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Estimates for bilinear Fourier multiplier operators and bilinear pseudo-differential operators

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The purpose of this article is to give a brief survey of some results on bilinear Fourier multiplier operators and bilinear pseudo-differential operators, which were recently obtained in the joint work of Naohito Tomita, Loukas Grafakos, and the author. The details will be published in [16], [25], and [26]. [1]

1 Linear operators

We recall some classical results on linear Fourier multiplier operators and linear pseudo-differential operators.

For $m \in L^\infty(\mathbb{R}^n)$, the linear Fourier multiplier operator $m(D)$ is defined by

$$m(D)f(x) = \int_{\mathbb{R}^n} e^{2\pi i x \xi} m(\xi) \widehat{f}(\xi)\, d\xi, \quad f \in \mathcal{S}(\mathbb{R}^n),$$

where $\widehat{f}$ denotes the Fourier transform defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \xi} dx.$$

The function m is called the *multiplier*.

For $0 < p < \infty$, we write $H^p(\mathbb{R}^n)$ to denote the usual Hardy space on $\mathbb{R}^n$ (see, e.g., [28, Chapter III]). We shall simply say that $m(D)$ is bounded in $H^p(\mathbb{R}^n)$ if there exists a constant $C_{m,n,p}$ such that the estimate

$$\|m(D)f\|_{H^p(\mathbb{R}^n)} \le C_{m,n,p} \|f\|_{H^p(\mathbb{R}^n)} \tag{1.1}$$

holds for all $f \in \mathcal{S}(\mathbb{R}^n) \cap H^p(\mathbb{R}^n)$. We want to find a simple sufficient condition on m for $m(D)$ to be bounded in $H^p(\mathbb{R}^n)$. A well-known criterion is the following.

Theorem 1.1. *If $m(\xi)$ is C^∞ away from the origin and satisfies the estimates*

$$|\partial_\xi^\alpha m(\xi)| \le C_\alpha |\xi|^{-|\alpha|} \tag{1.2}$$

for all α, then (1.1) *holds for all $0 < p < \infty$.*

In fact, for a fixed $0 < p < \infty$, the boundedness (1.1) holds if we assume (1.2) only for α up to certain order. Several sharp conditions generalizing (1.2) are known. One of such conditions is given in terms Sobolev norms.

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Definition 1.2. For $f \in \mathcal{S}'(\mathbb{R}^n)$, the Sobolev norm is defined by

$$\|f\|_{W^s(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} (1+|\xi|)^{2s} |\widehat{f}(\xi)|^2 \, d\xi \right)^{1/2}.$$

Taking a function $\Psi \in C_0^\infty(\mathbb{R}^n)$ such that $\operatorname{supp} \Psi \subset \mathbb{R}^n \setminus \{0\}$ and

$$\sum_{j \in \mathbb{Z}} \Psi(\xi/2^j) = 1, \quad \xi \in \mathbb{R}^n \setminus \{0\},$$

we define

$$A_s(m) = \sup_{j \in \mathbb{Z}} \|m(2^j \cdot)\Psi(\cdot)\|_{W^s(\mathbb{R}^n)}. \tag{1.3}$$

The following theorem was essentially proved by Hörmander [19, cf. Theorem 2.5].

Theorem 1.3 (Hörmander)**.** *If* $m \in L^\infty(\mathbb{R}^n)$ *and* $A_s(m) < \infty$ *with an* $s > n/2$*, then* (1.1) *holds for all* $1 < p < \infty$*.*

Generalization of this theorem to the case $0 < p \leq 1$ was given by Calderón and Torchinsky [6, Theorem 4.6].

Theorem 1.4 (Calderón-Torchinsky)**.** *If* $0 < p \leq 1$ *and if* $m \in L^\infty(\mathbb{R}^n)$ *satisfies* $A_s(m) < \infty$ *with an* $s > n(1/p - 1/2)$*, then* (1.1) *holds.*

It is known that Theorems 1.3 and 1.4 are 'sharp' in the sense that the numbers $n/2$ and $n(1/p - 1/2)$ can not be replaced by smaller numbers. We shall observe that these critical numbers are related to the boundedness of $m(D)$ for m satisfying

$$|\partial_\xi^\alpha m(\xi)| \leq C_\alpha (1+|\xi|)^{-b}. \tag{1.4}$$

Such an m is sometimes called exotic multiplier. Here we shall consider the general situation of pseudo-differential operators with exotic symbols. For a function $\sigma \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, the linear pseudo-differential operator $\sigma(X, D)$ is defined by

$$\sigma(X,D)f(x) = \int_{\mathbb{R}^n} e^{2\pi i x \xi} \sigma(x,\xi) \widehat{f}(\xi) \, d\xi, \quad x \in \mathbb{R}^n, \quad f \in \mathcal{S}(\mathbb{R}^n).$$

The function σ is called the *symbol* of the operator. As a generalization of m satisfying (1.4), we consider symbols $\sigma(x, \xi)$ that satisfy

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \leq C_{\alpha,\beta} (1+|\xi|)^{-b} \tag{1.5}$$

for all multi-indices α, β.

For $\sigma(X, D)$ with σ satisfying (1.5), the following basic L^2-boundedness was given by Calderón and Vaillancourt [7].

Theorem 1.5 (Calderón-Vaillancourt)**.** *If* σ *satisfies* (1.5) *with* $b = 0$*, then* $\sigma(X, D)$ *is bounded in* $L^2(\mathbb{R}^n)$*.*

Corresponding theorem for $L^p(\mathbb{R}^n)$ and $H^p(\mathbb{R}^n)$ were given by Coifman and Meyer [8], [9] (an independent proof was also given by the author [23], [24]), which reads as follows.

Theorem 1.6 (Coifman-Meyer). *If $0 < p < \infty$ and if σ satisfies* (1.5) *with $b = n|1/p - 1/2|$, then $\sigma(X, D)$ is bounded from $H^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.*

The number $n|1/p-1/2|$ of Theorem 1.6 is known to be sharp. We shall observe that the critical order $n|1/p-1/2|$ of Theorem 1.6 is related to the sharp differentiability condition of Theorem 1.4. To see this, simply notice that if m satisfies (1.4) for all α then it also satisfies (1.2) for $|\alpha| \le b$ (if b is not an integer, the derivatives must be interpreted in some generalized sense). Thus if the the number $n(1/p-1/2)$ in the theorem of Calderón-Torchinsky could be improved, then the number $n|1/p-1/2|$ in Theorem 1.6 would also be improved. But this is not the case.

In this article, we shall observe similar but somewhat different features for bilinear Fourier multipliers and bilinear pseudo-differential operators.

2 Bilinear Fourier multiplier operators

For $m \in L^\infty(\mathbb{R}^{2n})$, the bilinear Fourier multiplier operator T_m is defined by

$$T_m(f,g)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{2\pi i x(\xi+\eta)} m(\xi,\eta) \widehat{f}(\xi)\widehat{g}(\eta)\, d\xi d\eta, \quad x \in \mathbb{R}^n,$$

where f, $g \in \mathcal{S}(\mathbb{R}^n)$. The function m is called the *multiplier*. If we use the kernel $K = \mathcal{F}^{-1}m$ (inverse Fourier transform of m on $\mathbb{R}^{2n}$), we can write

$$T_m(f,g)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x-y_1, x-y_2) f(y_1) g(y_2)\, dy_1 dy_2, \quad x \in \mathbb{R}^n,$$

where the integral should be taken in an appropriate generalized sense if K is not integrable.

