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A characterization of real entire functions by polynomial approximation for exponential weights

By

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Abstract

R. S. Varga proved a characterization of entire functions of finite order in terms of polynomial approximation degree on $[-1,1]$. We will show an L^p version of Verga's result. Using this version, we discuss weighted polynomial approximation on $\mathbb{R}$ for entire functions of finite order.

§ 1. Introduction

Let $n \in \mathbb{N}$ and let $C(I)$ be the set of all real valued continuous functions on I, where $I = [-1,1]$. We define the degree of approximation for $f \in C(I)$ by

$$E_n(f) := \inf_{P \in \mathcal{P}_n} \|f - P\|_{L^\infty(I)},$$

where $\mathcal{P}_n$ denotes the class of all real polynomials of degree not more than n. In [1], S. Bernstein proved that f has an analytic extension to an entire function if and only if $\lim_{n\to\infty} E_n(f)^{1/n} = 0$. Note that the entire function discussed in this paper is a real one, that is, it is an analytic function on whole complex plane $\mathbb{C}$ whose value is real on $\mathbb{R} := (-\infty, \infty)$. R. S. Varga ([9]) considered the rate at which $E_n(f)^{1/n}$ tends to zero, and showed that $f \in C(I)$ satisfies

$$\limsup_{n\to\infty} \frac{n \log n}{\log(1/E_n(f))} = \lambda$$

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if and only if f has an analytic extension to an entire function of order λ. Recall that an entire function f is of order $\lambda = \lambda(f)$ if

$$\limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r} = \lambda,$$

where $M(r,f) := \max_{z\in\mathbb{C},|z|=r} |f(z)|$.

Let $0 < p \le \infty$. For a constant $a > 0$, put $I_a := [-a, a]$. We write $f \in L^p(I_a)$ if

$$\|f\|_{L^p(I_a)} := \left(\int_{-a}^{a} |f(t)|^p dt\right)^{1/p} < \infty$$

for $0 < p < \infty$ and $\|f\|_{L^\infty(I_a)} := \text{ess sup}_{x\in I_a}|f(x)| < \infty$. For $f \in L^p(I_a)$ and $n \in \mathbb{N}$, we set

$$E_{p,n}(f; I_a) := \inf_{P\in\mathcal{P}_n} \|f - P\|_{L^p(I_a)}.$$

Then the following L^p version of Varga's result is established. The proof is based on a reduction to Varga's one.

Theorem 1.1. *Let $0 < p \le \infty$ and $a > 0$. For a real valued function $f \in L^p(I_a)$, we put*

$$\rho^*_{p,a}(f) := \limsup_{n\to\infty} \frac{n \log n}{\log(1/E_{p,n}(f; I_a))}. \tag{1.1}$$

If $\rho^*_{p,a}(f) < \infty$, then f is equal to an entire function of order $\rho^*_{p,a}(f)$ on I_a almost everywhere. Conversely, if f is an entire function of order $\lambda(f) < \infty$, then its restriction to I_a satisfies (1.1) *for $\rho^*_{p,a}(f) = \lambda(f)$.*

We remark that Varga's result is the case $p = \infty$.

We next discuss the above result for approximations on the whole real line $\mathbb{R}$. Let $w(x) = \exp(-Q(x))$ be an exponential weight on $\mathbb{R}$ which belongs to a relevant class $\mathcal{F}(C^2+)$. For $w \in \mathcal{F}(C^2+)$, set $T(x) := xQ'(x)/Q(x)$. If T is bounded, then the weight w is called a Freud-type weight, and if T is unbounded, then w is called an Erdös-type weight. Since we assume that T is quasi-increasing, if w is Erdös-type, then $\lim_{x\to\infty} T(x) = \infty$ holds (for details see Section 3).

