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# On the Dirichlet problem of the biharmonic equation for the half-space

By

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**Abstract**

In this paper, we solve the first boundary value problem (Dirichlet problem) of the biharmonic equation for the half-space with respect to slowly growing and regular boundary functions. The relation between a particular solution and certain general solutions of this problem is discussed.

## § 1. Introduction

Let $n$ be a positive integer satisfying $n \geq 2$. Let $\mathbf{R}^{n+1}$ be the $(n+1)$-dimensional Euclidean space. A point in $\mathbf{R}^{n+1}$ is represented by

$$M = (X, y) = (x_1, \ldots, x_n, y)$$

with

$$|M| = (x_1^2 + \ldots + x_n^2 + y^2)^{\frac{1}{2}}.$$

By $\partial E$ and $\bar{E}$ we denote the boundary and the closure of a subset $E$ of $\mathbf{R}^{n+1}$, respectively. By $\mathbf{T}_{n+1}$ we denote the open half-space

$$\{M = (X, y) \in \mathbf{R}^{n+1} : y > 0\}.$$

Then $\partial\mathbf{T}_{n+1}$ is identified with $\mathbf{R}^n$ and the $n$-dimensional Lebesgue measure at $N \in \partial\mathbf{T}_{n+1}$ is denoted by $dN$. The sphere of radius $r$ centered at the origin of $\mathbf{R}^{n+1}$ is

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represented by $S_r^{n+1}$. When $g$ is a function defined on

$$\sigma_{n+1}(r) = \mathbf{T}_{n+1} \cap S_r^{n+1} \quad (r > 0),$$

we define the mean of $g$ as follows:

$$\mathcal{M}(g; r) = 2(s_{n+1} r^n)^{-1} \int_{\sigma_{n+1}(r)} g(M) d\sigma_M \quad (r > 0),$$

where $s_{n+1}$ is the surface area of $S_1^{n+1}$ (the $(n+1)$-dimensional unit sphere $\mathbf{S}^n$) and $d\sigma_M$ is the surface element on $S_r^{n+1}$ at $M \in \sigma_{n+1}(r)$.

Let the $(n+1)$-dimensional Laplace operator be defined by

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y^2},$$

and let the $p$th iterated Laplacian operator be defined by $\Delta^p = \Delta(\Delta^{p-1})$, $\Delta^0 = 1$, $(p = 1, 2, \cdots)$.

Let $m$ be a positive integer. For a subset $\Omega$ of $\mathbf{R}^{n+1}$, $C^m(\Omega)$ stands for the space of $m$-times continuously differentiable functions on $\Omega$ and $C(\Omega)$ denote the space of continuous functions on $\Omega$. A function $w \in C^4(\mathbf{T}_{n+1})$ is biharmonic in $\mathbf{T}_{n+1}$ if $\Delta^2 w \equiv 0$ in $\mathbf{T}_{n+1}$.

Let $f_0$ and $f_1$ be two functions defined on $\partial\mathbf{T}_{n+1}$. A solution of the first boundary value problem of the biharmonic equation for $\mathbf{T}_{n+1}$ with respect to $f_0$ and $f_1$ is a biharmonic function $w$ in $\mathbf{T}_{n+1}$ such that

$$\lim_{M \to N,\ M \in \mathbf{T}_{n+1}} w(M) = f_0(N), \qquad \lim_{M \to N,\ M \in \mathbf{T}_{n+1}} \frac{\partial w}{\partial y}(M) = f_1(N) \tag{1.1}$$

for every point $N \in \partial\mathbf{T}_{n+1}$.

Schot [8] gave a particular solution of the first boundary value problem of the biharmonic equation for $\mathbf{T}_{n+1}$ by using an iteratively defined system of two Dirichlet problems. Let $f_0$ and $f_1$ be given two functions which are sufficiently regular and bounded on $\partial\mathbf{T}_{n+1}$. Then

$$u(M) = \frac{2y^2}{s_{n+1}} \int_{\mathbf{R}^n} \left\{ \frac{(n+1)y}{|M-N|^{n+3}} f_0(N) - \frac{1}{|M-N|^{n+1}} f_1(N) \right\} dN \tag{1.2}$$

$(M = (X, y) \in \mathbf{T}_{n+1})$ is a solution of the first boundary value problem of the biharmonic equation with respect to $f_0$ and $f_1$ for $\mathbf{T}_{n+1}$.

Let $F_{l,n+1}$ $(l \geq 1)$ be the set of continuous functions $f(N)$ on $\partial\mathbf{T}_{n+1} = \mathbf{R}^n$ such that

$$\int_{\mathbf{R}^n} \frac{|f(N)|}{1 + |N|^{n+l+1}} dN < \infty. \tag{1.3}$$

We say that $h$ is a solution of the (classical) Dirichlet problem for $\mathbf{T}_{n+1}$ with respect to a continuous function $f$ on $\partial\mathbf{T}_{n+1}$, if $h$ is harmonic in $\mathbf{T}_{n+1}$ and

$$\lim_{M\to N,\ M\in\mathbf{T}_{n+1}} h(M) = f(N)$$

for every $N \in \partial\mathbf{T}_{n+1}$

With respect to the Dirichlet problem for the half-space $\mathbf{T}_{n+1}$, Yoshida [10] used the generalized Poisson integral $H_{l,n+1}f(M)$ with $f \in F_{l,n+1}$ and gave the following results.

