac{1}{8} \sum_{j=1}^{\infty} j^{-(3+\cos(\pi/4))/3} < \infty. \end{aligned} \tag{6.3}$$

On the other hand, for $x \in (j, j+1)$, $j = 1, 2, \cdots$,

$$Mf(x) \geq \int_j^{j+1} f(y)\,dy = \int_j^{j+1/8} |y|^{-1/3}\,dy \geq \frac{(j+1/8)^{-1/3}}{8} > \frac{(j+1)^{-1/3}}{8}. \tag{6.4}$$

Since $p(x) \leq 3$ for $x \in [j+1/4, j+3/4]$, $j = 1, 2, \cdots$,

$$\begin{aligned}\int_{\mathbb{R}} Mf(x)^{p(x)}\,dx &\geq \sum_{j=1}^{\infty} \int_j^{j+1} \left(\frac{(j+1)^{-1/3}}{8}\right)^{p(x)} dx \\ &\geq \sum_{j=1}^{\infty} \int_{j+1/4}^{j+3/4} \left(\frac{(j+1)^{-1/3}}{8}\right)^{3} dx \\ &\geq \frac{1}{2 \times 8^3} \sum_{j=1}^{\infty} (j+1)^{-1} = \infty. \end{aligned} \tag{6.5}$$

(6.3) and (6.5) disprove that M is bounded on $L^{p(\cdot)}(\mathbb{R})$. □

We remark that another example can be found in (9.2).

§ 7. Applications to density

We shall state and prove basic properties about density, which seem to have never explicitly appeared in any literature. As an application of what we have obtained, we consider a density condition. Here we place ourselves in the setting of domains. Let $\Omega \subset \mathbb{R}^n$ be an open set. We define

$$C^\infty_{\mathrm{comp}}(\Omega) := \{f \in C^\infty(\Omega) \,:\, \mathrm{supp}(f) \text{ is compact}\},$$

where $\mathrm{supp}(f) := \overline{\{x \in \Omega \,:\, f(x) \neq 0\}}$. We are interested in the condition that $C^\infty_{\mathrm{comp}}(\Omega)$ is dense in $L^{p(\cdot)}(\Omega)$.

Definition 7.1. Given a measurable function $p(\cdot) : \Omega \to [1, \infty]$, we define the Lebesgue space with variable exponent

$$L^{p(\cdot)}(\Omega) := \{f \,:\, \rho_p(f/\lambda) < \infty \text{ for some } \lambda > 0\},$$

where

$$\rho_p(f) := \int_{\Omega \setminus \Omega_\infty} |f(x)|^{p(x)} dx + \|f\|_{L^\infty(\Omega_\infty)},$$

and

$$\Omega_\infty = \{x \in \Omega : p(x) = \infty\}.$$

Moreover, define

$$\|f\|_{L^{p(\cdot)}(\Omega)} := \inf\left\{\lambda > 0 : \rho_p(f/\lambda) \leq 1\right\}.$$

Theorem 7.2. *If a variable exponent $p(\cdot) : \Omega \to [1, \infty]$ satisfies*

$$\operatorname*{ess\,sup}_{x \in \Omega \setminus \Omega_\infty} p(x) < \infty,$$

then the set

$$\mathcal{G} := \{g \in L^{p(\cdot)}(\Omega) \,:\, g \text{ is bounded}\} = L^{p(\cdot)}(\Omega) \cap L^\infty(\Omega)$$

is dense in $L^{p(\cdot)}(\Omega)$.

Proof. Take $f \in L^{p(\cdot)}(\Omega)$ arbitrarily and for each $j \in \mathbb{N}$ define

$$G_j := \{x \in \Omega \setminus \Omega_\infty \,:\, |x| < j\},$$
$$f_j(x) := \begin{cases} f(x) & (x \in G_j \cup \Omega_\infty,\ |f(x)| \leq j), \\ j\,\overline{f(x)}|f(x)|^{-1} & (x \in G_j \cup \Omega_\infty,\ |f(x)| > j), \\ 0 & (x \notin G_j \cup \Omega_\infty). \end{cases}$$

Then we see $f_j \in \mathcal{G}$ and that $|f_j| \leq \min\{j,\ |f|\}$. Thus, we are in the position of using the Lebesgue dominated convergence theorem and we obtain

$$\lim_{j \to \infty} \rho_p(f_j - f) = 0, \tag{7.1}$$

that is, $\lim_{j\to\infty} \|f_j - f\|_{L^{p(\cdot)}(\Omega)} = 0$. □

Theorem 7.3. *If a variable exponent $p(\cdot):\Omega\to[1,\infty)$ satisfies $p_+<\infty$, then the following hold:*

(1) *The set $C(\Omega)\cap L^{p(\cdot)}(\Omega)$ is dense in $L^{p(\cdot)}(\Omega)$.*

(2) *If Ω is an open set, then $C^\infty_{\mathrm{comp}}(\Omega)$ is dense in $L^{p(\cdot)}(\Omega)$.*

Proof. Take $f\in L^{p(\cdot)}(\Omega)$ and $\varepsilon>0$ arbitrarily.

We first prove (1). By virtue of Theorem 7.2, we can take a bounded function $g\in L^{p(\cdot)}(\Omega)$ so that $\|f-g\|_{L^{p(\cdot)}(\Omega)}<\varepsilon$. Now we use the Luzin theorem (cf. [12, 13]) to get a function $h\in C(\Omega)$ and an open set U such that

$$|U|<\min\left\{1,\ \left(\frac{\varepsilon}{2\|g\|_{L^\infty(\Omega)}}\right)^{p_+}\right\}, \tag{7.2}$$

that

$$\sup_{x\in\Omega}|h(x)|=\sup_{x\in\Omega\setminus U}|g(x)|\le\|g\|_{L^\infty(\Omega)}, \tag{7.3}$$

and that

$$g(x)=h(x)\text{ for all }x\in\Omega\setminus U. \tag{7.4}$$

By the triangle inequality, we have

$$\|g-h\|_{L^\infty(\Omega)}\le\|g\|_{L^\infty(\Omega)}+\|h\|_{L^\infty(\Omega)}\le 2\|g\|_{L^\infty(\Omega)}.$$

Since $t^{p(x)}\le\max(1,t^{p_+})$ holds for $t>0$, we obtain

$$\begin{aligned}\rho_p\left(\frac{g-h}{\varepsilon}\right)&=\int_\Omega\left|\frac{g(x)-h(x)}{\varepsilon}\right|^{p(x)}\chi_U(x)\,dx\\&\le|U|\cdot\max\left\{1,\ \left(\frac{2\|g\|_{L^\infty(\Omega)}}{\varepsilon}\right)^{p_+}\right\}\\&\le 1,\end{aligned}\tag{7.5}$$

namely, $\|g-h\|_{L^{p(\cdot)}(\Omega)}\le\varepsilon$. Therefore we have

$$\|f-h\|_{L^{p(\cdot)}(\Omega)}\le\|f-g\|_{L^{p(\cdot)}(\Omega)}+\|g-h\|_{L^{p(\cdot)}(\Omega)}<2\varepsilon.$$