Let $0 < p \le \infty$ again. We write $f \in L^p_w(\mathbb{R})$ if

$$\|wf\|_{L^p(\mathbb{R})} := \left(\int_{-\infty}^{\infty} |w(x)f(x)|^p dx\right)^{1/p} < \infty$$

for $0 < p < \infty$ and $\|wf\|_{L^\infty(\mathbb{R})} := \text{ess sup}_{x\in\mathbb{R}}|w(x)f(x)| < \infty$. For $f \in L^p_w(\mathbb{R})$ and $n \in \mathbb{N}$, we set

$$E_{p,n}(f; w) := \inf_{P\in\mathcal{P}_n} \|w \cdot (f - P)\|_{L^p(\mathbb{R})}. \tag{1.2}$$

The following theorem is a main result of this paper.

Theorem 1.2. *Let $w = \exp(-Q) \in \mathcal{F}(C^2+)$ and $0 < p \le \infty$. For a real valued function $f \in L^p_w(\mathbb{R})$, we define*

$$\rho_p(f) := \limsup_{n\to\infty} \frac{n \log n}{\log(1/E_{p,n}(f;w))}. \tag{1.3}$$

Then f is equal to an entire function with finite order λ on $\mathbb{R}$ almost everywhere if and only if $\rho_p(f)$ is finite. Furthermore $\rho_p(f) = 0$ if and only if $\lambda = 0$, and if $\rho_p(f) \neq 0$ then we have

$$\frac{1}{\lambda} - \frac{1}{A} \le \frac{1}{\rho_p(f)} \le \frac{1}{\lambda} - \frac{1}{B}, \tag{1.4}$$

where $T(x) := xQ'(x)/Q(x)$ and

$$A := \liminf_{|x|\to\infty} T(x), \quad B := \limsup_{|x|\to\infty} T(x). \tag{1.5}$$

Especially, if w is Erdös-type then $\lambda = \rho_p(f)$ holds true.

H. N. Mhaskar discussed the above result for a special Frued-type weight $w_\alpha(x) := \exp(-|x|^\alpha)$ $(\alpha \ge 2)$ in [5]. In his case, $T \equiv \alpha$ so that $1/\lambda - 1/\alpha = 1/\rho_p(f)$ holds. Note also that, when w is Erdös-type, then $\lim_{|x|\to\infty} T(x) = A = B = \infty$, so that (1.4) shows $\lambda = \rho_p(f)$.

This paper is organized as follows. We give a proof of Theorem 1.1 in Section 2. The definition of a class $\mathcal{F}(C^2+)$ is given in Section 3. We prepare some lemmas in Section 4. The proof of Theorem 1.2 is given in Section 5. We point out that basic and essential facts of the weighted polynomial approximation on $\mathbb{R}$ are proved by using logarithmic potential theory (cf. [7], see also [4] and [6]).

Throughout this paper, C will denote a positive constant whose value is not necessary the same at each occurrences; it may vary even within a line.

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§ 2. Proof of Theorem 1.1

We use the following lemmas. Recall that $I = I_1 = [-1, 1]$.

Lemma 2.1. *Let $P \in \mathcal{P}_n$. When $0 < p \le q \le \infty$, we have*

$$\|P\|_{L^p(I)} \le 2^{1/p-1/q} \|P\|_{L^q(I)}, \tag{2.1}$$

and when $0 < q \le p \le \infty$, we have

$$\|P\|_{L^p(I)} \le 2^{1-q/p} n^{2(1/q-1/p)} \|P\|_{L^q(I)}. \tag{2.2}$$

Proof. The inequality (2.1) is shown by Hölder's inequality. The second inequality (2.2) follows by use of the method of [6, Proof of Theorem 4.2.4]. $\square$

We also use Stirling's approximation.