If $f \in F_{l,n+1}$, then $H_{l,n+1}f(M)$ is a solution of the Dirichlet problem for $\mathbf{T}_{n+1}$ with $f$ satisfying

$$\mathcal{M}(y|H_{l,n+1}f|;r) = o(r^{l+2}) \quad (r\to\infty) \tag{1.4}$$

([10, Theorem 1]). If $h(M)$ is a solution of the Dirichlet problem for $\mathbf{T}_{n+1}$ with $f \in F_{l,n+1}$ $(l \geq 1)$ such that

$$\mathcal{M}(yh^{+};r) = o(r^{l+2}) \quad (r\to\infty),$$

then

$$h(M) = H_{l,n+1}f(M) + y\Pi(h)(M) \tag{1.5}$$

for every $M \in \mathbf{T}_{n+1}$, where $\Pi(h)(M)$ is a polynomial of $M = (x_1, x_2, \ldots, x_n, y) \in \mathbf{R}^{n+1}$ of degree at most $l-1$ and even with respect to the variable $y$ ([10, Theorem 2]).

In this paper, from these results, we shall show that a particular solution of the first boundary value problem of the biharmonic equation for $\mathbf{T}_{n+1}$ with respect to slowly growing regular boundary functions $f$ will be constructed by the generalized Poisson integrals $H_{l,n+1}f$ in (1.4) and generalize the result of (1.2) (Theorem 2.1). We shall also give the corresponding result to (1.5) (Theorem 2.2).

## § 2. Statements of results

In this section we use the following notations

$$B_r^m(Q) = \{P \in \mathbf{R}^m : |P-Q| < r\} \quad (Q\in\mathbf{R}^m, r>0)$$

and

$$B_r^m = \{P \in \mathbf{R}^m : |P| < r\} \quad (r>0).$$

Let $M$ and $N$ be two points in $\mathbf{T}_{n+1}$ and $\partial\mathbf{T}_{n+1}$, respectively. By $< M, N >$ we denote the usual inner product in $\mathbf{R}^{n+1}$. We note

$$|M-N|^{-n-1} = \sum_{k=0}^{\infty} c_{k,n+3}|N|^{-k-n-1}|M|^{k}L_{k,n+3}(\rho) \quad (|M|<|N|),$$

where

$$\rho = \frac{< M, N >}{|M||N|}, \qquad c_{k,n+3} = \binom{k+n}{k} \tag{2.1}$$

and $L_{k,n+3}$ is the $(n+3)$-dimensional Legendre polynomial of degree $k$. We remark $L_{k,n+3}(1) = 1$, $L_{k,n+3}(-1) = (-1)^k$, $L_{0,n+3} = 1$ and $L_{1,n+3}(t) = t$ (see Armitage [2, p. 55]).

Let $l$ be a non-negative integer. We set

$$V_{l,n+1}(M, N) = \begin{cases} \dfrac{2y}{s_{n+1}} \displaystyle\sum_{k=0}^{l-1} c_{k,n+3} |N|^{-k-n-1} |M|^k L_{k,n+3}(\rho) & (|N| \geq 1, \quad l \geq 1), \\ 0 & (|N| < 1, \quad l \geq 1), \\ 0 & (l = 0), \end{cases}$$

for any $M \in \mathbf{T}_{n+1}$ and any $N \in \partial \mathbf{T}_{n+1}$. The generalized *Poisson kernel* $K_{l,n+1}(M, N)$ $(M \in \mathbf{T}_{n+1}, N \in \partial \mathbf{T}_{n+1})$ is defined by

$$K_{l,n+1}(M, N) = K_{0,n+1}(M, N) - V_{l,n+1}(M, N) \quad (l \geq 0),$$

where

$$K_{0,n+1}(M, N) = \frac{2y}{s_{n+1}} |M - N|^{-n-1}$$

(Siegel [9, p.7] and Armitage [2, p.56]).

Let $l$ be a non-negative integer. By $G_{l,n+1}$ we denote the set of locally integrable functions $f$ on $\partial \mathbf{T}_{n+1} = \mathbf{R}^n$, if there exists $r_0 \geq 1$ satisfying

$$\int_{\mathbf{R}^n \setminus B^n_{r_0}} \frac{|f(N)|}{|N|^{n+l+1}} dN < \infty. \tag{2.2}$$

Let $l$ be a non-negative integer and $f \in G_{l,n+1}$. Then we shall see that the generalized Poisson integral $H_{l,n+1} f(M)$ with $f$;

$$H_{l,n+1} f(M) = \int_{\partial \mathbf{T}_{n+1}} K_{l,n+1}(M, N) f(N) dN$$

is a harmonic function on $\mathbf{T}_{n+1}$ ([10, Lemma 2]).