Next we assume that Ω is open and prove (2). Again we fix $\varepsilon>0$. For $f\in L^{p(\cdot)}(\Omega)$, take $h\in C(\Omega)$ such that $\|f-h\|_{L^{p(\cdot)}(\Omega)}<2\varepsilon$. Since $p_+<\infty$, we have $C^\infty_{\mathrm{comp}}(\Omega)\subset L^{p(\cdot)}(\Omega)$ and

$$\rho_p\left(\frac{h}{\varepsilon}\right)\le\max\{\varepsilon^{-p_+},\ \varepsilon^{-p_-}\}\rho_p(h)<\infty.$$

Thus if we take a bounded open set $G \subset \Omega$ so that

$$\rho_p\left(\frac{h\chi_{\Omega\setminus G}}{\varepsilon}\right) \le 1,$$

then we get

$$\|h - h\chi_G\|_{L^{p(\cdot)}(\Omega)} \le \varepsilon. \tag{7.6}$$

Now we take a polynomial $Q(x)$ so that

$$\sup_{x\in G} |h(x) - Q(x)| < \varepsilon \min\{1,\ |G|^{-1}\}$$

by using the Weierstrass theorem. Then, since $\min\{1, |G|^{-1}\}^{p(x)} \le \min\{1, |G|^{-1}\}$ for all $x \in G$, we have

$$\rho_p\left(\frac{h\chi_G - Q\chi_G}{\varepsilon}\right) \le |G| \min\{1,\ |G|^{-1}\} \le 1,$$

that is,

$$\|h\chi_G - Q\chi_G\|_{L^{p(\cdot)}(\Omega)} \le \varepsilon. \tag{7.7}$$

By virtue of $\rho_p\left(\frac{Q\chi_G}{\varepsilon}\right) < \infty$, we can take a small constant $a > 0$ so that

$$\rho_p\left(\frac{Q\chi_{G\setminus K_a}}{\varepsilon}\right) \le 1,$$

where K_a is a compact set defined by

$$K_a := \{x \in G \,:\, \mathrm{dist}(x, \partial G) \ge a\}.$$

Thus we obtain

$$\|Q\chi_G - Q\chi_{K_a}\|_{L^{p(\cdot)}(\Omega)} \le \varepsilon. \tag{7.8}$$

Now we fix a function $\varphi \in C^\infty_{\mathrm{comp}}(\Omega)$ such that

$$\mathrm{supp}(\varphi) \subset G, \quad 0 \le \varphi \le 1 \text{ on } G, \quad \varphi \equiv 1 \text{ on } K_a$$

to get

$$\|Q\chi_G - Q\varphi\|_{L^{p(\cdot)}(\Omega)} = \||Q||\chi_G - \varphi|\|_{L^{p(\cdot)}(\Omega)} \le \||Q||\chi_G - \chi_{K_a}|\|_{L^{p(\cdot)}(\Omega)} \le \varepsilon, \tag{7.9}$$

where the last inequality follows from (7.8). Combing (7.5), (7.6), (7.7) and (7.9), we have $Q\varphi \in C^\infty_{\rm comp}(\Omega)$ and

$$\begin{aligned}
&\|f - Q\varphi\|_{L^{p(\cdot)}(\Omega)} \\
&\le \|f-h\|_{L^{p(\cdot)}(\Omega)} + \|h - h\chi_G\|_{L^{p(\cdot)}(\Omega)} + \|h\chi_G - Q\chi_G\|_{L^{p(\cdot)}(\Omega)} + \|Q\chi_G - Q\varphi\|_{L^{p(\cdot)}(\Omega)} \\
&< 2\varepsilon + \varepsilon + \varepsilon + \varepsilon \\
&= 5\varepsilon.
\end{aligned}$$

Thus, the proof is therefore complete. □

§ 8. Appendix – the boundedness of M on $L^p(\mathbb{R}^n)$ –

As an appendix, we supply the proof of the boundedness of the Hardy-Littlewood maximal operator M on $L^p(\mathbb{R}^n)$.

Recall that, for a function $f \in L^1_{\rm loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the uncentered Hardy-Littlewood maximal function $Mf(x)$ is defined by

$$Mf(x) := \sup_{B \ni x} ⨍_B |f(y)|\,dy, \quad ⨍_B |f(y)|\,dy := \frac{1}{|B|}\int_B |f(y)|\,dy.$$

where the supremum is taken over all balls B containing x. Meanwhile, for a function $f \in L^1_{\rm loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the centered Hardy-Littlewood maximal function $M_{\rm centered}f(x)$ is defined by

$$M_{\rm centered}f(x) := \sup_{r>0} ⨍_{B(x,r)} |f(y)|\,dy. \tag{8.1}$$

Due to the estimate $M_{\rm centered}f(x) \le Mf(x) \le 2^n M_{\rm centered}f(x)$, most of the results for Mf carry over to those for $M_{\rm centered}f$. We do not allude to this point, unless there is not difference between Mf and $M_{\rm centered}f$. First, we check that Mf and $M_{\rm centered}f$ are both measurable functions. Our proof is simpler than that in the textbook [26].

Proposition 8.1. *Let $\lambda > 0$. Then the sets $E_\lambda := \{x \in \mathbb{R}^n : Mf(x) > \lambda\}$ and $E'_\lambda := \{x \in \mathbb{R}^n : M_{\rm centered}f(x) > \lambda\}$ are open.*

Proof. To prove this, we choose $x \in E_\lambda$ arbitrarily. Then by the definition (1.1) of Mf, we can find a ball B such that

$$x \in B, \ \frac{1}{|B|}\int_B |f(y)|\,dy > \lambda. \tag{8.2}$$

Then, by the definition of Mf, $B \subset E_\lambda$, and hence x is an interior point of E_λ. The point x being arbitrary, we see that E_λ is open.

We modify the above proof to obtain the proof for E'_λ. In view of the definition (8.1), the ball B in (8.2) must be centered at x, so that B assumes the form of $B = B(x, r)$ for some $r > 0$. By choosing κ slightly larger than 1, we have

$$\frac{1}{|B(x, \kappa r)|} \int_{B(x,r)} |f(y)|\, dy > \lambda.$$

Let $y \in B(x, (\kappa - 1)r)$. Then a geometric observation shows that $B(y, \kappa r) \supset B(x, r)$. Thus, it follows that

$$\frac{1}{|B(y, \kappa r)|} \int_{B(y,\kappa r)} |f(y)|\, dy \geq \frac{1}{|B(x, \kappa r)|} \int_{B(x,r)} |f(y)|\, dy > \lambda.$$

Hence, $B(x, (\kappa - 1)r) \subset E'_\lambda$. Since x is again arbitrary, it follows that E'_λ is an open set as well. □

Classically the boundedness of the Hardy-Littlewood maximal operator is shown as follows:

Theorem 8.2.