Proposition 2.2. *We have*

$$\lim_{m\to\infty} \frac{\sqrt{2\pi} m^{m+1/2} e^{-m}}{m!} = 1. \tag{2.3}$$

Now we give a proof of Theorem 1.1. We first assume $a = 1$. We will show that the quantity $\rho^*_{p,1}$ is independent of p, that is, $\rho^*_{p,1} = \rho^*_{\infty,1}$ holds for every $0 < p < \infty$. Let $0 < p \le \infty$ and $0 < q \le \infty$. Suppose $\rho^*_{p,1} < \infty$ and take $r > \rho^*_{p,1}$ arbitrarily. Since

$$\rho^*_{p,1} = \limsup_{n\to\infty} \frac{(n+1)\log(n+1)}{\log(1/E_{p,n}(f;I))},$$

there exists a large number N_1 such that for $n \ge N_1$, $E_{p,n}(f;I) \le (n+1)^{-(n+1)/r}$ holds.

We first consider the case $p \ge 1$ and $q \ge 1$. Take $R_n \in \mathcal{P}_n$ which satisfies $\|f - R_n\|_{L^p(I)} \le 2E_{p,n}(f;I)$, and set $P_0 = R_0$ and $P_n := R_n - R_{n-1}$ for $n \ge 1$. Since $\|f - R_n\|_{L^p(I)} + \|f - R_{n-1}\|_{L^p(I)} \le 2E_{p,n}(f;I) + 2E_{p,n-1}(f;I) \le 4E_{p,n-1}(f;I)$, we have

$$\|P_n\|_{L^p(I)} = \|f - R_n - (f - R_{n-1})\|_{L^p(I)} \le 4E_{p,n-1}(f;I) \le 4n^{-n/r}. \tag{2.4}$$

Take $N_2 \ge N_1$ such that $N_2^r > 2$. Then $n^{-n/r} < 2^{-n}$ for $n \ge N_2$ and hence (2.4) shows

$$\sum_{n=N_2}^{\infty} \|P_n\|_{L^p(I)} \le 4 \sum_{n=N_2}^{\infty} E_{p,n-1}(f;I) \le 4 \sum_{n=N_2}^{\infty} 2^{-n} < \infty,$$

that is, $f = \sum_{n=0}^{\infty} P_n$ converges in L^p-norm. From (2.4) and Lemma 2.1, we have

$$\|P_n\|_{L^q(I)} \leqslant M_n \|P_n\|_{L^p(I)} \leqslant 4M_n n^{-n/r} \tag{2.5}$$

where $M_n := 2n^{2|1/q-1/p|}$. Take $N_3 \ge N_2$ such that $(N_3^{1/N_3})^{2|1/q-1/p|} N_3^{-1/r} < 1/2$. Then

$$\begin{aligned}
\|f\|_{L^q(I)} &\le \sum_{k=0}^{\infty} \|P_k\|_{L^q(I)} \le \sum_{k=0}^{N_3-1} \|P_k\|_{L^q(I)} + \sum_{k=N_3}^{\infty} \|P_k\|_{L^q(I)} \\
&\le \sum_{k=0}^{N_3-1} \|P_k\|_{L^q(I)} + \sum_{k=N_3}^{\infty} 8k^{2|1/q-1/p|} k^{-k/r} \\
&\le \sum_{k=0}^{N_3-1} \|P_k\|_{L^q(I)} + 8 \sum_{k=N_3}^{\infty} \left((k^{1/k})^{2|1/q-1/p|} (k^{-1/r}) \right)^k \\
&\le \sum_{k=0}^{N_3-1} \|P_k\|_{L^q(I)} + C \sum_{k=N_3}^{\infty} 2^{-k} < \infty,
\end{aligned}$$

and hence $f \in L^q(I)$. Especially, by the case $q = \infty$, we may assume $f \in C(I)$. Similarly, using (2.5) and Proposition 2.2, we see that for $n \geq N_2$,

$$\begin{aligned}
&E_{q,n}(f;I) \leq \|f - \sum_{k=0}^{n} P_k\|_{L^q(I)} \leq \sum_{k=n+1}^{\infty} \|P_k\|_{L^q(I)} \\
&\leq n^{-n/r} \sum_{k=n+1}^{\infty} 8k^{2|1/q-1/p|} \left(\frac{n^n}{k^k}\right)^{1/r} \leq Cn^{-n/r} \sum_{k=n+1}^{\infty} k^{2|1/q-1/p|} \left(\frac{n!e^n/\sqrt{n}}{k!e^k/\sqrt{k}}\right)^{1/r} \\
&\leq Cn^{-n/r} \sum_{k=n+1}^{\infty} \left(\frac{k^{2r|1/q-1/p|}}{k(k-1)\cdots(n+1)}\right)^{1/r}.
\end{aligned}$$