**Theorem 2.1.** *Let $l$ $(l \geq 1)$ be an integer and $f_0 \in G_{l,n+1}$, $f_1 \in G_{l-1,n+1}$. Then*

$$W_{l,n+1}(f_0, f_1)(M) = H_{l,n+1} f_0(M) - y \frac{\partial}{\partial y} H_{l,n+1} f_0(M) + y H_{l-1,n+1} f_1(M)$$

*$(M = (X, y) \in \mathbf{T}_{n+1})$ is biharmonic on $\mathbf{T}_{n+1}$ satisfying*

$$\mathcal{M}(y |W_{l,n+1}(f_0, f_1)|; r) = O(r^{l+2}) \quad (r \to \infty), \tag{2.3}$$

$$\mathcal{M}(y^2 |\frac{\partial}{\partial y} W_{l,n+1}(f_0, f_1)|; r) = O(r^{l+2}) \quad (r \to \infty), \tag{2.4}$$

$$\mathcal{M}(y^3 |\Delta W_{l,n+1}(f_0, f_1)|; r) = O(r^{l+2}) \quad (r \to \infty). \tag{2.5}$$

*Furthermore, if* $f_0 \in G_{l,n+1} \cap C^2(\partial \mathbf{T}_{n+1})$ *and* $f_1 \in G_{l-1,n+1} \cap C(\partial \mathbf{T}_{n+1})$, *then* $W_{l,n+1}(f_0, f_1)(M)$ *is a solution of the first boundary value problem of the biharmonic equation for* $\mathbf{T}_{n+1}$ *with respect to* $f_0$ *and* $f_1$.

**Remark 2.1.** Applying Theorem 2.1 to $f_0 \in G_{1,n+1} \cap C^2(\partial \mathbf{T}_{n+1})$ and $-f_1 \in G_{0,n+1} \cap C(\partial \mathbf{T}_{n+1})$, we obtain following particular solution $W_{1,n+1}(f_0, -f_1)(M)$;

$$\begin{aligned} W_{1,n+1}(f_0, -f_1)(M) &= H_{1,n+1} f_0(M) - y \frac{\partial}{\partial y} H_{1,n+1} f_0(M) - y H_{0,n+1} f_1(M) \\ &= H_{0,n+1} f_0(M) - y \frac{\partial}{\partial y} H_{0,n+1} f_0(M) - y H_{0,n+1} f_1(M) \\ &= \frac{2y^2}{s_{n+1}} \int_{\mathbf{R}^n} \left\{ \frac{(n+1)y}{|M-N|^{n+3}} f_0(N) - \frac{1}{|M-N|^{n+1}} f_1(N) \right\} dN. \end{aligned}$$

This particular solution is similar to (1.2), and these boundary functions are not necessary bounded. So we generalize the result of Schot.

The next result concerns a type of uniqueness of solution of the first boundary value problem of the biharmonic equation for $\mathbf{T}_{n+1}$.

**Theorem 2.2.** *Let* $l$ $(l \geq 3)$ *be an integer and let* $f_0 \in G_{l,n+1} \cap C^2(\partial \mathbf{T}_{n+1})$ *and* $f_1 \in G_{l-1,n+1} \cap C(\partial \mathbf{T}_{n+1})$. *If* $w$ *is a solution of the first boundary value problem of the biharmonic equation for* $\mathbf{T}_{n+1}$ *with respect to* $f_0$ *and* $f_1$ *satisfying*

$$\mathcal{M}(yw^+; r) = O(r^{l+2}) \quad (r \to \infty), \tag{2.6}$$

$$\mathcal{M}\left(y^2 \left(\frac{\partial w}{\partial y}\right)^-; r\right) = O(r^{l+2}) \quad (r \to \infty), \tag{2.7}$$

$$\mathcal{M}(y^3 (\Delta w)^+; r) = O(r^{l+2}) \quad (r \to \infty), \tag{2.8}$$

*then*

$$w(X,y) = W_{l,n+1}(f_0, f_1)(X,y) + y^2 \sum_{j=0}^{[\frac{l}{2}]} \alpha_j y^{2j} \Delta^j P_{l-1}(X) + y^3 \sum_{j=0}^{[\frac{l}{2}]-1} \beta_j y^{2j} \Delta^j P_{l-2}(X)$$

*for every* $(X,y) \in \mathbf{T}_{n+1}$, *where* $P_k(X)$ *is a polynomial of* $X$ *of degree less than* $k+1$ $(k \in \{l-1,\ l-2\})$ *and*

$$\alpha_j = \begin{cases} (-1)^j \frac{2!(j+1)}{(2j+2)!} & (j = 0, 1, 2, \cdots, [\frac{l}{2}]), \\ 0 & (j = [\frac{l}{2}],\ l \text{ is even}), \end{cases}$$

$$\beta_j = (-1)^j \frac{3!(j+1)}{(2j+3)!} \quad (j = 0, 1, 2, \cdots, [\frac{l}{2}] - 1).$$

## § 3. Preliminaries

We require some preliminary results.

Let $k$ ($k \geq 1$) and $m$ ($\geq 0$) be two integers. We know that

$$\frac{d}{dt}L_{k,m}(t) = \alpha_{k,m}L_{k-1,m+2}(t), \tag{3.1}$$

where

$$\alpha_{k,m} = \frac{k(k+m-2)}{m-1}$$

(Müller [7, Lemma 13]). We also know that

$$|L_{k,m}(\rho)| \leq 1 \tag{3.2}$$

for any $\rho$ in (2.1), any non-negative integer $k$ and any positive integer $m \geq 2$ (see Armitage [2, Theorems C and D]).