(1) *The Hardy-Littlewood maximal operator M is of weak type $(1, 1)$, namely,*

$$|\{x \in \mathbb{R}^n : Mf(x) > \lambda\}| \leq C\, \lambda^{-1} \|f\|_{L^1(\mathbb{R}^n)}$$

holds for all $\lambda > 0$ and all $f \in L^1(\mathbb{R}^n)$.

(2) *If $1 < p \leq \infty$, then M is bounded on $L^p(\mathbb{R}^n)$, namely,*

$$\|Mf\|_{L^p(\mathbb{R}^n)} \leq C\, \|f\|_{L^p(\mathbb{R}^n)}$$

holds for all $f \in L^p(\mathbb{R}^n)$.

Before we proceed further, a couple of remarks may be in order.

Remark 3.

(1) If $p = \infty$, then Theorem 8.2 (2) is immediately proved by the definition of the norm $\|\cdot\|_{L^\infty(\mathbb{R}^n)}$.

(2) If $1 < p < \infty$, then M is of weak type (p, p), namely,

$$|\{x \in \mathbb{R}^n : Mf(x) > \lambda\}|^{1/p} \leq C\, \lambda^{-1} \|f\|_{L^p(\mathbb{R}^n)} \tag{8.3}$$

holds for all $\lambda > 0$ and all $f \in L^p(\mathbb{R}^n)$. This fact is easily checked by the Chebychev inequality and Theorem 8.2 (2).

(3) One of the important reasons why we are led to the weak $(1,1)$ inequality is that M always maps $L^1(\mathbb{R}^n)$ functions to non-integral function except the zero function. To explain why let us place ourselves in the case of $n = 1$. Then, a simple computation shows that $M(\chi_{[-1,1]}) \notin L^1(\mathbb{R})$ but that $\chi_{[-1,1]} \in L^1(\mathbb{R})$. By a similar reason, in $\mathbb{R}^n$, $Mf \notin L^1(\mathbb{R}^n)$ unless $f = 0$.

(4) The remark (3) above is also valid for the centered Hardy-Littlewood maximal operator.

In order to prove Theorem 8.2 we will use the following two lemmas.

Lemma 8.3 (Vitali's covering lemma). *Given a bounded set $E \subset \mathbb{R}^n$, we take a covering $\{B(x_j, r_j)\}_j$ of E. If $\{r_j\}_j$ is bounded, then there exists a disjoint subfamily $\{B(x_{j'}, r_{j'})\}$ such that $E \subset \bigcup_{j'} B(x_{j'}, 5r_{j'})$.*

We introduce some Japanese books, for example, Igari [12], Mizuta [27] and Sawano [38] for further information on the covering lemma. In [38] a covering lemma is presented as Theorem 2.2.8 but the condition $\sup_{\lambda\in\Lambda} r_\lambda < \infty$ was indispensable.

Lemma 8.4. *If $1 \le p < \infty$ and $f \in L^p(\mathbb{R}^n)$, then we have*

$$\int_{\mathbb{R}^n} |f(x)|^p \, dx = p \int_0^\infty t^{p-1} \left|\{x \in \mathbb{R}^n \, : \, |f(x)| > t\}\right| dt.$$

Proof. If we define the set $A := \{(x,s) \in \mathbb{R}^n \times [0,\infty) \, : \, |f(x)|^p > s\}$, then we get by Fubini's theorem,

$$\begin{aligned}
\int_{\mathbb{R}^n} |f(x)|^p \, dx &= \int_{\mathbb{R}^n} \left(\int_0^{|f(x)|^p} 1 \, ds \right) dx \\
&= \int_{\mathbb{R}^n} \left(\int_0^\infty \chi_A(x,s) \, ds \right) dx \\
&= \int_0^\infty \left(\int_{\mathbb{R}^n} \chi_A(x,s) \, dx \right) ds \\
&= \int_0^\infty \left|\{x \in \mathbb{R}^n \, : \, |f(x)|^p > s\}\right| ds.
\end{aligned}$$

In summary, we obtained

$$\int_{\mathbb{R}^n} |f(x)|^p \, dx = \int_0^\infty \left|\{x \in \mathbb{R}^n \, : \, |f(x)|^p > s\}\right| ds. \tag{8.4}$$

If we change variables, then we obtain

$$\int_{\mathbb{R}^n} |f(x)|^p \, dx = \int_0^\infty t^{p-1} \left|\{x \in \mathbb{R}^n \, : \, |f(x)| > t\}\right| dt.$$

This is the desired result. □

Proof of Theorem 8.2. We first prove (1). For every $\lambda > 0$ and $N \in \mathbb{N}$, we write

$$E_\lambda := \{x \in \mathbb{R}^n : Mf(x) > \lambda\} \quad \text{and} \quad E_{\lambda,N} := E_\lambda \cap B(0,N).$$

By the definition of $Mf(x)$, for each $x \in E_\lambda$ there exists a ball B_x such that $x \in B_x$ and that

$$\frac{1}{|B_x|}\int_{B_x} |f(y)|\,dy > \lambda.$$

We remark that $\{B_x\}_{x\in E_\lambda}$ is a covering of a bounded set $E_{\lambda,N}$ and that the radius of B_x is bounded, since $|B_x| \le \|f\|_{L^1(\mathbb{R}^n)}/\lambda$. By virtue of Vitali's covering lemma, there exists a disjoint subfamily

$$\{B_j := B_{x_j}\}_j \subset \{B_x\}_{x\in E_\lambda} \tag{8.5}$$

such that

$$E_{\lambda,N} \subset \bigcup_j 5B_j \quad \text{and} \quad \frac{1}{|B_j|}\int_{B_j} |f(y)|\,dy > \lambda.$$

Since $\{B_j\}_j$ is disjoint, we obtain

$$|E_{\lambda,N}| \le \left|\bigcup_j 5B_j\right| \le 5^n \sum_j |B_j| \le 5^n \sum_j \left(\lambda^{-1}\int_{B_j} |f(y)|\,dy\right) \le 5^n \lambda^{-1}\|f\|_{L^1(\mathbb{R}^n)}.$$

Moreover by $E_{\lambda,N} \subset E_{\lambda,N+1} \subset \cdots$ and $\bigcup_{N=1}^\infty E_{\lambda,N} = E_\lambda$, we have

$$|\{x \in \mathbb{R}^n : Mf(x) > \lambda\}| = \lim_{N\to\infty} |E_{\lambda,N}| \le 5^n\lambda^{-1}\|f\|_{L^1(\mathbb{R}^n)}.$$

Next we prove (2). Let $1 < p < \infty$. Take $a > 0$ arbitrarily and define

$$f^a(x) := \begin{cases} f(x) & (|f(x)| > a/2), \\ 0 & (|f(x)| \le a/2), \end{cases} \qquad f_a(x) := f(x) - f^a(x) \quad (x \in \mathbb{R}^n).$$

Since

$$Mf(x) \le M(f^a)(x) + M(f_a)(x) \le M(f^a)(x) + \frac{a}{2} \quad (x \in \mathbb{R}^n),$$

we have

$$\{x \in \mathbb{R}^n : Mf(x) > a\} \subset \{x \in \mathbb{R}^n : M(f^a)(x) > a/2\}.$$