Now we take K such that $K \geq 2r|1/q - 1/p|$. Then

$$\begin{aligned}
&\sum_{k=n+1}^{\infty} \left(\frac{k^{2r|1/q-1/p|}}{k(k-1)\cdots(n+1)}\right)^{1/r} \\
&\leq \sum_{k=n+1}^{K+n+1} \left(\frac{k^K}{k(k-1)\cdots(n+1)}\right)^{1/r} + \sum_{k=K+n+2}^{\infty} \left(\frac{k^K}{k(k-1)\cdots(n+1)}\right)^{1/r} \\
&\leq K(K+n+1)^{K/r} + 2^{K/r} \sum_{k=K+n+2}^{\infty} 2^{-(k-K-n-1)/r} \leq Cn^{K/r},
\end{aligned}$$

and hence

$$E_{q,n}(f;I) \leq Cn^{(-n+K)/r}. \tag{2.6}$$

When $p \geq 1$ and $0 < q < 1$, we repeat the above estimates for $\|f\|_{L^q(I)}^q$ and $E_{q,n}(f;I)^q$ instead of $\|f\|_{L^q(I)}$ and $E_{q,n}(f;I)$, then we also have (2.6). Therefore, from the definition (1.1),

$$\begin{aligned}
\rho_{q,1}^* &= \limsup_{n\to\infty} \frac{n \log n}{\log(1/E_{q,n}(f;I))} \leq \limsup_{n\to\infty} \frac{n \log n}{\log(1/Cn^{(K-n)/r})} \\
&= r \limsup_{n\to\infty} \frac{1}{1 - (r \log C)/n \log n - K/n} = r.
\end{aligned}$$

Since $r > \rho_{p,1}^*$ is arbitrary, we see $\rho_{q,1}^* \leq \rho_{p,1}^*$.

When $0 < p < 1$, considering the estimates of $\|f\|_{L^p(I)}^p$ instead of $\|f\|_{L^p(I)}$, we also have $\rho_{q,1}^* \leq \rho_{p,1}^*$. Since $0 < p \leq \infty$ and $0 < q \leq \infty$ are arbitrary, $\rho_{q,1}^* = \rho_{p,1}^*$ follows. Especially, $\rho_{p,1}^* = \rho_{\infty,1}^*$ for all $0 < p \leq \infty$. Hence the result of Varga gives us

$$\begin{aligned}
&f \in L^p(I), \quad \rho_{p,1}^* < \infty \Longleftrightarrow f \in L^\infty(I), \quad \rho_{\infty,1}^* = \rho_{p,1}^* < \infty \\
&\Longleftrightarrow f \text{ is an entire function of order } \rho_{\infty,1}^* = \rho_{p,1}^* < \infty.
\end{aligned}$$

Next, we show the case of $a > 0$. Set $f_a(x) := f(ax)$. By a change of variable, we have $E_{p,n}(f;I_a) = a^{1/p} E_{p,n}(f_a;I)$, which implies $\rho_{p,a}^*(f) = \rho_{p,1}^*(f_a)$. On the other hand, since $M(ar, f) = M(r, f_a)$, we see $\lambda(f) = \lambda(f_a)$. Hence the assertion for the case $a > 0$ follows from the case $a = 1$. This completes the proof of Theorem 1.1.

§ 3. Definition of class $\mathcal{F}(C^2+)$

We say that $f : (0, \infty) \mapsto (0, \infty)$ is quasi-increasing if there exists $C > 0$ such that $f(x) \le Cf(y)$ for any $0 < x < y$.