**Lemma 3.1.** *Let $k$ be a positive integer. Let $M \in \mathbf{R}^{n+1}$ and $\rho$ in (2.1). Then*

$$\left|\frac{\partial}{\partial y}L_{k,n+3}(\rho)\right| \leq \alpha_{k,n+3}|M|^{-1}\ (k \geq 1), \tag{3.3}$$

$$\left|\frac{\partial^2}{\partial y^2}L_{k,n+3}(\rho)\right| \leq \alpha_{k,n+3}(4+\alpha_{k-1,n+5})|M|^{-2}\ (k \geq 2). \tag{3.4}$$

*Proof.* From (3.1) we have

$$\begin{aligned}
\frac{\partial}{\partial y}L_{k,n+3}(\rho) &= -\alpha_{k,n+3}y|M|^{-2}\rho L_{k-1,n+5}(\rho) \qquad (k \geq 1),\\
\frac{\partial^2}{\partial y^2}L_{k,n+3}(\rho) &= -\alpha_{k,n+3}|M|^{-2}\rho L_{k-1,n+5}(\rho)\\
&\qquad +3\alpha_{k,n+3}y^2|M|^{-4}\rho L_{k-1,n+5}(\rho)\\
&\qquad +\alpha_{k,n+3}\alpha_{k-1,n+5}y^2|M|^{-4}\rho^2 L_{k-2,n+7}(\rho) \qquad (k \geq 2).
\end{aligned}$$

Hence from (3.2) we obtain (3.3) and (3.4). □

**Lemma 3.2.** *Let $l$ be a non-negative integer. For any $M \in \mathbf{T}_{n+1}$ and any $N \in \partial\mathbf{T}_{n+1}$ satisfying $2|M| \geq |N|$ and $|N| \geq 1$, we have*

$$y|V_{l,n+1}(M,N)| \leq a_0|N|^{-l-n}|M|^{l+1}, \tag{3.5}$$

$$y^2\left|\frac{\partial}{\partial y}V_{l,n+1}(M,N)\right| \leq a_1|N|^{-l-n}|M|^{l+1}, \tag{3.6}$$

$$y^3\left|\frac{\partial^2}{\partial y^2}V_{l,n+1}(M,N)\right| \leq a_2|N|^{-l-n}|M|^{l+1}, \tag{3.7}$$

*where $a_j$ ($j = 0,1,2$) are constants depending only on $l$ and $n+1$.*

*Proof.* Here we shall prove (3.6) only. Take any $M \in \mathbf{T}_{n+1}$ and any $N \in \partial\mathbf{T}_{n+1}$ satisfying $2|M| \geq |N|$ and $|N| \geq 1$. Then from (3.2) and (3.3) we have

$$\begin{aligned}
\left|\frac{\partial}{\partial y}V_{l,n+1}(M,N)\right| &\leq \frac{2}{s_{n+1}}\sum_{k=0}^{l-1} c_{k,n+3}|N|^{-k-n-1}|M|^k|L_{k,n+3}(\rho)| \\
&\quad + \frac{2y^2}{s_{n+1}}\sum_{k=0}^{l-1} c_{k,n+3}k|N|^{-k-n-1}|M|^{k-2}|L_{k,n+3}(\rho)| \\
&\quad + \frac{2y}{s_{n+1}}\sum_{k=0}^{l-1} c_{k,n+3}|N|^{-k-n-1}|M|^k\left|\frac{\partial}{\partial y}L_{k,n+3}(\rho)\right| \\
&\leq \frac{2}{s_{n+1}}|N|^{-n-1}\sum_{k=0}^{l-1} c_{k,n+3}2^{-k}\left(\frac{2|M|}{|N|}\right)^k \\
&\quad + \frac{2y^2}{s_{n+1}}|N|^{-n-1}|M|^{-2}\sum_{k=0}^{l-1} c_{k,n+3}k2^{-k}\left(\frac{2|M|}{|N|}\right)^k \\
&\quad + \frac{2y}{s_{n+1}}|N|^{-n-1}|M|^{-1}\sum_{k=0}^{l-1} c_{k,n+3}\alpha_{k,n+3}2^{-k}\left(\frac{2|M|}{|N|}\right)^k \\
&\leq |N|^{-l-n}|M|^{l-1}\frac{2^l}{s_{n+1}}\sum_{k=0}^{l-1} c_{k,n+3}(1+k+\alpha_{k,n+3})2^{-k}.
\end{aligned}$$

If we put

$$a_1 = \frac{2^l}{s_{n+1}}\sum_{k=0}^{l-1} c_{k,n+3}(1+k+\alpha_{k,n+3})2^{-k},$$

then we obtain (3.6).