The weak $(1,1)$ inequality gives us

$$|\{x \in \mathbb{R}^n : Mf(x) > a\}| \le |\{x \in \mathbb{R}^n : M(f^a)(x) > a/2\}| \le C \cdot \frac{2}{a} \cdot \|f^a\|_{L^1(\mathbb{R}^n)}.$$

By virtue of Lemma 8.4 we get

$$\begin{aligned}\int_{\mathbb{R}^n}(Mf(x))^p\,dx &= p\int_0^\infty a^{p-1}\,|\{x\in\mathbb{R}^n\,:\,Mf(x)>a\}|\,da\\ &\le Cp\int_0^\infty a^{p-2}\|f^a\|_{L^1(\mathbb{R}^n)}\,da\\ &= Cp\int_{\mathbb{R}^n}\left(\int_0^\infty a^{p-2}|f^a(y)|\,da\right)dy.\end{aligned}$$

Now we note that

$$\int_0^\infty a^{p-2}|f^a(y)|\,da=\int_0^{2|f(y)|}a^{p-2}|f(y)|\,da=\frac{1}{p-1}(2|f(y)|)^{p-1}|f(y)|=\frac{2^{p-1}}{p-1}|f(y)|^p.$$

Consequently we have

$$\int_{\mathbb{R}^n}(Mf(x))^p\,dx\le\frac{C2^{p-1}p}{p-1}\int_{\mathbb{R}^n}|f(y)|^p\,dy.$$

Thus, the proof is therefore complete. □

Recall that M_{centered} is the centered Hardy-Littlewood maximal operator generated by balls. Here and below we write $M_{\text{centered,balls}}$ for definiteness. The following result is known about the mollifier.

Lemma 8.5 ([11, Section 2]). *Let $\psi\in L^1(\mathbb{R}^n)$ be a radial decreasing function. Define*

$$\psi_t(x):=t^{-n}\psi(x/t)\quad(t>0).$$

Then we have that for all $t>0$ and all $f\in L^1_{\text{loc}}(\mathbb{R}^n)$,

$$|\psi_t*f(x)|\le\|\psi\|_{L^1}M_{\text{centered,balls}}f(x). \tag{8.6}$$

The function ψ_t*f is often called a mollifier.

Let $M_{\text{centered,cubes}}$ be the centered Hardy-Littlewood maximal operator generated by cubes. Since the volume of unit ball is $\pi^{n/2}/\Gamma(1+n/2)$, we have

$$M_{\text{centered,balls}}f(x)\le\Gamma(1+n/2)2^n\pi^{-n/2}M_{\text{centered,cubes}}f(x).$$

Thus, if we use Lemma 8.5, then we obtain

$$|\psi_t*f(x)|\le\|\psi\|_{L^1}M_{\text{centered,balls}}f(x)\le\Gamma(1+n/2)2^n\pi^{-n/2}\|\psi\|_{L^1}M_{\text{centered,cubes}}f(x).$$

But the next lemma shows that the bound can be improved for large n.

Lemma 8.6. *Let $\psi \in L^1(\mathbb{R}^n)$ be a radial decreasing function. Define*

$$\psi_t(x) := t^{-n}\psi(x/t) \quad (t > 0).$$

Then we have that for all $t > 0$ and all $f \in L^1_{\rm loc}(\mathbb{R}^n)$,

$$|\psi_t * f(x)| \le 2^{2n}\|\psi\|_{L^1} M_{\rm centered,cubes}f(x).$$

Proof. Taking $K > 0$ arbitrarily, we have

$$\begin{aligned}
&\int_{[-K,K]^n} \left|t^{-n}\psi(x/t)f(x-y)\right| dy \\
&= \sum_{(k_1,\ldots,k_n)\in\mathbb{N}^n} \int_{\{x\,:\,2^{-k_j}K<|x_j|\le 2^{-k_j+1}K,\, j=1,\cdots,n\}} t^{-n}\left|\psi(x/t)f(x-y)\right| dy \\
&\le \sum_{(k_1,\ldots,k_n)\in\mathbb{N}^n} t^{-n}\left|\psi(2^{-k_1}K/t,\ldots,2^{-k_n}K/t)\right| \\
&\qquad\qquad \int_{-2^{-k_1+1}K}^{2^{-k_1+1}K} \cdots \int_{-2^{-k_n+1}K}^{2^{-k_n+1}K} |f(x-y)|\, dy.
\end{aligned}$$

By using the operator $M_{\rm centered,cubes}$, we obtain

$$\begin{aligned}
&\int_{[-K,K]^n} \left|t^{-n}\psi(x/t)f(x-y)\right| dy \\
&\le \sum_{(k_1,\ldots,k_n)\in\mathbb{N}^n} t^{-n}\left|\psi(2^{-k_1}K/t,\ldots,2^{-k_n}K/t)\right| \\
&\qquad \times (2K)^n 2^{-(k_1+\ldots+k_n)+n} M_{\rm centered,cubes}f(x) \\
&= 2^{3n} \sum_{(k_1,\ldots,k_n)\in\mathbb{N}^n} (2^{-k_1-1}K/t)\cdots(2^{-k_n-1}K/t)\left|\psi(2^{-k_1}K/t,\ldots,2^{-k_n}K/t)\right| \\
&\qquad\qquad \times M_{\rm centered,cubes}f(x) \\
&\le 2^{3n} \sum_{(k_1,\ldots,k_n)\in\mathbb{N}^n} \left(\int_{2^{-k_1-1}K/t}^{2^{-k_1}K/t} \cdots \int_{2^{-k_n-1}K/t}^{2^{-k_n}K/t} |\psi(y)|\, dy\right) M_{\rm centered,cubes}f(x) \\
&= 2^{3n}\left(\int_{[0,K/(2t)]^n} |\psi(y)|\, dy\right) M_{\rm centered,cubes}f(x) \\
&\le 2^{2n}\|\psi\|_{L^1} M_{\rm centered,cubes}f(x),
\end{aligned}$$

that is,

$$|\psi_t * f(x)| \le \int_{\mathbb{R}^n} \left|t^{-n}\psi(x/t)f(x-y)\right| dy \le 2^{2n}\|\psi\|_{L^1} M_{\rm centered,cubes}f(x).$$

Thus, the proof is complete. □

§ 9. Open problems

Here we state open problems about the boundedness of the Hardy-Littlewood maximal operator M on variable Lebesgue spaces. To this end, we need to consider a couple of conditions. We first formulate the conditions and then we propose open problems. Related to these open problems, we shall state a known result for the boundedness of the Hardy-Littlewood maximal function (Theorem 9.6) and we improve it in Theorem 9.7.

First of all, we recall that Cruz-Uribe, Diening and Fiorenza proved the following weak-type results.