Definition 3.1. Following [4], we write $w \in \mathcal{F}(C^2+)$ if a weight w is the form $w(x) = \exp(-Q(x))$, where Q is a real, nonnegative, continuous and even function on $\mathbb{R}$, and satisfies the following conditions:

(a) $Q'(x)$ is continuous in $\mathbb{R}$, with $Q(0) = 0$.
(b) $Q''(x)$ exists and is positive in $\mathbb{R} \setminus \{0\}$.
(c) $\lim_{x\to\infty} Q(x) = \infty$.
(d) The function

$$T(x) := \frac{xQ'(x)}{Q(x)}, \quad x \neq 0 \tag{3.1}$$

is quasi-increasing in $(0, \infty)$ and $T(x) \ge \Lambda$ for $x \in \mathbb{R} \setminus \{0\}$ with some constant $\Lambda > 1$.
(e) There exists $C > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \le C\frac{|Q'(x)|}{Q(x)}, \quad a.e.\ x \in \mathbb{R}.$$

and there also exist a compact subinterval J containing 0 of $\mathbb{R}$, and $C > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \ge C\frac{|Q'(x)|}{Q(x)}, \quad a.e.\ x \in \mathbb{R} \setminus J.$$

We recall some examples (cf. [3], [4]). The weight $w_\alpha(x) := \exp(-|x|^\alpha)$ $(\alpha > 1)$ is Freud-type. For $\ell \in \mathbb{N}$, $\alpha > 0$ and $\beta \ge 0$ with $\alpha + \beta > 1$, we set $Q_{\ell,\alpha,\beta}(x) := |x|^\beta\{\exp_\ell(|x|^\alpha) - \exp_\ell(0)\}$, where $\exp_\ell(x) := \exp(\exp(\ldots(\exp x)\ldots))$ (ℓ times). Then $w := \exp(-Q_{\ell,\alpha,\beta})$ belongs to $\mathcal{F}(C^2+)$ and it is an Erdös-type weight. Also $Q_\alpha(x) = (1 + |x|)^{|x|^\alpha} - 1$, $\alpha > 1$ defines an Erdös-type weight.

We need the Mhaskar-Rakhmanov-Saff numbers $\{a_n\}$ for $w = \exp(-Q) \in \mathcal{F}(C^2+)$. For each $n \in \mathbb{N}$, a_n is a positive root of the equation

$$n = \frac{2}{\pi}\int_0^1 \frac{a_n u Q'(a_n u)}{(1-u^2)^{1/2}}du.$$

Since Q' is a positive increasing function on $(0, \infty)$, we see easily $\lim_{n\to\infty} a_n = \infty$. The following estimates are shown in [8, Proposition 3]: There exists $C > 0$ such that for all $n \in \mathbb{N}$,

$$a_n \le Cn^{1/\Lambda}, \tag{3.2}$$

where $\Lambda > 1$ is the constant in Defintion 3.1 (d). Furthermore, if w is Erdös-type, then for any fixed $\eta > 0$, there exists $C > 0$ such that for every $n \in \mathbb{N}$,

$$a_n \leq C n^{\eta}. \tag{3.3}$$

§ 4. Lemmas

We prepare some lemmas which are used in the proof of Theorem 1.2. Fix $w = \exp(-Q) \in \mathcal{F}(C^2+)$ and let $\{a_n\}$ be the Mhaskar-Rakhmanov-Saff numbers for w.

Lemma 4.1 ([4, Theorem 1.9]). *Let $0 < p \leq \infty$. Then there exists a constant $C > 0$ such that for every $n \in \mathbb{N}$ and every $P \in \mathcal{P}_n$, we have*

$$\|wP\|_{L^p(\mathbb{R})} \leq C \|wP\|_{L^p(|x| \leq a_n)}. \tag{4.1}$$

Lemma 4.2 ([2, p.11]). *Let f be an entire function and $\sum_{k=0}^{\infty} d_k z^k$ be its power series expansion. Then the order of f is given by*

$$\lambda(f) = \limsup_{n \to \infty} \frac{n \log n}{\log(1/|d_n|)}. \tag{4.2}$$