Similary we can prove (3.5) from (3.2), and (3.7) is proved from (3.2), (3.3) and (3.4). □

**Lemma 3.3.** *Let $l$ be a non-negative integer. For any $M \in \mathbf{T}_{n+1}$ and any $N \in \partial\mathbf{T}_{n+1}$ satisfying $2|M| < |N|$ and $|N| \geq 1$, we have*

$$y|K_{l,n+1}(M,N)| \leq b_0|N|^{-l-n-1}|M|^{l+2}, \tag{3.8}$$

$$y^2\left|\frac{\partial}{\partial y}K_{l,n+1}(M,N)\right| \leq b_1|N|^{-l-n-1}|M|^{l+2}, \tag{3.9}$$

$$y^3\left|\frac{\partial^2}{\partial y^2}K_{l,n+1}(M,N)\right| \leq b_2|N|^{-l-n-1}|M|^{l+2}, \tag{3.10}$$

*where $b_j$ $(j = 0, 1, 2)$ are constants depending only on $l$ and $n+1$.*

*Proof.* Take any $M \in \mathbf{T}_{n+1}$ and any $N \in \partial\mathbf{T}_{n+1}$ satisfying $2|M| < |N|$ and

$|N| \geq 1$. Then we note that

$$K_{l,n+1}(M,N) = \frac{2y}{s_{n+1}} \sum_{k=l}^{\infty} c_{k,n+3} |N|^{-k-n-1} |M|^k L_{k,n+3}(\rho).$$

By a similar argument of the proof of Lemma 3.2, put

$$b_1 = \frac{2^{l+1}}{s_{n+1}} \sum_{k=l}^{\infty} c_{k,n+3}(1 + k + \alpha_{k,n+3}) 2^{-k}.$$

Then we easily see that $b_1$ is finite and so we obtain (3.9). Similarly we obtain (3.8) and (3.10). □

## § 4. Proof of Theorem 2.1

Let $l$ be a non-negative integer. For a function $f \in G_{l,n+1}$ we define

$$\psi(r) = \int_{B_{2r}^n \setminus B_1^n} \frac{|f(N)|}{|N|^{l+n}} dN \qquad (r > 0).$$

We know that

(4.1) $$\psi(r) = o(r) \qquad (r \to \infty)$$

(see Yoshida [10, p.606]).

**Lemma 4.1.** *Let $l$ be a non-negative integer and $f \in G_{l,n+1}$. Then*

(4.2) $$\mathcal{M}(y|H_{l,n+1}f|; r) = O(r^{l+2}) \qquad (r \to \infty),$$

(4.3) $$\mathcal{M}(y^2|\frac{\partial}{\partial y} H_{l,n+1}f|; r) = O(r^{l+2}) \qquad (r \to \infty),$$

(4.4) $$\mathcal{M}(y^3|\frac{\partial^2}{\partial y^2} H_{l,n+1}f|; r) = O(r^{l+2}) \qquad (r \to \infty).$$

*Proof.* Here we shall prove (4.3) only. Suppose that $2r > 1$. We see first that

$$\mathcal{M}(y^2|\frac{\partial}{\partial y} H_{l,n+1}f|; r) \leq I_1 + I_2 + I_3,$$

where

$$I_1 = \frac{2}{s_{n+1} r^n} \int_{\sigma_{n+1}(r)} \left\{ \int_{\mathbf{R}^n \setminus B_{2r}^n} |f(N)| y^2 \left| \frac{\partial}{\partial y} K_{l,n+1}(M,N) \right| dN \right\} d\sigma_M,$$

$$I_2 = \frac{2}{s_{n+1} r^n} \int_{\sigma_{n+1}(r)} \left\{ \int_{B_{2r}^n} |f(N)| y^2 \left| \frac{\partial}{\partial y} K_{0,n+1}(M,N) \right| dN \right\} d\sigma_M,$$

$$I_3 = \frac{2}{s_{n+1}r^n}\int_{\sigma_{n+1}(r)}\left\{\int_{B_{2r}^n\setminus B_1^n}|f(N)|y^2\left|\frac{\partial}{\partial y}V_{l,n+1}(M,N)\right|dN\right\}d\sigma_M.$$

From Lemma 3.3 we have

$$I_1 \le b_1 r^{l+2}\int_{\mathbf{R}^n\setminus B_{2r}^n}\frac{|f(N)|}{|N|^{l+n+1}}dN.$$

Since $f \in G_{l,n+1}$, there exist $A > 0$ and $r_0$ such that

$$\int_{\mathbf{R}^n\setminus B_{2r}^n}\frac{|f(N)|}{|N|^{l+n+1}}dN < A \qquad (r > r_0).$$

Hence we obtain

$$I_1 = O(r^{l+2}) \qquad (r\to\infty).$$

We remark that

$$\mathcal{M}(yK_{0,n+1}(\cdot,N);r) \le \frac{2}{(n+1)s_{n+1}}r^{-n+1}$$

for any $N \in \mathbf{R}^n$ and any $r > 0$ (see Kuran [6, Lemma 2] and Helms [5, p.109; Example 2]). Let

$$I_2' = \frac{2}{s_{n+1}r^n}\int_{\sigma_{n+1}(r)}\left\{\int_{B_{2r}^n}|f(N)|yK_{0,n+1}(M,N)dN\right\}d\sigma_M.$$