Proposition 9.1 ([1]). *If a variable exponent $p(\cdot) : \mathbb{R}^n \to [1,\infty]$ satisfies $1 = p_- \le p_+ \le \infty$ and $1/p(\,\cdot\,) \in LH(\mathbb{R}^n)$, then M is of weak type $(p(\,\cdot\,), p(\,\cdot\,))$, namely, for all $\lambda > 0$ and all $f \in L^{p(\,\cdot\,)}(\mathbb{R}^n)$ we have the inequality*

$$\left\|\chi_{\{Mf(x)>\lambda\}}\right\|_{L^{p(\,\cdot\,)}(\mathbb{R}^n)} \le C\lambda^{-1} \|f\|_{L^{p(\,\cdot\,)}(\mathbb{R}^n)} .$$

In the case of $\Omega = \mathbb{R}^n$ Diening [8, Theorem 8.1] has proved the following equivalence. Below $\mathcal{Y}$ consists of all families of disjoint open cubes.

Proposition 9.2 ([8]). *Given a variable exponent $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, the next four conditions are equivalent:*

(D1) $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$.

(D2) $p'(\cdot) \in \mathcal{B}(\mathbb{R}^n)$.

(D3) *There exists a constant $q \in (1, p_-)$ such that $p(\cdot)/q \in \mathcal{B}(\mathbb{R}^n)$.*

(D4) *For all $Y \in \mathcal{Y}$ and all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ we have*

$$\left\| \sum_{Q\in Y} |f|_Q \chi_Q \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

If we take an arbitrary open cube Q and put $Y := \{Q\}$ and $f := f\,\chi_Q$ in Proposition 9.2 (D4), then we get a weaker condition

(A1) For all open cubes Q and all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ we have

$$|f|_Q \|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C \|f\,\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Lemma 9.3. *Condition* (A1) *is equivalent to the following* (A2) *called the Muckenhoupt condition for a variable exponent $p(\,\cdot\,)$:*

(A2)

$$\sup_Q \frac{1}{|Q|}\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_Q\|_{L^{p'(\cdot)}(\mathbb{R}^n)} < \infty, \tag{9.1}$$

where the supremum is taken over all open cubes Q.

We define the class $\mathcal{A}(\mathbb{R}^n)$ as the set of all variable exponents $p(\cdot)$ satisfying (A1) or (A2). We can easily see that $\mathcal{B}(\mathbb{R}^n) \subset \mathcal{A}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$.

Proof of Lemma 9.3. Take an open cube Q and $f \in L^{p(\cdot)}(\mathbb{R}^n)$ arbitrarily. We first prove (A1) $\Rightarrow$ (A2). Let us assume (A1). Using the associate norm, we obtain

$$\begin{aligned}
&\frac{1}{|Q|}\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_Q\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\\
&\le C \cdot \frac{1}{|Q|}\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \sup\left\{\int_{\mathbb{R}^n} |f(x)\chi_Q(x)|\,dx \,:\, \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le 1\right\}\\
&= C \sup\left\{|f|_Q\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \,:\, \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le 1\right\}\\
&\le C \sup\left\{\|f\,\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \,:\, \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le 1\right\}\\
&\le C,
\end{aligned}$$

namely, (A2) holds. Next we prove (A2) $\Rightarrow$ (A1). Assume (A2). By virtue of the generalized Hölder inequality we get

$$\begin{aligned}
|f|_Q\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} &= \frac{1}{|Q|}\int_Q |f(y)|\,dy \cdot \|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\le C \cdot \frac{1}{|Q|}\|f\,\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_Q\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\le C\,\|f\,\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Therefore (A1) is true. □

We can conjecture the following equivalence for variable exponents similar to the Muckenhoupt A_p weights.

Open Problem 9.4. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Get some conditions for $p(\cdot)$ so that three conditions* (C1), (C2) *and* (C3) *are equivalent:*

(C1) *$p(\cdot) \in \mathcal{A}(\mathbb{R}^n)$.*

(C2) *$p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$.*

(C3) *The Hardy-Littlewood maximal operator M is of weak type $(p(\cdot), p(\cdot))$.*

Remark 4. Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Some facts on Open Problem 9.4 are known.

(1) We can easily show (C3) $\Rightarrow$ (C1). Assume (C3) and take an open cube Q and $f \in L^{p(\cdot)}(\mathbb{R}^n)$ arbitrarily. If $|f|_Q = 0$, then (A1) holds immediately. We have only to consider the case $|f|_Q > 0$. If we take $\lambda = |f|_Q/2$, then $C\,M(f\,\chi_Q)(x) > \lambda$ $(x \in Q)$ because of $|f|_Q \chi_Q(x) \le C\,M(f\,\chi_Q)(x)$. Thus we have

$$\begin{aligned} |f|_Q \|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\le |f|_Q \left\|\chi_{\{C\,M(f\,\chi_Q)(x)>\lambda\}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\le |f|_Q \cdot C\lambda^{-1} \left\|f\,\chi_Q\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &= C\left\|f\,\chi_Q\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

This implies that $p(\cdot)$ satisfies (A1).

(2) In the case of that $p(\cdot)$ equals to a constant outside a ball, Kopaliani [17] has proved that (C1) $\Rightarrow$ (C2) holds, namely, three conditions (C1), (C2) and (C3) are equivalent.

(3) In the case of that $p(\cdot)$ is radial decreasing, i.e.,

$$p(x) \ge p(y) \quad (|x| \le |y|),$$

Lerner [23] has proved (C1) $\Rightarrow$ (C3) is true, namely, two conditions (C1) and (C3) are equivalent.

(4) In the case of $n \ge 2$, Kopaliani [18] has proved (C1) $\Rightarrow$ (C3) is not always true by giving the following counter example: Take constants $1 < p_1 < p_2 < \infty$ and define a function $k \in C^\infty(\mathbb{R})$ so that

$$k(t) \begin{cases} = p_1 & (t \notin [0,3]), \\ = p_2 & (1 \le t \le 2), \\ \in (p_1, p_2) & (t \in [0,1] \cup [2,3]). \end{cases}$$

Then the variable exponent

$$p(x) := k(x_1) \quad (x = (x_1, \ldots, x_n) \in \mathbb{R}^n) \tag{9.2}$$

satisfies (C1), but does not satisfy (C3). Of course, this example is not radially decreasing.

In the case of $p_- = 1$, we can conjecture the problem corresponding to the Muckenhoupt A_1 weights. This is also still open.

Open Problem 9.5. *Let $p(\cdot) : \mathbb{R}^n \to [1,\infty]$ such that $1 = p_- \le p_+ < \infty$. Obtain some conditions on $p(\cdot)$ so that the following two conditions are equivalent:*

(C1) $p(\,\cdot\,) \in \mathcal{A}(\mathbb{R}^n)$.

(C3) *The Hardy-Littlewood maximal operator M is of weak type $(p(\,\cdot\,), p(\,\cdot\,))$.*

As is mentioned above, Kopaliani [17] has proved the next theorem. Lerner [23] has given an alternative proof of the theorem.

Theorem 9.6 ([17, 23])**.** *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Assume the following conditions:*

(C0) *$p(\cdot)$ equals to a constant outside a ball.*

(C1) $p(\cdot) \in \mathcal{A}(\mathbb{R}^n)$.

Then we have $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$.

We remark that the condition (C0) can be relaxed.