Lemma 4.3 ([4, Theorem 10.3]). *There exists a constant $C > 0$ such that for every $n \in \mathbb{N}$ and $P \in \mathcal{P}_n$, when $0 < p \leq q \leq \infty$, we have*

$$\|wP\|_{L^p(\mathbb{R})} \leq C a_n^{1/p - 1/q} \|wP\|_{L^q(\mathbb{R})}, \tag{4.3}$$

and when $0 < q \leq p \leq \infty$, we have

$$\|wP\|_{L^p(\mathbb{R})} \leq C \left(\frac{n\sqrt{T(a_n)}}{a_n} \right)^{1/q - 1/p} \|wP\|_{L^q(\mathbb{R})}. \tag{4.4}$$

Lemma 4.4 ([4, Theorem 1.15, Corollary 1.16]). *Let $0 < p \leq \infty$. Then, there exists a constant $C > 0$ such that for every $n \in \mathbb{N}$ and $P \in \mathcal{P}_n$,*

$$\|wP'\|_{L^p(\mathbb{R})} \leq C \frac{n\sqrt{T(a_n)}}{a_n} \|wP\|_{L^p(\mathbb{R})}. \tag{4.5}$$

Lemma 4.5 ([4, Lemma 3.4 (3.18),(3.17)]). *There exists a constant $C > 1$ such that for every $n \in \mathbb{N}$,*

$$\frac{1}{C} \frac{n}{\sqrt{T(a_n)}} \leq Q(a_n) \leq \frac{Cn}{\sqrt{T(a_n)}}. \tag{4.6}$$

Lemma 4.6 ([4, Lemma 3.7 (3.38)]). *There exist $0 < \varepsilon \leq 2$ and $C > 0$ such that for every $n \in \mathbb{N}$, $T(a_n) \leq C n^{2-\varepsilon}$.*

In the following Lemmas, we further assume that $w \in \mathcal{F}(C^2+)$ is Freud-type. The constants A and B are those that are defined in (1.5). Note that $1 < A \le B$.

Lemma 4.7. *Let w be Freud-type. For any $\varepsilon > 0$, there exists $C > 1$ such that for every $n \in \mathbb{N}$,*

$$\frac{1}{C}\, n^{1/(B+\varepsilon)} \le a_n \le C n^{1/(A-\varepsilon)}. \tag{4.7}$$

Proof. Let $\varepsilon > 0$. By the definition of A and B, there exists a constant N such that for $x \ge N$, we have $(A-\varepsilon)/x \le Q'(x)/Q(x) \le (B+\varepsilon)/x$, so integrating this against dx on $[N, x]$, $(x/N)^{A-\varepsilon} \le Q(x)/Q(N) \le (x/N)^{B+\varepsilon}$. Hence if $a_n > N$, then we see $C_1 a_n^{A-\varepsilon} \le Q(a_n) \le C_2 a_n^{B+\varepsilon}$, where $C_1 = Q(N)N^{\varepsilon - A}$ and $C_2 = Q(N)N^{-\varepsilon-B}$. Since $T(x) > 1$ is bounded, Lemma 4.5 implies $a_n^{A-\varepsilon}/C \le n \le C a_n^{B+\varepsilon}$. These inequalities also hold for n with $a_n \le N$ if we take C larger. Hence (4.7) follows. □

Lemma 4.8 (cf. [5, Lemma 2]). *Let w be Freud-type and let $\{p_k\}_{k=0}^{\infty}$ be the sequence of orthonormal polynomials for the weight w^2. For $f \in L^2_w(\mathbb{R})$, we define Fourier-type coefficients $\{b_k\}$ by*

$$b_k := \int_{-\infty}^{\infty} f(t)p_k(t)w(t)^2 dt, \quad k \in \mathbb{N} \cup \{0\}. \tag{4.8}$$