In this section, we consider the multipliers m that satisfy, in certain weak sense, the conditions

$$|\partial_\xi^{\alpha_1}\partial_\eta^{\alpha_2} m(\xi,\eta)| \le C_{\alpha_1,\alpha_2}(|\xi|+|\eta|)^{-|\alpha_1|-|\alpha_2|}, \tag{2.1}$$

and we will be concerned about the following boundedness of T_m between Lebesgue or Hardy spaces:

$$T_m : H^p(\mathbb{R}^n) \times H^q(\mathbb{R}^n) \to L^r(\mathbb{R}^n), \quad 1/p + 1/q = 1/r.$$

The restriction $1/p+1/q = 1/r$ is natural since in the most simplest case $m(\xi,\eta) = 1$ we have $T_m(f,g)(x) = f(x)g(x)$ for which Hölder's inequality $\|fg\|_{L^r} \le \|f\|_{L^p}\|g\|_{L^q}$ holds only for $1/p + 1/q = 1/r$. We always adopt the convention that

$$H^p(\mathbb{R}^n) = L^p(\mathbb{R}^n) \quad \text{if} \quad 1 < p \le \infty.$$

In the case $p = q = r = \infty$, we shall consider $L^\infty \times L^\infty \to BMO$ instead of $L^\infty \times L^\infty \to L^\infty$. We write

$$\|T_m\|_{H^p \times H^q \to L^r}$$

to denote the smallest A, possibly infinity, that satisfies $\|T_m(f,g)\|_{L^r} \le A\|f\|_{H^p}\|g\|_{H^q}$ for all $f \in \mathcal{S} \cap H^p$ and $g \in \mathcal{S} \cap H^q$. We define $\|T_m\|_{L^\infty \times L^\infty \to BMO}$ in the same way by replacing the norms $\|\cdot\|_{H^p}$, $\|\cdot\|_{H^q}$, $\|\cdot\|_{L^r}$ by $\|\cdot\|_{L^\infty}$, $\|\cdot\|_{L^\infty}$, $\|\cdot\|_{BMO}$, respectively.

For smooth multipliers satisfying (2.1), the basic result reads as follows.

Theorem 2.1. *If m satisfies* (2.1) *for all multi-indices α_1, α_2, then $\|T_m\|_{H^p \times H^q \to L^r} < \infty$ for all $0 < p, q, r \le \infty$ satisfying $1/p + 1/q = 1/r > 0$ and $\|T_m\|_{L^\infty \times L^\infty \to BMO} < \infty$.*

Theorem 2.1 is due to Coifman and Meyer [8], [9], [10] (the case p, q, $r > 1$), Kenig and Stein [21] (the case $1/2 < r \le 1$), and Grafakos and Kalton [15] (full range p, q, $r > 0$).

As in the case of linear Fourier multiplier operators, to assure the boundedness of the bilinear operator T_m, it is sufficient to assume the condition (2.1) for derivatives up to certain order. In the papers cited above, the authors are mostly assuming (2.1) for $|\alpha_1| + |\alpha_2| \le 2n+1$. We shall consider the problem to find weak differentiability conditions of the type (2.1) that assure the boundedness of T_m.

Before going into the problem, we shall see that the bilinear Fourier multiplier operators naturally appear in several problems in analysis. We shall see this in the following two examples.

Example 1. As a first example, we consider the Cauchy integral, which was also the basic motivation of the study of Coifman and Meyer [8], [9], [10]. The Cauchy integral is defined by

$$I_A f(x) = \int_{\mathbb{R}} \frac{f(y)}{x - y + i(A(x) - A(y))} dy,$$

where A is a real-valued function on $\mathbb{R}$ with $A' \in L^\infty$. One of the way to study this integral is to write it as

$$I_A f(x) = \sum_{k=0}^{\infty} (-i)^k \int_{\mathbb{R}} \frac{(A(x) - A(y))^k}{(x-y)^{k+1}} f(y) dy.$$

The term corresponding to $k = 0$ is nothing but the Hilbert transform:

$$Hf(x) = \int_{\mathbb{R}} \frac{1}{x-y} f(y) dy = -\pi i \int_{\mathbb{R}} e^{2\pi i x \xi} \mathrm{sign}\,(\xi) \widehat{f}(\xi) d\xi,$$

which is the typical example of $m(D)$ of Theorem 1.1. The term corresponding to $k = 1$ is, except for the factor $-i$,

$$C_A f(x) = \int_{\mathbb{R}} \frac{A(x) - A(y)}{(x-y)^2} f(y) dy.$$

This is called Calderón's commutator, and is an example of bilinear Fourier multiplier operator. In fact, by writing $A(x) - A(y) = \int_y^x a(z) dz$, $a = A'$, we have

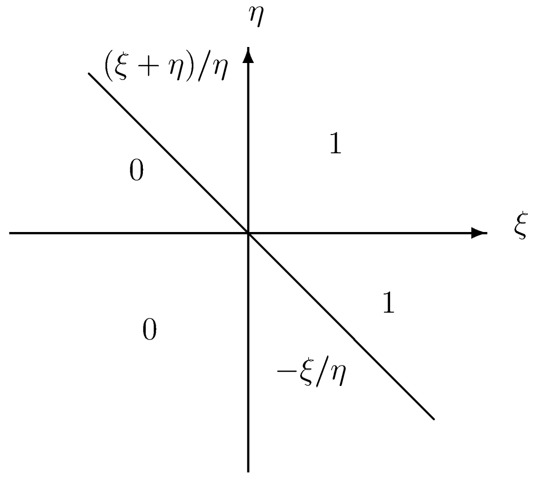
$$C_A f(x) = \iint \widehat{f}(\xi) \widehat{a}(\eta) m(\xi, \eta) e^{2\pi i x (\xi + \eta)} d\xi d\eta = T_m(f, a)(x)$$

with

$$m(\xi, \eta) = -\pi i \int_0^1 \mathrm{sign}\,(\xi + t\eta) dt$$

$$= -\pi i \begin{cases} 0 & \text{if} \quad \xi \le 0,\ \xi+\eta \le 0, \\ (\xi+\eta)/\eta & \text{if} \quad \xi \le 0,\ \xi+\eta > 0, \\ -\xi/\eta & \text{if} \quad \xi > 0,\ \xi+\eta \le 0, \\ 1 & \text{if} \quad \xi > 0,\ \xi+\eta > 0. \end{cases}$$

The following is the picture of this multiplier.



Except for the origin, the multiplier $m(\xi, \eta)$ is Lipschitz continuous and hence has the first order derivative in the classical sense. If we use the differentiability in L^2 sense, we see the following: if $\Psi \in C_0^\infty(\mathbb{R}^2)$ and $\operatorname{supp}\Psi$ does not contain the origin, then $m\Psi \in W^s(\mathbb{R}^2)$ for $s < 3/2$. The theorem to be given below (Theorem 2.3) covers such multipliers and shows, in particular, that C_A is bounded in $L^p(\mathbb{R})$, $1 < p < \infty$, if $A' \in L^\infty$. (For the Cauchy integral and Calderón's commutator, many approaches are known. A recent approach is given by Muscalu [27].)