Theorem 9.7. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Assume the following conditions:*

(C0)′ $p(\cdot) \in LH_\infty(\mathbb{R}^n)$.

(C1) $p(\cdot) \in \mathcal{A}(\mathbb{R}^n)$.

Then we have $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$.

We will use the following lemmas, which we admit, in order to prove the theorem. The first lemma is due to Kopaliani [17] and Lerner [23]. The second one is known as the classical weak type inequality for M (cf. [40]). The third lemma follows from the definition of the norm.

Lemma 9.8 ([17, 23])**.** *Let $E \subset \mathbb{R}^n$ be a measurable set such that $|E| > 0$ and $p(\cdot) \in \mathcal{A}(\mathbb{R}^n)$. Then we have that for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$,*

$$\|(Mf)\chi_E\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le c(p,n)\, \max\left\{1,\ \|\chi_E\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{\frac{1}{(p')_-}}\right\} \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where $c(p,n) > 0$ is a constant depending only on n and $p(\cdot)$.

Lemma 9.9 ([40])**.** *We have that for all $\alpha > 0$ and all measurable functions f,*

$$|\{x \in \mathbb{R}^n \,:\, Mf(x) > \alpha\}| \le C\,\alpha^{-1} \int_{\{|2f|>\alpha\}} |f(x)|\,dx.$$

Lemma 9.10. *Let $p(\cdot) : \Omega \to [1,\infty]$ be a variable exponent.*

(1) *If $\|f\|_{L^{p(\,\cdot\,)}(\Omega)} \le 1$, then we have $\rho_p(f) \le \|f\|_{L^{p(\,\cdot\,)}(\Omega)} \le 1$.*

(2) *Conversely if* $\rho_p(f) \le 1$*, then* $\|f\|_{L^{p(\cdot)}(\Omega)} \le 1$ *holds.*

Proof of Theorem 9.7. It suffices to prove that

$$\|Mf\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C$$

for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ with $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = 1$. We have that

$$\|Mf\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le \|(Mf)\chi_{\{|f|>1\}}\|_{L^{p(\cdot)}(\mathbb{R}^n)} + \|(Mf)\chi_{\{|f|\le 1\}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

By Lemma 4.2, we conclude

$$\|(Mf)\chi_{\{|f|\le 1\}}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C.$$

On the other hand, by virtue of Lemma 9.9 and 9.10 (1) we obtain

$$\begin{aligned}
|\{x \in \mathbb{R}^n \,:\, Mf(x) > 1\}| &\le C \int_{\{|2f|>1\}} |f(x)|\,dx \\
&= \frac{C}{2} \int_{\{|2f|>1\}} |2f(x)|\,dx \\
&\le \frac{C}{2} \int_{\{|2f|>1\}} |2f(x)|^{p(x)}\,dx \\
&\le C \int_{\mathbb{R}^n} |f(x)|^{p(x)}\,dx \\
&= C.
\end{aligned}$$

Therefore, by $|f| \le Mf$ and Lemma 9.8, we get

$$\begin{aligned}
\|(Mf)\chi_{\{|f|>1\}}\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\le \|(Mf)\chi_{\{Mf>1\}}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\le c(p,n) \max\left\{1,\ \|\chi_{\{Mf>1\}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{\frac{1}{(p')_-}}\right\} \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\le C.
\end{aligned}$$

□

§ 10. Application to density – Sobolev spaces with variable exponent –

In this section we give alternative proofs for two theorems on density as applications of some results in previous sections.

Recall that the Schwartz class is defined by

$$\mathcal{S}(\mathbb{R}^n) := \{u \in C^\infty(\mathbb{R}^n) \,:\, \sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta u(x)| < \infty \text{ for all } \alpha,\, \beta \in {\mathbb{N}_0}^n\}.$$

The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is topologized by the family $\{p_N\}_{N\in\mathbb{N}}$, where

$$p_N(\varphi) = \sum_{|\alpha|\le N} \sup_{x\in\mathbb{R}^n} (1+|x|)^N |D^\alpha \varphi(x)|.$$

As the topological dual, $\mathcal{S}'(\mathbb{R}^n)$ is defined and usually it is equipped with the weak-* topology.

Given a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\alpha \in {\mathbb{N}_0}^n$, we define the derivative $D^\alpha f$ in the weak sense by

$$\int_{\mathbb{R}^n} D^\alpha f(x) u(x)\, dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} f(x) D^\alpha u(x)\, dx \quad (u \in \mathcal{S}(\mathbb{R}^n)).$$

Definition 10.1. Let $s \in \mathbb{N}$ and $X(\mathbb{R}^n) \subset L^1_{\mathrm{loc}}(\mathbb{R}^n)$ be a subspace equipped with a norm $\|\cdot\|_X$. Suppose that for every $f \in X(\mathbb{R}^n)$ there exists $N \in \mathbb{N}$ such that

$$\left| \int_{\mathbb{R}^n} f(x)\varphi(x)\, dx \right| \le N\, p_N(\varphi) \quad (\varphi \in \mathcal{S}(\mathbb{R}^n)).$$

The Sobolev space $X_s(\mathbb{R}^n)$ and its norm are defined respectively by

$$X_s(\mathbb{R}^n) := \left\{ f \in X(\mathbb{R}^n) \,:\, D^\alpha f \in X(\mathbb{R}^n) \text{ for all } \alpha \in {\mathbb{N}_0}^n,\, |\alpha| \le s \right\},$$
$$\|f\|_{X_s} := \sum_{|\alpha|\le s} \|D^\alpha f\|_X.$$

The above is a very general framework. Here we provide an example. We let $X = L^{p(\cdot)}(\mathbb{R}^n)$. Assume that $p(\cdot)$ satisfies (1.3) and (1.4) as well as $1 < p_- \le p_+ < \infty$. Then in [31], we proved that

$$\|f\|_{L^{p(\cdot)}} \sim \left\| \left(\sum_{j=-\infty}^{\infty} |\mathcal{F}^{-1}[\varphi(2^{-j}\cdot)\mathcal{F}f]|^2 \right)^{1/2} \right\|_{L^{p(\cdot)}}, \tag{10.1}$$

where $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfies

$$\mathrm{supp}(\varphi) \subset B(8) \setminus B(1) \text{ and } \sum_{j=-\infty}^{\infty} \varphi(2^{-j}\xi) \equiv \chi_{\mathbb{R}^n\setminus\{0\}}(\xi).$$

Thus, by using the vector-valued boundedness of the Hardy-Littlewood maximal operator, we have

$$\|f\|_{L^{p(\cdot)}_s} \sim \left\| \left(\sum_{j=-\infty}^{\infty} (1+2^{2js}) |\mathcal{F}^{-1}[\varphi(2^{-j}\cdot)\mathcal{F}f]|^2 \right)^{1/2} \right\|_{L^{p(\cdot)}}.$$

Note that

$$f = \sum_{j=-\infty}^{\infty} \varphi_j(D) f$$

takes place in $\mathcal{S}'(\mathbb{R}^n)$.

We remark that (10.1) is a consequence of the extrapolation result in [3]. We refer to [15, 19] for related results.