If $\beta > \rho_2(f)$, then for any $\varepsilon > 0$, there exists $C > 0$ such that for n large enough,

$$\sum_{k=n}^{\infty} |b_k| \|w p_k^{(n)}\|_{L^\infty(\mathbb{R})} \le C^{n+1}(n!)^{1-1/(B+\varepsilon)-1/\beta} \tag{4.9}$$

Proof. Let $\varepsilon > 0$, and let $B^* := 1 - 1/(B+\varepsilon) - 1/\beta$. We recall (1.3) for $p = 2$. By Proposition 2.2,

$$\rho_2(f) = \limsup_{n\to\infty} \frac{(n+1)\log(n+1)}{\log(1/E_{2,n}(f;w))} = \limsup_{n\to\infty} \frac{\log(n+1)!}{\log(1/E_{2,n}(f;w))},$$

so that if $\rho_2(f) < \beta$, there exists $N > 1$ such that for $n \ge N$, $E_{2,n}(f;w) \le ((n+1)!)^{-1/\beta}$. Since $f = \sum_{k=0}^{\infty} b_k p_k$, for every $P \in \mathcal{P}_{n-1}$, we see

$$\|w\cdot(f-P)\|^2_{L^2(\mathbb{R})} = \|w \cdot (\sum_{k=0}^{\infty} b_k p_k - P)\|^2_{L^2(\mathbb{R})} = \sum_{k=n}^{\infty} b_k^2 + \|w\tilde{P}\|^2_{L^2(\mathbb{R})},$$

where $\tilde{P} = P - \sum_{k=0}^{n-1} b_k p_k \in \mathcal{P}_{n-1}$. This implies $E_{2,n-1}(f;w) = \inf_{P\in\mathcal{P}_{n-1}} \|w\cdot(f - P)\|^2_{L^2(\mathbb{R})} = (\sum_{k=n}^{\infty} b_k^2)^{1/2}$, and hence for every $n \ge N$, we have

$$|b_n| \le E_{2,n-1}(f;w) \le (n!)^{-1/\beta}. \tag{4.10}$$

Let $n \geq N$. By Lemma 4.4 for $p_n^{(n-1)} \in \mathcal{P}_1$,

$$\|wp_n^{(n)}\|_{L^\infty(\mathbb{R})} \leq C\frac{\sqrt{T(a_1)}}{a_1}\|wp_n^{(n-1)}\|_{L^\infty(\mathbb{R})} \leq C\|wp_n^{(n-1)}\|_{L^\infty(\mathbb{R})}.$$

Since $p_k^{(n)} \in \mathcal{P}_{k-n}$ for $k \geq n+1$, by (4.4) with $q = 2,\ p = \infty$ and the boundedness of T we have for $n \geq N$,

$$\begin{aligned}
\sum_{k=n}^{\infty} |b_k|\|wp_k^{(n)}\|_{L^\infty(\mathbb{R})} &\leq \|b_n|\|wp_n^{(n-1)}\|_{L^\infty(\mathbb{R})} + \sum_{k=n+1}^{\infty} |b_k|\|wp_k^{(n)}\|_{L^\infty(\mathbb{R})} \\
&\leq C\left(|b_n|\|wp_n^{(n-1)}\|_{L^2(\mathbb{R})} + \sum_{k=n+1}^{\infty} |b_k|\left(\frac{k-n}{a_{k-n}}\right)^{1/2}\|wp_k^{(n)}\|_{L^2(\mathbb{R})}\right) \\
&\leq C\left(|b_n|\|wp_n^{(n-1)}\|_{L^2(\mathbb{R})} + \sum_{k=n+1}^{\infty} |b_k|\left(\frac{k-n}{a_{k-n}}\right)\|wp_k^{(n)}\|_{L^2(\mathbb{R})}\right)
\end{aligned}$$

because $(k-n)/a_{k-n} \geq C$ holds (use (3.2)). By repeating application of Lemma 4.4, the above is dominated by

$$\begin{aligned}
&C^2\left(|b_n|\left(\frac{2}{a_2}\right)\|wp_n^{(n-2)}\|_{L^2(\mathbb{R})} + \sum_{k=n+1}^{\infty} |b_k|\left(\frac{(k-n)(k+1-n)}{a_{k-n}a_{k+1-n}}\right)\|wp_k^{(n-1)}\|_{L^2(\mathbb{R})}\right) \\
&\quad \leq \cdots \leq C^n\left(|b_n|\left(\frac{n!}{a_n\cdots a_2}\right) + C\sum_{k=n+1}^{\infty} |b_k|\left(\frac{k(k-1)\cdots(k-n)}{a_k a_{k-1}\cdots a_{k-n}}\right)\right).
\end{aligned}$$