Example 2. As the second example, we consider the inequality

$$\|D^s(fg)\|_{L^p} \lesssim \|D^s f\|_{L^p}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|D^s g\|_{L^p},$$

where $D^s(f) = \mathcal{F}^{-1}(|\xi|^s \widehat{f}(\xi))$. This inequality and its variants are called the Kato-Ponce inequalities (see [20]). In a proof of the above inequality, we take a function $\psi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ satisfying $\sum_{j=-\infty}^{\infty} \psi(2^{-j}\xi) = 1$ and decompose f and g as

$$f = \sum_{j=-\infty}^{\infty} \psi(2^{-j}D)f = \sum_{j=-\infty}^{\infty} f_j, \quad g = \sum_{j=-\infty}^{\infty} \psi(2^{-j}D)g = \sum_{j=-\infty}^{\infty} g_j.$$

We have

$$fg = \sum_{j,k\in\mathbb{Z}} f_j g_k = \sum_{j<k-10} + \sum_{|j-k|\le 10} + \sum_{j>k+10}.$$

To estimate $D^s(fg)$, the essential part comes from $|j-k| \le 10$, typically from $j = k$, where we are led to consider the following bilinear multiplier:

$$m(\xi, \eta) = \sum_{j\ge 0} 2^{-sj} \sum_{k\in\mathbb{Z}} \Theta(2^{-k}(\xi+\eta))\Psi_1(2^{-(j+k)}\xi)\Psi_2(2^{-(j+k)}\eta),$$

where Ψ_1, Ψ_2, and Θ are smooth functions supported in some annulus. It is straightforward to verify that m satisfies condition (2.1) for $|\alpha_1|+|\alpha_2|<s$ but not for larger $|\alpha_1|+|\alpha_2|$. For more details, see [14] and [16, Introduction and Appendix B].

Now, we come back to the problem of finding weak differentiability conditions for bilinear Fourier multipliers. We want to find conditions similar to those in Hörmander's theorem or in the Calderón-Torchinsky theorem. In this direction, there are some recent results. Tomita [29] proved that $\|T_m\|_{L^p\times L^q\to L^r}<\infty$, $1<p,q,r<\infty$, if

$$\sup_{j\in\mathbb{Z}}\|m(2^j\cdot)\Psi(\cdot)\|_{W^s(\mathbb{R}^{2n})}<\infty \tag{2.2}$$

with an $s>n$. Grafakos and Si [17] proved that Tomita's result can be extended to $r\le 1$ if we strengthen the assumption (2.2) by using L^λ-based Sobolev space, $1<\lambda\le 2$.

In this article, we consider the problem in a different formulation; to measure the smoothness of multipliers, we use, instead of the usual Sobolev norm on $\mathbb{R}^{2n}$, the product type Sobolev norm. In this setting, we can obtain 'sharp' differentiability conditions, and the result implies some improvements of the results of Tomita and Grafakos-Si.

We begin with the following definition.

Definition 2.2 (Product type Sobolev norm)**.** For s_1, $s_2>0$ and for functions $F\in L^2(\mathbb{R}^{2n})$, we define

$$\|F\|_{W^{(s_1,s_2)}(\mathbb{R}^{2n})}=\left(\iint_{\mathbb{R}^n\times\mathbb{R}^n}(1+|x_1|)^{2s_1}(1+|x_2|)^{2s_2}|\mathcal{F}^{-1}F(x_1,x_2)|^2\,dx_1dx_2\right)^{1/2}.$$

We take a function Ψ such that

$$\begin{aligned}&\Psi\in C_0^\infty(\mathbb{R}^{2n}),\quad \operatorname{supp}\Psi\subset\mathbb{R}^{2n}\setminus\{0\},\\&\sum_{j\in\mathbb{Z}}\Psi(\xi/2^j,\eta/2^j)=1\quad(\forall(\xi,\eta)\in\mathbb{R}^{2n}\setminus\{0\}),\end{aligned}\tag{2.3}$$

and, for $m\in L^\infty(\mathbb{R}^{2n})$, we define

$$A_{(s_1,s_2)}(m)=\sup_{j\in\mathbb{Z}}\|m(2^j\xi,2^j\eta)\Psi(\xi,\eta)\|_{W^{(s_1,s_2)}(\mathbb{R}^{2n})}.$$

We consider the estimate

$$\|T_m\|_{H^p\times H^q\to L^r}\lesssim A_{(s_1,s_2)}(m),\tag{2.4}$$

where $0<p,q,r\le\infty$ and $1/p+1/q=1/r$. In the case $p=q=r=\infty$, we consider the estimate for $L^\infty\times L^\infty\to BMO$ in place of $H^p\times H^q\to L^r$.

The following is the main result of [16] and [25].

Theorem 2.3 ([16], [25])**.** *Let $0<p,q,r\le\infty$ and $1/p+1/q=1/r$. If*

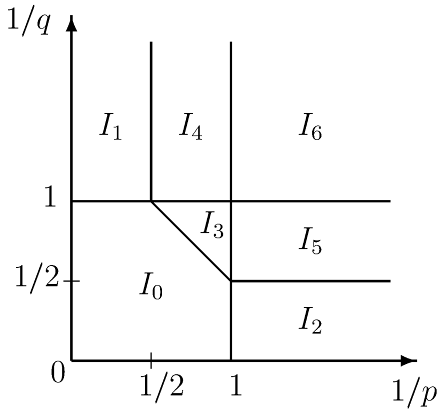
$$\begin{aligned}&s_1>\max\{n/2,n(1/p-1/2)\},\quad s_2>\max\{n/2,n/q-n/2\},\\&s_1+s_2>n/r-n/2,\end{aligned}\tag{2.5}$$

then the estimate (2.4) *holds, where* $H^p \times H^q \to L^r$ *is replaced by* $L^\infty \times L^\infty \to BMO$ *when* $p = q = r = \infty$*. Conversely, if* (2.4) *with the convention that* $H^p \times H^q \to L^r$ *is replaced by* $L^\infty \times L^\infty \to BMO$ *when* $p = q = r = \infty$ *holds, then*

$$\begin{aligned} &s_1 \geq \max\{n/2, n(1/p - 1/2)\}, \quad s_2 \geq \max\{n/2, n/q - n/2\}, \\ &s_1 + s_2 \geq n/r - n/2. \end{aligned} \tag{2.6}$$

Thus, in terms of the product type norm $A_{(s_1,s_2)}$, the condition (2.5) or (2.6) is the sharp condition for (2.4) (the equality cases of (2.6) are open).

To see easily the various conditions of Theorem 2.3, we divide the region of $(1/p, 1/q)$ into 7 regions $I_0, \ldots, I_6$ as in the following figure.



The assumptions of (2.5) are written as follows:

$$\begin{array}{ll}
s_1 > n/2, \quad s_2 > n/2 & \text{if} \quad (1/p, 1/q) \in I_0; \\
s_1 > n/2, \quad s_2 > n/q - n/2 & \text{if} \quad (1/p, 1/q) \in I_1; \\
s_1 > n/p - n/2, \quad s_2 > n/2 & \text{if} \quad (1/p, 1/q) \in I_2; \\
\begin{cases} s_1 > n/2, \quad s_2 > n/2, \\ s_1 + s_2 > n/p + n/q - n/2 \end{cases} & \text{if} \quad (1/p, 1/q) \in I_3; \\
\begin{cases} s_1 > n/2, \quad s_2 > n/q - n/2, \\ s_1 + s_2 > n/p + n/q - n/2 \end{cases} & \text{if} \quad (1/p, 1/q) \in I_4; \\
\begin{cases} s_1 > n/p - n/2, \quad s_2 > n/2, \\ s_1 + s_2 > n/p + n/q - n/2 \end{cases} & \text{if} \quad (1/p, 1/q) \in I_5; \\
\begin{cases} s_1 > n/p - n/2, \quad s_2 > n/q - n/2, \\ s_1 + s_2 > n/p + n/q - n/2 \end{cases} & \text{if} \quad (1/p, 1/q) \in I_6.
\end{array}$$

Notice that the condition $s_1 + s_2 > n/p + n/q - n/2$ is necessary only in the regions I_3, I_4, I_5, and I_6.

Similar but partial results for multilinear Fourier multiplier operators are also given in [16].