Since

$$\left|y\frac{\partial}{\partial y}K_{0,n+1}(M,N)\right| \le (n+2)K_{0,n+1}(M,N)$$

for any $M \in \mathbf{T}_{n+1}$ and any $N \in \mathbf{R}^n$, it follows that $I_2 \le (n+2)I_2'$. We have

$$\begin{aligned}
I_2' &= \int_{B_{2r}^n}|f(N)|\mathcal{M}(yK_{0,n+1}(\cdot,N);r)dN \\
&\le \frac{2r^{-n+1}}{(n+1)s_{n+1}}\int_{B_{2r}^n}|f(N)|dN \\
&= \frac{2r^{-n+1}}{(n+1)s_{n+1}}\int_{B_{2r}^n\setminus B_1^n}|f(N)|dN + \frac{2r^{-n+1}}{(n+1)s_{n+1}}\int_{B_1^n}|f(N)|dN \\
&= \frac{2^{l+n+1}r^{l+1}}{(n+1)s_{n+1}}\int_{B_{2r}^n\setminus B_1^n}\frac{|f(N)|}{(2r)^{l+n}}dN + \frac{2r^{-n+1}}{(n+1)s_{n+1}}\int_{B_1^n}|f(N)|dN \\
&\le \frac{2^{l+n+1}}{(n+1)s_{n+1}}r^{l+1}\psi(r) + \frac{2r^{-n+1}}{(n+1)s_{n+1}}\int_{B_1^n}|f(N)|dN.
\end{aligned}$$

And so (4.1) gives $I_2' = o(r^{l+2})$ $(r\to\infty)$. Hence $I_2 = o(r^{l+2})$ $(r\to\infty)$.

We see from Lemma 3.2 that

$$I_3 \le a_1 r^{l+1}\psi(r)$$

and hence we have from (4.1) $I_3 = o(r^{l+2}) \quad (r \to \infty)$. Thus we obtain (4.3).
Similary we can prove (4.2) and (4.4). □

**Lemma 4.2** (Gilbarg and Trudinger [4, Theorem4.8]). *Let $\Omega$ be an open subset of $\mathbf{R}^{n+1}$. If $u(M)$ is a harmonic function on $\Omega$, then*

$$\sup_{M\in\Omega} y^k \left|\frac{\partial^k u}{\partial y^k}\right| \le C_{n+1} \sup_{M\in\Omega} |u(M)|, \quad (k=0,1,2),$$

*where $C_{n+1}$ is a constant depending only on $n+1$.*

For any point $N$ in $\partial\mathbf{T}_{n+1}$ and a positive number $\delta$, we set

$$U_{N,\delta} = B_\delta^{n+1}(N) \cap \mathbf{T}_{n+1}, \quad W_{N,\delta} = B_\delta^{n+1}(N) \cap \partial\mathbf{T}_{n+1}.$$

**Lemma 4.3** (Gilbarg and Trudinger [4, Lemma 6.18]). *Let $N$ be any point in $\partial\mathbf{T}_{n+1}$ and let $\delta$ be a positive number. If $\varphi \in C^2(\overline{U_{N,\delta}})$ and $v \in C(\overline{U_{N,\delta}}) \cap C^2(U_{N,\delta})$ is a function satisfying $\Delta v = 0$ in $U_{N,\delta}$, $v = \varphi$ on $W_{N,\delta}$, then $v \in C^2(U_{N,\delta} \cup W_{N,\delta})$.*

*Proof of Theorem 2.1.* We see from Yoshida [10, Lemma 2] that $H_{l,n+1}f_0(M)$ and $H_{l-1,n+1}f_1(M)$ are harmonic on $\mathbf{T}_{n+1}$. And so $H_{l,n+1}f_0(M)$, $y\frac{\partial}{\partial y}H_{l,n+1}f_0(M)$ and $yH_{l-1,n+1}f_1(M)$ are biharmonic on $\mathbf{T}_{n+1}$. Hence $W_{l,n+1}(f_0,f_1)(M)$ is biharmonic on $\mathbf{T}_{n+1}$.

Since we have

$$\begin{aligned}\mathcal{M}(y|W_{l,n+1}(f_0,f_1)|;r) \le\ & \mathcal{M}(y|H_{l,n+1}f_0|;r)\\ & +\mathcal{M}(y^2|\frac{\partial}{\partial y}H_{l,n+1}f_0|;r)\\ & +r\mathcal{M}(y|H_{l-1,n+1}f_1|;r),\end{aligned}$$

$$\begin{aligned}\mathcal{M}(y^2|\frac{\partial}{\partial y}W_{l,n+1}(f_0,f_1)|;r) \le\ & \mathcal{M}(y^3|\frac{\partial^2}{\partial y^2}H_{l,n+1}f_0|;r)\\ & +r\mathcal{M}(y|H_{l-1,n+1}f_1|;r)\\ & +r\mathcal{M}(y^2|\frac{\partial}{\partial y}H_{l-1,n+1}f_1|;r),\end{aligned}$$

$$\begin{aligned}\mathcal{M}(y^3|\Delta W_{l,n+1}(f_0,f_1)|;r) \le\ & 2\mathcal{M}(y^3|\frac{\partial^2}{\partial y^2}H_{l,n+1}f_0|;r)\\ & +2r\mathcal{M}(y^2|\frac{\partial}{\partial y}H_{l-1,n+1}f_1|;r),\end{aligned}$$

(2.3), (2.4) and (2.5) hold in view of Lemma 4.1.