Now we state two theorems on density. The following result is proved by Diening.

Theorem 10.2 (Diening [7]). *If $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, then $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ is dense in $L^{p(\cdot)}_s(\mathbb{R}^n)$.*

Recall that the set $LH_0(\Omega)$ consists of all locally log-Hölder continuous functions.

Theorem 10.3 (Cruz-Uribe–Fiorenza [2]). *If $p(\cdot) \in LH_0(\mathbb{R}^n)$ and $1 \le p_- \le p_+ < \infty$, then $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ is dense in $L^{p(\cdot)}_s(\mathbb{R}^n)$.*

We will give alternative proofs of the two theorems above. In order to prove Theorem 10.2 we invoke the next theorem due to Nakai–Tomita–Yabuta [32].

Theorem 10.4 ([32]). *Assume the following four conditions:*

(1) *$\chi_B \in X(\mathbb{R}^n)$ for all open balls $B \subset \mathbb{R}^n$.*

(2) *If $g \in X(\mathbb{R}^n)$ and $|f| \le |g|$ a.e. $\mathbb{R}^n$, then $f \in X(\mathbb{R}^n)$.*

(3) *If $g \in X(\mathbb{R}^n)$, $|f_j| \le |g|$ $(j = 1, 2, \dots)$ a.e. $\mathbb{R}^n$ and $\lim_{j\to\infty} f_j = 0$ a.e. $\mathbb{R}^n$, then $\lim_{j\to\infty} \|f_j\|_X = 0$.*

(4) *The Hardy-Littlewood maximal operator M is bounded on $X(\mathbb{R}^n)$.*

Then $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ is dense in $X_s(\mathbb{R}^n)$.

We give a proof of Theorem 10.4 later for convenience. Theorem 10.2 is a direct consequence of Theorem 10.4.

We admit the next theorem, which follows from the definition.

Theorem 10.5. *Let $p(\cdot) : \Omega \to [1, \infty]$ be a variable exponent and $f_j \in L^{p(\cdot)}(\Omega)$ $(j = 1, 2, 3, \dots)$.*

(1) *If $\lim_{j\to\infty} \|f_j\|_{L^{p(\cdot)}(\Omega)} = 0$, then $\lim_{j\to\infty} \rho_p(f_j) = 0$ holds.*

(2) *Following two conditions* (A) *and* (B) *are equivalent:*

(A) $\operatorname{ess\,sup}_{x\in\Omega\setminus\Omega_\infty} p(x) < \infty$.

(B) *If* $\lim_{j\to\infty} \rho_p(f_j) = 0$, *then* $\lim_{j\to\infty} \|f_j\|_{L^{p(\cdot)}(\Omega)} = 0$ *holds.*

Proof of Theorem 10.2. We suppose $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ and apply Theorem 10.4 with $X = L^{p(\cdot)}(\mathbb{R}^n)$. Theorem 10.4 (1), (2) and (4) are obviously true. We shall check (3). If $g \in L^{p(\cdot)}(\mathbb{R}^n)$, $|f_j| \le |g|$ $(j = 1, 2, \dots)$ a.e. $\mathbb{R}^n$ and $\lim_{j\to\infty} f_j = 0$ a.e. $\mathbb{R}^n$, then we have

$$\rho_p(f_j) = \int_{\mathbb{R}^n} |f_j(x)|^{p(x)}\, dx \le \int_{\mathbb{R}^n} |g(x)|^{p(x)}\, dx, \quad |g|^{p(\cdot)} \in L^1(\mathbb{R}^n).$$

Thus by the Lebesgue dominated convergence theorem we obtain

$$\lim_{j\to\infty} \rho_p(f_j) = \int_{\mathbb{R}^n} \lim_{j\to\infty} |f_j(x)|^{p(x)}\, dx = 0.$$

Therefore we get $\lim_{j\to\infty} \|f_j\|_{L^{p(\cdot)}} = 0$ by Theorem 10.5. □

Now we prove Theorem 10.4. Note that the assumptions (1) and (2) imply that $C^\infty_{\mathrm{comp}}(\mathbb{R}^n) \subset X_s(\mathbb{R}^n)$. We will use the following lemma.

Lemma 10.6. *Define*

$$X_{s,\mathrm{comp}}(\mathbb{R}^n) := \{f \in X_s(\mathbb{R}^n) \,:\, \mathrm{supp}(f) \text{ is compact}\}$$

and assume the condition (3) *of Theorem* 10.4. *Then,* $X_{s,\mathrm{comp}}(\mathbb{R}^n)$ *is dense in* $X_s(\mathbb{R}^n)$.

Proof. Take a cut-off function $\zeta \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ so that

$$0 \le \zeta \le 1, \quad \zeta(x) = \begin{cases} 1 & (|x| \le 1), \\ 0 & (|x| > 2). \end{cases}$$

Given a function $f \in X_s(\mathbb{R}^n)$, we define

$$f_j(x) := f(x)\zeta(x/j) \quad (j \in \mathbb{N}).$$

Then we have $f_j \in X_{s,\mathrm{comp}}(\mathbb{R}^n)$ and by condition (3),

$$\lim_{j\to\infty} \|f - f_j\|_{X_s} = 0.$$

Thus, the proof is complete. □

Proof of Theorem 10.4. First note that (1) and (2) imply that $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ is a subset of $X(\mathbb{R}^n)$. Fix a non-negative and radial decreasing function $\psi \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ such that $\|\psi\|_{L^1} = 1$ and define $\psi_t(x) := t^{-n}\psi(x/t)$ $(t > 0)$. By virtue of Lemma 10.6, we shall prove

$$\lim_{t\to 0} \|f - \psi_t * f\|_{X_s} = 0 \quad \text{for all } f \in X_{s,\mathrm{comp}}(\mathbb{R}^n). \tag{10.2}$$

Remark that

$$D^\alpha(\psi_t * f)(x) = \int_{\mathbb{R}^n} (D^\alpha f)(x-y)\psi_t(y)\,dy$$

for every $\alpha \in {\mathbb{N}_0}^n$ with $|\alpha| \le s$. Thus if we prove

$$\lim_{t\to 0} \|f - \psi_t * f\|_X = 0 \quad \text{for all } f \in X(\mathbb{R}^n) \text{ with compact support,} \tag{10.3}$$

then (10.2) is obtained. Take $f \in X(\mathbb{R}^n)$ with compact support. Then Lemma 8.5 gives us the estimate

$$|\psi_t * f(x)| \le Mf(x)$$

and due to condition (4) we see that $Mf \in X(\mathbb{R}^n)$. On the other hand, we have that $\lim_{t\to 0}(f - \psi_t * f) = 0$ a.e. $\mathbb{R}^n$. Therefore, by virtue of condition (3), we conclude that $\lim_{t\to 0} \|f - \psi_t * f\|_X = 0$. □

Next we give a proof of Theorem 10.3. In order to prove the theorem we will use the following lemmas.