3 Bilinear Pseudo-differential operators

We next consider bilinear pseudo-differential operators and want to find a theorem corresponding to Theorem 1.6. We begin with the definition of bilinear pseudo-differential operators.

Definition 3.1. For a function $\sigma = \sigma(x, \xi, \eta)$ on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$, we define the bilinear pseudo-differential operator T_σ by

$$T_\sigma(f, g)(x) = \iint_{\mathbb{R}^n \times \mathbb{R}^n} e^{2\pi i x(\xi+\eta)} \sigma(x, \xi, \eta) \widehat{f}(\xi) \widehat{g}(\eta) \, d\xi d\eta, \quad x \in \mathbb{R}^n,$$

where $f, g \in \mathcal{S}(\mathbb{R}^n)$. The function σ is called the *symbol* of the operator.

We shall consider the following class of symbols and operators.

Definition 3.2. For $m \in \mathbb{R}$, we define the symbol class $BS^m_{0,0}$ as the set of all C^∞ functions $\sigma = \sigma(x, \xi, \eta)$ on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ that satisfy

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x, \xi, \eta)| \le C_{\alpha,\beta,\gamma} (1 + |\xi| + |\eta|)^m$$

for all multi-indices α, β, γ. We write the corresponding class of bilinear pseudo-differential operators as

$$\mathrm{Op}(BS^m_{0,0}) = \{T_\sigma \mid \sigma \in BS^m_{0,0}\}.$$

Bényi-Bernicot-Maldonado-Naibo-Torres [1] considered the symbol class $BS^m_{\rho,\delta}$ for $0 \le \rho, \delta \le 1$ and showed that symbolic calculus in the corresponding operator class $\mathrm{Op}(BS^m_{\rho,\delta})$ works in a similar way as in the linear case. In this section, we restrict our study to the case $\rho = \delta = 0$.

In contrast to the Calderón-Vaillancourt theorem (Theorem 1.5), operators in the bilinear class $\mathrm{Op}(BS^0_{0,0})$ do not have good boundedness. In fact, Bényi and Torres [4] proved the following.

Theorem 3.3 ([4])**.** *There exists a symbol $\sigma \in BS^0_{0,0}$ such that T_σ is not bounded in $L^p \times L^q \to L^r$ for any $1 \le p,\ q,\ r < \infty$, $1/p + 1/q = 1/r$.*

Recently, Michalowski, Rule, and Staubach [22] proved that operators of class $\mathrm{Op}(BS^m_{0,0})$ are bounded in $L^2 \times L^2 \to L^1$ if $m < -n/2$. Generalizing this, Bényi, Bernicot, Maldonado, Naibo, and Torres [1] proved the following.

Theorem 3.4 ([22], [1])**.** *Operators of class $\mathrm{Op}(BS^m_{0,0})$ are bounded in $L^p \times L^q \to L^r$, $1 \le p, q, r \le \infty$, $1/p + 1/q = 1/r$, if*

$$m < \widetilde{m}(p, q) = -n \left(\max\left\{ \frac{1}{2}, \frac{1}{p}, \frac{1}{q}, 1 - \frac{1}{r} \right\} \right).$$

Our purpose is to refine Theorem 3.4. We consider the problem in the full range $0 < p,\ q,\ r \le \infty$ by replacing some L^p spaces by the local Hardy spaces h^p or by *bmo* (the definitions of these spaces will be given below) and we completely determine the values of m for which the operators of class $\mathrm{Op}(BS^m_{0,0})$ are bounded between h^p, L^p, and *bmo*. In particular, as for the boundedness in $L^p \times L^q \to L^r$, $1/p+1/q = 1/r$

in the range $1 \le p,\, q,\, r \le \infty$, we will show that $\widetilde{m}(p,q)$ given in Theorem 3.4 are sharp and also show that the boundedness still holds in the critical case $m = \widetilde{m}(p,q)$ except for $(p,q) = (1,\infty)$, $(\infty,1)$, (∞,∞).

In order to give our results in a precise form, we recall the definitions of h^p and *bmo*.

Definition 3.5 ([14])**.** Let $0 < p \le 1$ and take a $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \phi(x)\, dx \neq 0$. For $f \in \mathcal{S}'(\mathbb{R}^n)$, we define

$$\|f\|_{h^p(\mathbb{R}^n)} = \left\{ \int_{\mathbb{R}^n} \left(\sup_{0<t<1} |(\phi_t * f)(x)| \right)^p dx \right\}^{1/p},$$

where $\phi_t(x) = t^{-n}\phi(x/t)$. The set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $\|f\|_{h^p(\mathbb{R}^n)} < \infty$ is denoted by $h^p(\mathbb{R}^n)$.

It is known that $h^p(\mathbb{R}^n)$ does not depend on the choice of ϕ. Obviously $h^p(\mathbb{R}^n) \supset H^p(\mathbb{R}^n)$. The Schwartz class $\mathcal{S}(\mathbb{R}^n)$ is a dense subspace of $h^p(\mathbb{R}^n)$.

Definition 3.6. For locally integrable functions f on $\mathbb{R}^n$, we define

$$\|f\|_{bmo(\mathbb{R}^n)} = \sup_{|Q|\le 1} \frac{1}{|Q|} \int_Q |f(x) - f_Q|\, dx + \sup_{|Q|=1} \frac{1}{|Q|} \int_Q |f(x)|\, dx,$$

where Q denotes cubes in $\mathbb{R}^n$ and $f_Q = |Q|^{-1} \int_Q f(y)dy$. The class $bmo(\mathbb{R}^n)$ is defined to be the set of all locally integrable functions f on $\mathbb{R}^n$ such that $\|f\|_{bmo(\mathbb{R}^n)} < \infty$.

Obviously $bmo(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$. It is known that the dual spaces of $h^1(\mathbb{R}^n)$ is $bmo(\mathbb{R}^n)$ ([14, Corollary 1, p. 36]).

We use the following notation:

$$X^p = X^p(\mathbb{R}^n) = \begin{cases} h^p(\mathbb{R}^n) & \text{if} \quad 0 < p \le 1, \\ L^p(\mathbb{R}^n) & \text{if} \quad 1 < p < \infty, \\ bmo(\mathbb{R}^n) & \text{if} \quad p = \infty. \end{cases}$$

If $0 < p, q, r \le \infty$ and $1/p + 1/q = 1/r$, we define

$$\|T_\sigma\|_{X^p \times X^q \to X^r} = \sup\{\|T_\sigma(f,g)\|_{X^r} \mid f \in \mathcal{S},\ g \in \mathcal{S},\ \|f\|_{X^p} = \|g\|_{X^q} = 1\}.$$

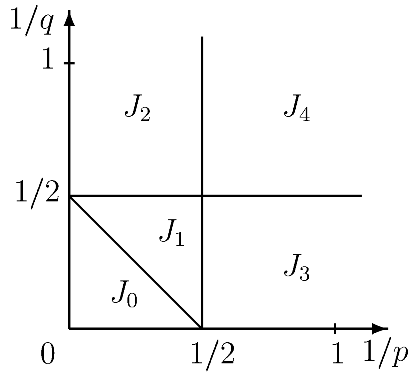
If $\|T_\sigma\|_{X^p \times X^q \to X^r} < \infty$, then, with a slight abuse of terminology, we say that T_σ is bounded in $X^p \times X^q \to X^r$.

The following is our main theorem.