To prove (1.1), it is enough to show that

$$\lim_{M\to N, M\in \mathbf{T}_{n+1}} y\frac{\partial}{\partial y}H_{l,n+1}f_0(M)=0, \tag{4.5}$$

$$\lim_{M\to N, M\in \mathbf{T}_{n+1}} y\frac{\partial}{\partial y}H_{l-1,n+1}f_1(M)=0, \tag{4.6}$$

$$\lim_{M\to N, M\in \mathbf{T}_{n+1}} y\frac{\partial^2}{\partial y^2}H_{l,n+1}f_0(M)=0, \tag{4.7}$$

for every point $N\in\partial\mathbf{T}_{n+1}$.

Let $N$ be any fixed point on $\partial\mathbf{T}_{n+1}$. We put

$$u(M)=H_{l,n+1}f_0(M)-f_0(N)\quad (M\in\mathbf{T}_{n+1}).$$

Let $\varepsilon$ be any positive number. Since $H_{l,n+1}f_0(M)$ is a solution of the Dirichlet problem for $\mathbf{T}_{n+1}$ with $f_0$ ([10, Theorem 1]), we take a positive number $\delta$ such that

$$|u(M)|<\varepsilon\quad (M\in B_\delta^{n+1}(N)\cap\mathbf{T}_{n+1}).$$

By Lemma 4.2, we have

$$\sup_{M\in U_{N,\delta}} y\left|\frac{\partial}{\partial y}H_{l,n+1}f_0(M)\right|\le C_{n+1}\varepsilon.$$

Hence we obtain (4.5). In the similar way to (4.5) we also obtain (4.6).

To prove (4.7), we set

$$\varphi(X,y)=f_0(X)\quad ((X,y)\in\mathbf{T}_{n+1}\cup\partial\mathbf{T}_{n+1}),$$

$$v(X,y)=\begin{cases} H_{l,n+1}f_0(X,y) & ((X,y)\in\mathbf{T}_{n+1}),\\ f_0(X) & ((X,0)\in\partial\mathbf{T}_{n+1}).\end{cases}$$

By Lemma 4.3, we have (4.7). □

## § 5. Proof of Theorem 2.2

If $N\in\partial\mathbf{T}_{n+1}$ and $a$ and $r$ are positive numbers, we put

$$D(N;\ a,\ r)=\{M=(X,y)\in\mathbf{T}_{n+1}:\ y=a,\ |(X,0)-N|<r\}.$$

A function $g:\mathbf{T}_{n+1}\to\mathbf{R}$ will be said to be locally convergent in mean to 0 on $\partial\mathbf{T}_{n+1}$ if for each point $N$ of $\partial\mathbf{T}_{n+1}$ there is a positive number $r$ such that

$$\int_{D(N;\ a,\ r)}|g(X,a)|dX\to 0\quad (a\to 0+).$$

**Remark 5.1.** From Lebesgue's bounded convergence theorem, if $u$ is continuous in $\mathbf{T}_{n+1}$ and has the properties
(i) for each point $(X,0) \in \partial\mathbf{T}_{n+1}$, there exists a positive number $r = r(X)$ such that $u$ is bounded in $\mathbf{T}_{n+1} \cap B_r^{n+1}(X,0)$,
(ii) for each point $(X,0)$ of $\partial\mathbf{T}_{n+1}$,

$$\lim_{y \to 0+} u(X,y) = 0,$$

then $u$ is locally convergent in mean to 0 on $\partial\mathbf{T}_{n+1}$ (Armitage [1, p.44]).

Let $p$ be a positive integer. For a subset $\Omega$ of $\mathbf{T}_{n+1}$, we say that a function $w \in C^{2p}(\Omega)$ is $p$-harmonic in $\Omega$ if $\Delta^p w \equiv 0$ in $\Omega$.

**Lemma 5.1** (Armitage [1, Theorem 5]). *Let $w$ be $p$-harmonic in $\mathbf{T}_{n+1}$. If each function*

$$w,\ \frac{\partial w}{\partial y}, \cdots,\ \frac{\partial^{p-1} w}{\partial y^{p-1}}$$

*is locally convergent in mean to 0 on $\partial\mathbf{T}_{n+1}$, then $w$ has the $p$-harmonic continuation $w^*$*

$$w^*(X,y) = \begin{cases} w(X,y) & (y > 0), \\ 0 & (y = 0), \\ v(X,-y) & (y < 0), \end{cases}$$

*where*

$$v(X,y) = \sum_{k=0}^{p-1} \frac{1}{(k!)^2} (-y)^{k+p} \Delta^k \{y^{k-p} w(X,y)\}.$$

When $g$ is a function on $S_r^{n+1}$, we define

$$\tilde{\mathcal{M}}(g;r) = (s_{n+1} r^n)^{-1} \int_{S_r^{n+1}} g(M) d\sigma_M.$$

*Proof of Theorem 2.2.* We put

$$u = w - W_{l,n+1}(f_0, f_1).$$

For each point $N \in \partial\mathbf{T}_{n+1}$, we have

$$\text{(5.1)} \qquad \lim_{M \in \mathbf{T}_{n+1},\ M \to N} u(M) = 0, \qquad \lim_{M \in \mathbf{T}_{n+1},\ M \to N} \frac{\partial u}{\partial y}(M) = 0.$$