Lemma 10.7. *If a variable exponent $p(\cdot) : \mathbb{R}^n \to [1,\infty]$ satisfies $p_+ < \infty$, then the set*

$$L^\infty_{\mathrm{comp}}(\mathbb{R}^n) := \{f \text{ is bounded and compactly supported}\}$$

is dense in $L^{p(\cdot)}(\mathbb{R}^n)$.

Proof. Take $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $\varepsilon > 0$ arbitrarily. By Theorem 7.2 we can take a bounded function $g \in L^{p(\cdot)}(\mathbb{R}^n)$ so that $\|f - g\|_{L^{p(\cdot)}} < \varepsilon$. Now we define $g_j := g\chi_{B(0,j)} \in L^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ $(j \in \mathbb{N})$. Then, since $p_+ < \infty$, the Lebesgue dominated convergence theorem implies that

$$\lim_{j\to\infty} \rho_p(g - g_j) = 0. \tag{10.4}$$

Thus there exists $J \in \mathbb{N}$ such that $\|g - g_j\|_{L^{p(\cdot)}} < \varepsilon$ for all $j \ge J$. Namely we get

$$\|f - g_j\|_{L^{p(\cdot)}} \le \|f - g\|_{L^{p(\cdot)}} + \|g - g_j\|_{L^{p(\cdot)}} < 2\varepsilon.$$

Thus, the proof is complete. □

Lemma 10.8. *Let $\psi \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$. If $p(\cdot) \in LH_0(\mathbb{R}^n)$ and $1 \le p_- \le p_+ < \infty$, then, for all $N \in \mathbb{N}$, for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ supported on $B(0,N)$ and for all $t > 0$,*

$$\|\psi_t * f\|_{L^{p(\cdot)}} \le C_N \|f\|_{L^{p(\cdot)}},$$

in particular, $\psi_t * f \in L^{p(\cdot)}(\mathbb{R}^n)$.

The proof of Lemma 10.8 is based on the next lemma.

Lemma 10.9. *Let $p(\cdot) \in LH_0(\mathbb{R}^n)$ and $1 \le p_- \le p_+ < \infty$. Then there exists a constant $C > 0$ such that*

$$\left(\fint_{B(x,t)} |f(y)|\, dy \right)^{p(x)} \le C \left(\fint_{B(x,t)} |f(y)|^{p(y)}\, dy + 1 \right) \tag{10.5}$$

for all $t > 0$ and all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ with $\|f\|_{L^{p(\cdot)}} \le 1$.

Proof. In (4.1), we defined

$$I := I(x,t) = \fint_{B(x,t)} |f(y)|\, dy, \quad J := J(x,t) = \fint_{B(x,t)} |f(y)|^{p(y)}\, dy.$$

By Lemma 4.1, then we have

$$I \le C J^{1/p(x)} + 1.$$

If we insert the definition of I and J, then we have the desired result. □

Proof of Lemma 10.8. Assume $\|f\|_{L^{p(\cdot)}} = 1$ and the support of f is included in $B(0,N)$. Let $t \in (0,1]$. Then the support of $\psi_t * f$ is included in $B(0, N+2)$. We write $\rho_p(\psi_t * f)$ out in full:

$$\rho_p(\psi_t * f) = \int_{B(0,N+2)} \left| \int_{\mathbb{R}^n} t^{-n} \psi((x-y)/t) f(y)\, dy \right|^{p(x)} dx.$$

Applying (10.5) we obtain

$$\begin{aligned}
\rho_p(\psi_t * f) &\le C \int_{B(0,N+2)} \left(\fint_{B(x,t)} |f(y)|\, dy \right)^{p(x)} dx \\
&\le C \int_{B(0,N+2)} \left(1 + \fint_{B(x,t)} |f(y)|^{p(y)}\, dy \right) dx \\
&= C|B(0,N+2)| \\
&\qquad + \frac{C}{|B(0,t)|} \int_{B(0,N+2)} \left(\int_{\mathbb{R}^n} \chi_{\{|x-y|<t\}}(x,y) |f(y)|^{p(y)}\, dy \right) dx \\
&\le C(|B(0,N+2)| + 1).
\end{aligned}$$

Therefore by Lemma 9.10 we get $\|\psi_t * f\|_{L^{p(\cdot)}} \le C_N$. □

Proof of Theorem 10.3. Take a non-negative and radial decreasing function $\psi \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ so that $\|\psi\|_{L^1} = 1$. By Lemma 10.6, it is enough to prove that

$$\lim_{t \to 0} \|f - \psi_t * f\|_{L^{p(\cdot)}_s} = 0,$$

for all $f \in L^{p(\cdot)}_s(\mathbb{R}^n)$ with compact suport. Since

$$D^\alpha(\psi_t * f)(x) = \int_{\mathbb{R}^n} (D^\alpha f)(x-y)\psi_t(y)\,dy$$

for every $\alpha \in \mathbb{N}_0{}^n$ with $|\alpha| \le s$, it is also enough to prove that

$$\lim_{t\to 0} \|f - \psi_t * f\|_{L^{p(\cdot)}} = 0 \tag{10.6}$$

for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ with compact suport.

Now, let $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $\operatorname{supp} f \subset B(0,N)$. Since $L^\infty_{\rm comp}(\mathbb{R}^n)$ is dense in $L^{p(\cdot)}(\mathbb{R}^n)$ by Lemma 10.7, for $\epsilon > 0$ we can take a function $g \in L^\infty_{\rm comp}(\mathbb{R}^n)$ such that $\|f-g\|_{L^{p(\cdot)}(\mathbb{R}^n)} < \epsilon/(2(C_N+1))$, where C_N is the constant in Lemma 10.8. In this case we may assume that $\operatorname{supp}(f-g) \subset B(0,N)$. Then, using Lemma 10.8, we have that, for $t \in (0,1]$,

$$\begin{aligned}\|\psi_t * f - f\|_{L^{p(\cdot)}} &\le \|\psi_t * f - \psi_t * g\|_{L^{p(\cdot)}} + \|\psi_t * g - g\|_{L^{p(\cdot)}} + \|g-f\|_{L^{p(\cdot)}} \\ &\le C_N\|f-g\|_{L^{p(\cdot)}} + \|\psi_t * g - g\|_{L^{p(\cdot)}} + \|g-f\|_{L^{p(\cdot)}} \\ &\le \epsilon/2 + \|\psi_t * g - g\|_{L^{p(\cdot)}}.\end{aligned}$$

We note that $\psi_t * g(x) \to g(x)$ a.e. x as $t \to 0$. From $g \in L^\infty_{\rm comp}(\mathbb{R}^n)$ it follows that $\|\psi_t * g\|_{L^\infty} \le \|g\|_{L^\infty}$ and that $\operatorname{supp}\psi_t * g$ is included in $B(0,N+2)$ for $0<t<1$. Hence the Lebesgue dominated convergence theorem gives us $\lim_{t\to 0}\rho_p(g-\psi_t * g) = 0$. Consequently we can take $0 < t_\varepsilon < 1$ so that $\|f - \psi_t * f\|_{L^{p(\cdot)}} < \varepsilon$ holds whenever $0 < t \le t_\varepsilon$. □

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