Theorem 3.7 ([26])**.** *Let $m \in \mathbb{R}$, $0 < p, q, r \le \infty$, and $1/p + 1/q = 1/r$. Then all the operators of class* $\mathrm{Op}(BS^m_{0,0})$ *are bounded in $X^p(\mathbb{R}^n) \times X^q(\mathbb{R}^n) \to X^r(\mathbb{R}^n)$ if and only if*

$$m \le m(p,q) = -n \left(\max\left\{ \frac{1}{2}, \frac{1}{p}, \frac{1}{q}, 1 - \frac{1}{r}, \frac{1}{r} - \frac{1}{2} \right\} \right).$$

To see the various values of $m(p,q)$ of this theorem, we divide the region of $(1/p, 1/q)$ into 5 regions $J_0, \ldots, J_4$ as follows:



Then

$$m(p,q)=\begin{cases} n/r-n & \text{if } (1/p,1/q)\in J_0;\\ -n/2 & \text{if } (1/p,1/q)\in J_1;\\ -n/q & \text{if } (1/p,1/q)\in J_2;\\ -n/p & \text{if } (1/p,1/q)\in J_3;\\ n/2-n/r & \text{if } (1/p,1/q)\in J_4,\end{cases}$$

where $1/r = 1/p + 1/q$. Observe that $\widetilde{m}(p,q)$ of Theorem 3.7 coincides with $m(p,q)$ in the region $1/p + 1/q \le 1$.

We shall briefly explain some ideas to prove Theorem 3.7.

First we explain the idea to prove the 'only if' part of the theorem. Here we use Khintchine's inequality: if $0 < p < \infty$, then

$$\left(\sum_m |c_m|^2\right)^{1/2} \approx \left\{E\left[\left|\sum_m r_m c_m\right|^p\right]\right\}^{1/p}$$

for all $\{c_m\} \subset \mathbb{C}$ with $\sum_m |c_m|^2 < \infty$, where $\{r_m\}$ denotes independent and identically distributed random variables on a probability space with

$$\mathrm{Prob}\,\{r_m = 1\} = \mathrm{Prob}\,\{r_m = -1\} = \frac{1}{2}$$

and $E[\cdots]$ denotes the expectation. (For this inequality, see, *e.g.*, [30, Chapter V, Section 8].)

Applying Khintchine's inequality to $c_m = a_m e^{2\pi i m x}$, we obtain the following.

Lemma 3.8. *If* $0 < p < \infty$*, then*

$$\left(\sum_m |a_m|^2\right)^{1/2} \approx \left\{E\left[\int_{\mathbb{T}^n}\left|\sum_m r_m a_m e^{2\pi i m x}\right|^p dx\right]\right\}^{1/p}$$

for all $\{a_m\} \subset \mathbb{C}$ *with* $\sum_m |a_m|^2 < \infty$.

This implies the following.

Corollary 3.9. *If $\sum_m |a_m|^2 < \infty$ and $0 < p < \infty$, then there exist a sequence of $\pm$ signs such that*

$$\Big\| \sum_m \pm a_m e^{2\pi i m x} \Big\|_{L^p(\mathbb{T}^n)} \le C_p \Big(\sum_m |a_m|^2 \Big)^{1/2}, \tag{3.1}$$

and also there exist a sequence of $\pm$ signs such that

$$\Big(\sum_m |a_m|^2 \Big)^{1/2} \le C_p \Big\| \sum_m \pm a_m e^{2\pi i m x} \Big\|_{L^p(\mathbb{T}^n)}. \tag{3.2}$$

The inequalities in this corollary may be considered as an 'improvement' of Parseval's identity or the Hausdorf-Young inequality. In fact, if we don't use $\pm$ sign, then Parseval's identity and the Hausdorf-Young inequality give

$$\Big\| \sum_m a_m e^{2\pi i m x} \Big\|_{L^p(\mathbb{T}^n)} \le \Big(\sum_m |a_m|^q \Big)^{1/q}, \quad 1 \le q \le 2 \le p \le \infty,\ 1/p + 1/q = 1,$$

and these are the best we can expect. If, however, we have freedom to choose $\pm$ signs as in (3.1), then we have inequalities with p and q in a wider range. Similar observation may be possible for (3.2).

Now we give a brief sketch of the proof of the following assertion: if $2 \le p < \infty$, $2 \le q < \infty$, $1/p + 1/q = 1/r \ge 1/2$, and if all the operators of class $\mathrm{Op}(BS^m_{0,0})$ are bounded in $L^p \times L^q \to L^r$, then $m \le -n/2$.

Take a $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that $\widehat{\psi} = 1$ on $[-1/4, 1/4]^n$, and $\operatorname{supp} \widehat{\psi} \subset [-1/2, 1/2]^n$. For a sequence of complex numbers $\{c_{k,\ell}\}_{k,\ell \in \mathbb{Z}^n}$ satisfying $\sup_{k,\ell \in \mathbb{Z}^n} |c_{k,\ell}| \le 1$, we consider

$$\sigma(\xi, \eta) = \sum_{k,\ell \in \mathbb{Z}^n} c_{k,\ell} (1 + |k| + |\ell|)^m \widehat{\psi}(\xi - k) \widehat{\psi}(\eta - \ell).$$

Then

$$\big| \partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi, \eta) \big| \le C_{\alpha,\beta} (1 + |\xi| + |\eta|)^m,$$

with $C_{\alpha,\beta}$ independent of $\{c_{k,\ell}\}$. By the assumption and by the closed graph theorem, there exists a constant C independent of $\{c_{k,\ell}\}$ such that

$$\|T_\sigma(f, g)\|_{L^r} \le C \|f\|_{L^p} \|g\|_{L^q}. \tag{3.3}$$

Take a $\theta \in \mathcal{S}$ such that $\widehat{\theta} = 1$ on $[-1/8, 1/8]^n$, and $\operatorname{supp} \widehat{\theta} \subset [-1/4, 1/4]^n$. Let $\epsilon > 0$. By Corollary 3.9, we can take $\alpha_k = \pm 1$ and $\beta_\ell = \pm 1$ so that

$$f(x) = \sum_{k \in \mathbb{Z}^n,\, k \neq 0} \alpha_k |k|^{-n/2-\epsilon} e^{2\pi i k x} \theta(x) \in L^p(\mathbb{R}^n),$$
$$g(x) = \sum_{\ell \in \mathbb{Z}^n,\, \ell \neq 0} \beta_\ell |\ell|^{-n/2-\epsilon} e^{2\pi i \ell x} \theta(x) \in L^q(\mathbb{R}^n).$$

For these f and g, we have

$$T_\sigma(f, g)(x) = \sum_{k,\ell \neq 0} \alpha_k \beta_\ell c_{k,\ell} |k|^{-n/2-\epsilon} |\ell|^{-n/2-\epsilon} (1 + |k| + |\ell|)^m e^{2\pi i (k+\ell) x} \theta(x)^2.$$

Take arbitrary $\gamma_\nu = \pm 1$ ($\nu \in \mathbb{Z}^n$) and choose $c_{k,\ell}$ so that $\alpha_k \beta_\ell c_{k,\ell} = \gamma_{k+\ell}$. Then

$$\begin{aligned} T_\sigma(f,g)(x) &= \sum_{\nu \in \mathbb{Z}^n} \gamma_\nu a_\nu e^{2\pi i \nu x} \theta(x)^2, \\ a_\nu &= \sum_{\substack{k,\ell \neq 0, \\ k+\ell=\nu}} |k|^{-n/2-\epsilon} |\ell|^{-n/2-\epsilon} (1+|k|+|\ell|)^m. \end{aligned}$$

By calculation, we have

$$a_\nu \approx (1+|\nu|)^{m-2\epsilon}.$$

The inequality (3.3) implies

$$\left\| \sum_{\nu \in \mathbb{Z}^n} \gamma_\nu a_\nu e^{2\pi i \nu x} \theta(x)^2 \right\|_{L^r} \leq C$$

with C independent of $\gamma_\nu = \pm 1$. Hence by Khintchine's inequality (Lemma 3.8) we have $\sum_\nu |a_\nu|^2 \leq C$, which is possible only when $m - 2\epsilon < -n/2$. Letting $\epsilon \to 0$, we obtain $m \leq -n/2$.