Since $u$ and $\frac{\partial u}{\partial y}$ are both locally convergent in mean to 0 on $\partial\mathbf{T}_{n+1}$ from Remark 5.1, Lemma 5.1 shows that $u$ has a biharmonic continuation. So we have

$$\lim_{M \in \mathbf{T}_{n+1}\ M \to N} y \frac{\partial^2 u}{\partial y^2}(M) = 0 \quad (N \in \partial\mathbf{T}_{n+1}).$$

From

$$\frac{\partial(yu)}{\partial y} = u + y\frac{\partial u}{\partial y}, \quad \frac{\partial^2(yu)}{\partial y^2} = 2\frac{\partial u}{\partial y} + y\frac{\partial^2 u}{\partial y^2},$$

we see that each function $yu$, $\frac{\partial(yu)}{\partial y}$, $\frac{\partial^2(yu)}{\partial y^2}$ is locally convergent in mean to 0 on $\partial\mathbf{T}_{n+1}$. Since $u$ is biharmonic on $\mathbf{T}_{n+1}$, $yu$ is 3-harmonic on $\mathbf{T}_{n+1}$. So Lemma 5.1 shows that $yu$ has a 3-harmonic continuation $g$ to $\mathbf{R}^{n+1}$ such that

$$g(X,y) = \begin{cases} yu(X,y) & (y>0), \\ 0 & (y=0), \\ v(X,-y) & (y<0), \end{cases}$$

where

$$v(X,y) = yu(X,y) - 2y^2\frac{\partial u}{\partial y}(X,y) + y^3\Delta u(X,y).$$

Since

$$\begin{aligned} \tilde{\mathcal{M}}(g^+;r) &= \mathcal{M}(yu^+;r) + \mathcal{M}(v^+;r) \\ &\le 2\mathcal{M}(yu^+;r) + 2\mathcal{M}(y^2\left(\frac{\partial u}{\partial y}\right)^- ;r) + \mathcal{M}(y^3(\Delta u)^+;r), \end{aligned}$$

it follows from (2.3), (2.4) and (2.5) that

$$\liminf_{r\to\infty} \frac{\tilde{\mathcal{M}}(g^+;r)}{r^{l+2}} < \infty.$$

Hence by [3] $g(M)$ is a polynomial of $M = (x_1, x_2, \cdots, x_n, y)$ of degree at most $l+2$, and so $u$ is a polynomial of degree at most $l+1$. We put

$$u(X,y) = \sum_{j=0}^{l+1} P_{l-j+1} y^j,$$

where $P_{l-j+1} = P_{l-j+1}(X)$ is a polynomial of $X = (x_1, x_2, \cdots, x_n)$ of degree at most $l-j+1$. And we write $\Delta_X = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$. From

$$\begin{aligned} \Delta^2 u &= \sum_{j=0}^{l+1} y^j \Delta_X^2 P_{l-j+1} + 2\sum_{j=2}^{l+1} j(j-1)y^{j-2}\Delta_X P_{l-j+1} \\ &\quad + \sum_{j=4}^{l+1} j(j-1)(j-2)(j-3)y^{j-4}P_{l-j+1} \\ &= y^{l-1}\Delta_X^2 P_2 + y^{l-2}\Delta_X^2 P_3 \\ &\quad + \sum_{j=0}^{l-3} y^j \left\{ \Delta_X^2 P_{l-j+1} + 2\frac{(j+2)!}{j!}\Delta_X P_{l-j-1} + \frac{(j+4)!}{j!}P_{l-j-3} \right\} \end{aligned}$$

and the biharmonicity of $u$ on $\mathbf{T}_{n+1}$, we see that

$$\Delta_X^2 P_{l-j+1} + 2\frac{(j+2)!}{j!}\Delta_X P_{l-j-1} + \frac{(j+4)!}{j!}P_{l-j-3} \equiv 0 \tag{5.2}$$

for each $j = 0, 1, \cdots, l-3$.

It follows from (5.1) that $P_{l+1} \equiv 0$ and $P_l \equiv 0$, and so we have

$$\Delta_X^2 P_{l+1}(X) \equiv 0 \quad \text{and} \quad \Delta_X^2 P_l(X) \equiv 0. \tag{5.3}$$

Thus from (5.2) and (5.3) we see that

$$P_{l-(2k+1)} = (-1)^k \frac{2!(k+1)}{(2k+2)!}\Delta_X^k P_{l-1}, \quad P_{l-(2k+2)} = (-1)^k \frac{3!(k+1)}{(2k+3)!}\Delta_X^k P_{l-2}$$

$(k = 0, 1, 2, \cdots)$. If we put

$$\alpha_j = \begin{cases} (-1)^j \frac{2!(j+1)}{(2j+2)!} & (j = 0, 1, 2, \cdots, [\frac{l}{2}]), \\ 0 & (j = [\frac{l}{2}],\ l \text{ is even}), \end{cases}$$

$$\beta_j = (-1)^j \frac{3!(j+1)}{(2j+3)!} \quad (j = 0, 1, 2, \cdots, [\frac{l}{2}] - 1),$$

then we obtain

$$u = y^2 \sum_{j=0}^{[\frac{l}{2}]} \alpha_j y^{2j} \Delta^j P_{l-1}(X) + y^3 \sum_{j=0}^{[\frac{l}{2}]-1} \beta_j y^{2j} \Delta^j P_{l-2}(X).$$

Thus we have the conclusion. □

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