Next we shall briefly sketch the proof of the 'if' part of Theorem 3.7.

To prove the 'if' part of Theorem 3.7, we prove the following three estimates.

(1) $\sigma \in BS^{-n/2}_{0,0} \Rightarrow T_\sigma : L^2 \times L^2 \to h^1$.

(2) $\sigma \in BS^{-n/p}_{0,0}$, $0 < p < 1 \Rightarrow T_\sigma : h^p \times bmo \to h^p$.

(3) $\sigma \in BS^{-n/p}_{0,0}$, $0 < p < 1 \Rightarrow T_\sigma : h^p \times L^2 \to h^r$, $1/p + 1/2 = 1/r$.

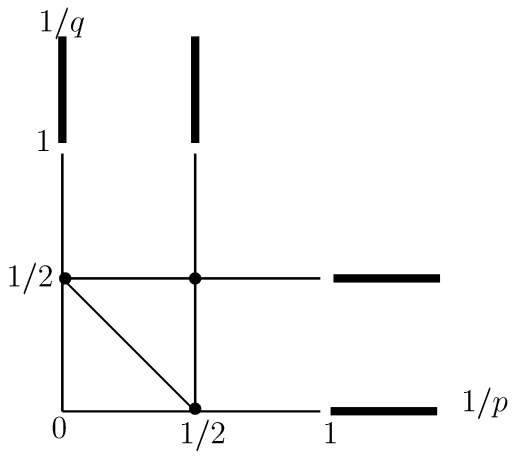
By obvious symmetry, the conclusions of (2) and (3) can be replaced by $bmo \times h^p \to h^p$ or $L^2 \times h^p \to h^r$, respectively. By duality, (1) implies the following:

$$(1') \ \sigma \in BS^{-n/2}_{0,0} \Rightarrow T_\sigma : L^2 \times bmo \to L^2 \ \text{ and } \ bmo \times L^2 \to L^2.$$

These estimates combined with interpolation and duality arguments will yield the whole 'if' part of Theorem 3.7. See the picture below, where the points $(1/2, 1/2)$, $(1/2, 0)$, $(0, 1/2)$ and the four half lines

$$\begin{aligned} &\{(1/p, 1/q) \mid 1 < 1/p < \infty,\ 1/q = 0\}, \\ &\{(1/p, 1/q) \mid 1 < 1/p < \infty,\ 1/q = 1/2\}, \\ &\{(1/p, 1/q) \mid 1/p = 0,\ 1 < 1/q < \infty\}, \\ &\{(1/p, 1/q) \mid 1/p = 1/2,\ 1 < 1/q < \infty\}, \end{aligned}$$

are the places where we prove the estimates directly.



Notice that we do not directly prove the estimates $h^1 \times bmo \to h^1$, $bmo \times h^1 \to h^1$, and $bmo \times bmo \to bmo$; these will be derived with the aid of interpolation and duality from $(1')$ and (2). (this procedure goes back to [23]).

(1) $\sigma \in BS^{-n/2}_{0,0} \Rightarrow T_\sigma : L^2 \times L^2 \to h^1$.

Here, instead of $T_\sigma : L^2 \times L^2 \to h^1$, we shall sketch the proof of the estimate $T_\sigma : L^2 \times L^2 \to L^1$; to replace L^1 by h^1 requires only a slight modification. To prove this estimate, we assume that m is supported on $\{|\xi| + |\eta| \geq 2\}$ since compactly supported m is easy to handle. By using appropriate partition of unity, we may also assume that m is supported in a small cone in $\mathbb{R}^{2n}$.

First, suppose $\sigma \in BS^{-n/2}_{0,0}$ and

$$\operatorname{supp} \sigma \subset \{|\xi|/8 \leq |\eta| \leq 8|\xi|\} \cap \{|\xi| + |\eta| \geq 2\}. \tag{3.4}$$

Using a Ψ satisfying (2.3), we decompose σ as

$$\sigma(x, \xi, \eta) = \sum_{j=1}^{\infty} \sigma(x, \xi, \eta)\Psi(2^{-j}\xi, 2^{-j}\eta) = \sum_{j=1}^{\infty} \sigma_j(x, \xi, \eta). \tag{3.5}$$

By (3.4), $|\xi| \approx |\eta| \approx 2^j$ on $\operatorname{supp} \sigma_j$. Hence we can take a $\psi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ such that

$$T_{\sigma_j}(f, g) = T_{\sigma_j}(f_j, g_j), \quad f_j = \psi(2^{-j}D)f, \quad g_j = \psi(2^{-j}D)g. \tag{3.6}$$

Thus

$$T_\sigma(f, g) = \sum_{j=1}^{\infty} T_{\sigma_j}(f_j, g_j).$$

To estimate each term in the right hand side, we use the following lemma.

Lemma 3.10. *Let N be a sufficiently large positive integer and suppose $\sigma(x, \xi, \eta)$ satisfy*

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x, \xi, \eta)| \leq r^{-n/2} \mathbf{1}\{|\xi| \leq r\} \tag{3.7}$$

for $|\alpha|, |\beta|, |\gamma| \leq N$ with some $r > 0$. Then

$$\|T_\sigma(f, g)\|_{L^1} \leq C\|f\|_{L^2}\|g\|_{L^2}. \tag{3.8}$$

Here N and C can be taken depending only on n and independent of $r > 0$.

Proof of Lemma 3.10. We write

$$\begin{aligned} T_\sigma(f,g)(x) &= \int_{\mathbb{R}^n} e^{2\pi i x\xi} \left(\int_{\mathbb{R}^n} e^{2\pi i x\eta} \sigma(x,\xi,\eta) \widehat{g}(\eta)\, d\eta \right) \widehat{f}(\xi)\, d\xi \\ &= \int_{\mathbb{R}^n} e^{2\pi i x\xi} \tau(g;x,\xi) \widehat{f}(\xi)\, d\xi = \tau(g;X,D)f(x). \end{aligned} \tag{3.9}$$

The following pointwise estimate is easy to prove:

$$|T_\sigma(f,g)(x)| = |\tau(g;X,D)f(x)| \lesssim \Big(\sum_{|\alpha|\le N} \int_{\mathbb{R}^n} |\partial_\xi^\alpha \tau(g;x,\xi)|^2\, d\xi \Big)^{1/2} \widetilde{f}(x), \tag{3.10}$$

$$\widetilde{f}(x) = \left(\int_{\mathbb{R}^n} \frac{|f(y)|^2}{(1+|x-y|)^{2N}}\, dy \right)^{1/2}. \tag{3.11}$$

This inequality and Schwarz inequality give

$$\begin{aligned} \|T_\sigma(f,g)\|_{L^1} &\lesssim \Big(\sum_{|\alpha|\le N} \iint_{\mathbb{R}^n\times\mathbb{R}^n} |\partial_\xi^\alpha \tau(g;x,\xi)|^2\, dxd\xi \Big)^{1/2} \|\widetilde{f}\|_{L^2} \\ &\approx \Big(\sum_{|\alpha|\le N} \iint_{\mathbb{R}^n\times\mathbb{R}^n} |\partial_\xi^\alpha \tau(g;x,\xi)|^2\, dxd\xi \Big)^{1/2} \|f\|_{L^2}. \end{aligned}$$

The assumption (3.7) and the Calderón-Vaillancourt theorem (Theorem 1.5) give

$$\int_{\mathbb{R}^n} |\partial_\xi^\alpha \tau(g;x,\xi)|^2 dx \lesssim r^{-n} \mathbf{1}\{|\xi|\le r\} \|g\|_{L^2}^2.$$

Hence integration with respect to ξ gives

$$\sum_{|\alpha|\le N} \iint_{\mathbb{R}^n\times\mathbb{R}^n} |\partial_\xi^\alpha \tau(g;x,\xi)|^2\, dxd\xi \lesssim \int_{\mathbb{R}^n} r^{-n} \mathbf{1}\{|\xi|\le r\}\, d\xi\, \|g\|_{L^2}^2 \approx \|g\|_{L^2}^2.$$

Combining the estimates, we obtain (3.8). (The idea of using the formula (3.9) and the inequalities (3.10)-(3.11) is due to [22].)

We come back to the proof of $T_\sigma : L^2\times L^2 \to L^1$ for $\sigma \in BS^{-n/2}_{0,0}$ satisfying (3.4). For $\sigma \in BS^{-n/2}_{0,0}$, σ_j satisfies the assumption of Lemma 3.10 with $r\approx 2^j$. Hence Lemma 3.10 yields

$$\|T_{\sigma_j}(f_j,g_j)\|_{L^1} \lesssim \|f_j\|_{L^2}\|g_j\|_{L^2}.$$

Taking sum over j and using Schwarz inequality, we obtain

$$\begin{aligned} \|T_\sigma(f,g)\|_{L^1} &\le \sum_j \|T_{\sigma_j}(f_j,g_j)\|_{L^1} \lesssim \sum_j \|f_j\|_{L^2}\|g_j\|_{L^2} \\ &\le \Big(\sum_j \|f_j\|_{L^2}^2 \Big)^{1/2} \Big(\sum_j \|g_j\|_{L^2}^2 \Big)^{1/2} \lesssim \|f\|_{L^2}\|g\|_{L^2}, \end{aligned}$$

which is the desired estimate.

Next, consider $\sigma \in BS^{-n/2}_{0,0}$ such that

$$\operatorname{supp} \sigma \subset \{|\xi| \le |\eta|/2\} \cap \{|\xi| + |\eta| \ge 2\}. \tag{3.12}$$

We again decompose σ as in (3.5). In the present case, (3.6) no longer holds; instead, we have

$$T_{\sigma_j}(f,g) = T_{\sigma_j}(f,g_j), \qquad g_j = \psi(2^{-j}D)g,$$

with some $\psi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$. Here, to simplify the argument, we assume that $\sigma(x,\xi,\eta)$ does not depend on x, thus $\sigma(x,\xi,\eta) = \sigma(\xi,\eta)$. From (3.12), it follows that $|\xi+\eta| \approx 2^j$ on $\operatorname{supp} \sigma_j$, which implies that the support of the Fourier transform of $T_{\sigma_j}(f,g)$ is included in $\{\zeta \mid B^{-1}2^j \le |\zeta| \le B2^j\}$ for some $B > 1$ (here we used the assumption $\sigma(x,\xi,\eta) = \sigma(\xi,\eta)$).

We use the following lemma, which is well known in the Littlewood-Paley theory for Hardy spaces.

Lemma 3.11. *Suppose $\{F_j\} \subset \mathcal{S}'$ and suppose there exists $B > 1$ such that $\operatorname{supp} \widehat{F_j} \subset \{\zeta \in \mathbb{R}^n : B^{-1}2^j \le |\zeta| \le B2^j\}$. Then*

$$\Big\| \sum_j F_j \Big\|_{H^1} \lesssim \Big\| \Big(\sum_j |F_j|^2 \Big)^{1/2} \Big\|_{L^1}.$$

By this lemma, we have

$$\begin{aligned} \|T_\sigma(f,g)\|_{L^1} &= \Big\| \sum_{j=1}^\infty T_{\sigma_j}(f,g_j) \Big\|_{L^1} \lesssim \Big\| \sum_{j=1}^\infty T_{\sigma_j}(f,g_j) \Big\|_{H^1} \\ &\lesssim \Big\| \Big(\sum_{j=1}^\infty |T_{\sigma_j}(f,g_j)|^2 \Big)^{1/2} \Big\|_{L^1}. \end{aligned}$$

With the aid of this inequality, the argument is again reduced to the estimate of L^2 norms of $T_{\sigma_j}(f,g_j)$; we omit the rest of the argument. The idea of using Littlewood-Paley theorem to reduce the estimate to L^2 norms of functions goes back to Tomita [29].

To prove the estimates of T_m in $h^p \times bmo \to h^p$ and in $h^p \times L^2 \to h^r$, we use the basic result of atomic decomposition in h^p.

Definition 3.12. For $0 < p \le 1$, a function a on $\mathbb{R}^n$ is called an *h^p-atom of first kind* if there exists a cube $Q = Q_a$ with $|Q| \le 1$ such that

$$\operatorname{supp} a \subset Q, \quad \|a\|_{L^\infty} \le |Q|^{-1/p}, \tag{3.13}$$

and

$$\int_{\mathbb{R}^n} x^\alpha \, a(x)\, dx = 0, \qquad |\alpha| \le [n/p - n],$$

where $[n/p-n]$ denotes the integer part of $n/p-n$. A function a on $\mathbb{R}^n$ is called an *h^p-atom of second kind* if there exists a cube $Q = Q_a$ with $|Q| = 1$ satisfying (3.13). Both kind of atoms are simply called *h^p-atoms*.

Lemma 3.13 ([14, Lemma 5]). *Let $0 < p \le 1$ and $f \in \mathcal{S}'(\mathbb{R}^n)$. Then $f \in h^p(\mathbb{R}^n)$ if and only if f can be written as $f = \sum_{i=1}^{\infty} \lambda_i a_i$ with $\{a_i\}$ a sequence of h^p-atoms, $\{\lambda_i\}$ a sequence of non-negative real numbers such that $\sum_{i=1}^{\infty} \lambda_i^p < \infty$, and the series $\sum_{i=1}^{\infty} \lambda_i a_i$ converging in $\mathcal{S}'(\mathbb{R}^n)$. Furthermore,*

$$\|f\|_{h^p} \approx \inf \left(\sum_{i=1}^{\infty} |\lambda_i|^p \right)^{1/p},$$

where the infimum is taken over all representations of f.

(2) $\sigma \in BS_{0,0}^{-n/p}$, $0 < p < 1 \Rightarrow T_\sigma : h^p \times bmo \to h^p$.

By virtue of Lemma 3.13, the boundedness $T_\sigma : h^p \times bmo \to h^p$ follows if we prove the estimate

$$\|T_\sigma(a, g)\|_{h^p} \lesssim \|g\|_{bmo}$$

for all h^p-atoms a. For sufficiently larger M, we have

$$\|F\|_{h^p} \lesssim \|(1 + |\cdot|)^M F\|_{L^2}.$$

Thus it is sufficient to prove the weighted L^2-estimate

$$\|(1 + |\cdot|)^M T_\sigma(a, g)\|_{L^2} \lesssim \|g\|_{bmo}.$$

This estimate can be proved, with the aid of the basic estimate (1′), in almost parallel way as in the linear case given in [23]; we omit the details.

(3) $\sigma \in BS_{0,0}^{-n/p}$, $0 < p < 1 \Rightarrow T_\sigma : h^p \times L^2 \to h^r$, $1/p + 1/2 = 1/r$.

Here again, instead of $h^p \times L^2 \to h^r$, we shall sketch the proof of the estimate of $h^p \times L^2 \to L^r$; to replace L^r by h^r requires only a slight modification.

To prove the boundedness $T_\sigma : h^p \times L^2 \to L^r$, we prove the following: for any h^p-atom a, there exists a function u such that

$$|T_\sigma(a, g)(x)| \lesssim u(x) \left(\int \frac{|g(y)|^2}{(1 + |x - y|)^{2M}} dy \right)^{1/2} = u(x) \widetilde{g}(x) \tag{3.14}$$

and $\|u\|_{L^p} \lesssim 1$. From this we can derive the desired estimate in the following way. We decompose $f \in h^p$ as $f = \sum_{j=0}^{\infty} \lambda_j a_j$, where a_j are h^p-atoms, λ_j are non-negative real numbers, and $\sum_{j=0}^{\infty} \lambda_j^p \approx \|f\|_{h^p}^p$. We take the function u_j corresponding to a_j as mentioned above. Then

$$|T_\sigma(f, g)](x)| \le \sum_j \lambda_j |T_\sigma(a_j, g)| \lesssim \sum_j \lambda_j u_j(x) \widetilde{g}(x).$$

Hölder's inequality gives

$$\|T_\sigma(f, g)\|_{L^r} \lesssim \left\| \sum_j \lambda_j u_j \right\|_{L^p} \|\widetilde{g}\|_{L^2}.$$

We have $\|\widetilde{g}\|_{L^2} = c\|g\|_{L^2}$ and

$$\left\| \sum_j \lambda_j u_j \right\|_{L^p} \le \left(\sum_j \lambda_j^p \|u_j\|_{L^p}^p \right)^{1/p} \lesssim \left(\sum_j \lambda_j^p \right)^{1/p} \approx \|f\|_{h^p}.$$

Thus $\|T_\sigma(f,g)\|_{L^r} \lesssim \|f\|_{h^p}\|g\|_{L^2}$ as desired.

Here we omit the proof of (3.14), which is similar to the argument given in [25, Section 4]. Notice that the L^r-norm-estimate

$$\|T_\sigma(a,g)(x)\|_{L^r} \lesssim \|g\|_{L^2}$$

for h^p-atoms a is not sufficient to get the boundedness $T_\sigma : h^p \times L^2 \to L^r$. In fact, in the atomic decomposition of f, we only have control of $\sum_{j=0}^{\infty} \lambda_j^p \approx \|f\|_{h^p}^p$, and we can not estimate

$$\left\| \sum_{j=0}^{\infty} \lambda_j T_\sigma(a_j,g) \right\|_{L^r}^r \le \sum_{j=0}^{\infty} \left\| \lambda_j T_\sigma(a_j,g) \right\|_{L^r}^r \lesssim \sum_{j=0}^{\infty} \lambda_j^r \|g\|_{L^2}^r$$

for $r < p$.

References

[1] Á. Bényi, F. Bernicot, D. Maldonado, V. Naibo, and R. Torres, On the Hörmander classes of bilinear pseudodifferential operators II, preprint.

[2] Á. Bényi, D. Maldonado, V. Naibo, and R. Torres, On the Hörmander classes of bilinear pseudodifferential operators, Integral Equations Operator Theory 67 (2010), 341–364.

[3] Á. Bényi and R. Torres, Symbolic calculus and the transpose of bilinear pseudodifferential operators, Comm. PDE **28** (2003), 1161–1181.

[4] Á. Bényi and R. Torres, Almost orthogonality and a class of bounded bilinear pseudodifferential operators, Math. Res. Lett. 11 (2004), 1–12.

[5] A. P. Calderón and A. Torchinsky, Parabolic maximal functions associated with a distribution, Adv. in Math. 16 (1975), 1–64.

[6] A. P. Calderón and A. Torchinsky, Parabolic maximal functions associated with a distribution, II, Adv. in Math. 24 (1977), 101–171.

[7] A. P. Calderón and R. Vaillancourt, A class of bounded pseudo-differential operators, Proc. Nat. Acad. Sci. U.S.A. 69 (1972), 1185–1187.

[8] R. R. Coifman and Y. Meyer, On commutators of singular integrals and bilinear singular integrals, Trans. Amer. Math. Soc. **212** (1975), 315–331.

[9] R. R. Coifman and Y. Meyer, Au delà des opérateurs pseudo-différentiels, Astérisque **57** (1978), 1–185.

[10] R. R. Coifman and Y. Meyer, Commutateurs d'intégrales singulières et opérateurs multilinéaires, Ann. Inst. Fourier (Grenoble) **28** (1978), 177–202.

[11] C. Fefferman and E. M. Stein, H^p spaces of several variables, Acta Math. **129** (1972), 137–193.

[12] M. Fujita and N. Tomita, Weighted norm inequalities for multilinear Fourier multipliers, Trans. Amer. Math. Soc. 364 (2012), 6335–6353.

[13] D. Goldberg, A local version of Hardy spaces, Duke Math. J. 46 (1979), 27–42.

[14] L. Grafakos, Multilinear operators in harmonic analysis and partial differential equations, Harmonic Analysis and Nonlinear Partial Differential Equations, RIMS Kôkyûroku Bessatsu B33 (2012), 11–27.

[15] L. Grafakos and N. Kalton, Multilinear Calderón-Zygmund operators on Hardy spaces, Collect. Math. **52** (2001), 169–179.

[16] L. Grafakos, A. Miyachi, and N. Tomita, On multilinear Fourier multipliers of limited smoothness, to appear in Canadian J. Math.

[17] L. Grafakos and Z. Si, The Hörmander multiplier theorem for multilinear operators, J. Reine Angew. Math. 668 (2012), 133–147.

[18] L. Grafakos and R. Torres, Multilinear Calderón-Zygmund theory, Adv. Math. 165 (2002), 124–164.

[19] L. Hörmander, Estimates for translation invariant operators in L^p spaces, Acta Math. **104** (1960), 93–140.

[20] T. Kato and G. Ponce, Commutator estimates and the Euler and Navier-Stokes equations, Comm. Pure Appl. Math. 41 (1988), 891–907.

[21] C. Kenig and E. M. Stein, Multilinear estimates and fractional integrals, Math. Res. Lett. **6** (1999), 1–15.

[22] N. Michalowski, D. Rule, and W. Staubach, Multilinear pseudodifferential operators beyond Calderón-Zygmund operators, preprint.

[23] A. Miyachi, On some Fourier multipliers for $H^p(\mathbb{R}^n)$, J. Fac. Sci., Univ. Tokyo, Sec. IA, 27 (1980), 157–179.

[24] A. Miyachi, Estimates for pseudo-differential operators of class $S_{0,0}$, Math. Nachr. 133 (1987), 135–154.

[25] A. Miyachi and N. Tomita, Minimal smoothness conditions for bilinear Fourier multipliers, to appear in Rev. Mat. Iberoam.

[26] A. Miyachi and N. Tomita, Calderón-Vaillancourt type theorem for bilinear pseudo-differential operators, to appear in Indiana Univ. Math. J.

[27] C. Muscalu, Calderón commutators and the Cauchy integral on Lipschitz curves revisited I. First commutator and generalizations/ II. Cauchy integral and generalizations/ III. Polydisc extensions, arXiv:1201.3845v1 [math.CA], 18 Jan 2012/ arXiv:1201.3850v1 [math.CA], 18 Jan 2012/ arXiv:1201.3855v1 [math.CA], 18 Jan 2012.

[28] E. M. Stein, Harmonic Analysis, Real Variable Methods, Orthogonality, and Oscillatory Integrals, Princeton University Press, Princeton, 1993.

[29] N. Tomita, A Hörmander type multiplier theorem for multilinear operators, J. Funct. Anal. 259 (2010), 2028–2044.

[30] A. Zygmund, Trigonometric Series, Vol. I, Second edition, Cambridge Univ. Press, Cambridge, 1959.

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