Here we use the fact that $\|wp_k\|_{L^2(\mathbb{R})} = 1$ for every $k \geq 0$, because $\{p_k\}$ are orthnormal polynomials for w^2. It follows from Lemma 4.7 and (4.10) that the last term in the above is dominated by

$$C^n(n!)^{B^*} + C^{n+1}\sum_{k=n+1}^{\infty}(k!)^{B^*}((k-n)!)^{-B^*-1/\beta}. \tag{4.11}$$

When $B^* \leq 0$, (4.11) is estimated as

$$\begin{aligned}
&C^n(n!)^{B^*} + C^{n+1}\sum_{k=n+1}^{\infty}\left(\frac{k!}{(k-n)!}\right)^{B^*}((k-n)!)^{-1/\beta} \\
&\quad \leq C^n(n!)^{B^*} + C^{n+1}(n!)^{B^*}\sum_{k=n+1}^{\infty}((k-n)!)^{-1/\beta} \leq C^{n+1}(n!)^{B^*},
\end{aligned}$$

and hence (4.9) follows. Let $B^* > 0$. We rewrite (4.11) as follows:

$$C^n(n!)^{B^*} + C^{n+1}(n!)^{B^*}2^{nB^*}\sum_{k=n+1}^{\infty}\left(\frac{k!}{(k-n)!n!}\frac{1}{2^k}\right)^{B^*}2^{(k-n)B^*}\frac{1}{((k-n)!)^{1/\beta}}.$$

By Hölder's inequality for $p = 1/B^*,\ q = 1/(1-B^*)$,

$$\sum_{k=n+1}^{\infty}\left(\frac{k!}{(k-n)!n!}\frac{1}{2^k}\right)^{B^*} 2^{(k-n)B^*}\frac{1}{((k-n)!)^{1/\beta}}$$
$$\leq \left(\sum_{k=n+1}^{\infty}\frac{k!}{(k-n)!n!}\frac{1}{2^k}\right)^{B^*}\left(\sum_{k=n+1}^{\infty}2^{(k-n)B^*/(1-B^*)}\frac{1}{((k-n)!)^{1/\beta(1-B^*)}}\right)^{1-B^*}.$$

Since $\sum_{\ell=0}^{\infty}((n+\ell)!/(n!\ell!))x^{n+\ell} = x^n/(1-x)^{n+1}$ by the binomial theorem, we see

$$\sum_{k=n+1}^{\infty}\frac{k!}{(k-n)!n!}\frac{1}{2^k} \leq \sum_{\ell=0}^{\infty}\frac{(n+\ell)!}{\ell!n!}\frac{1}{2^{\ell+n}} = 2$$

and we also see

$$\sum_{k=n+1}^{\infty}2^{(k-n)B^*/(1-B^*)}\frac{1}{((k-n)!)^{1/\beta(1-B^*)}} = \sum_{\ell=1}^{\infty}2^{\ell B^*/(1-B^*)}\frac{1}{(\ell!)^{1/\beta(1-B^*)}} < \infty.$$

Hence (4.11) is bounded by

$$C^n(n!)^{B^*} + C^{n+1}(n!)^{B^*}2^{nB^*}C \leq C^{n+1}(n!)^{B^*},$$

so that (4.9) follows. □

§ 5. Proof of Theorem 1.2

The process of the proof of Theorem 1.2 is as follows: Let $w \in \mathcal{F}(C^2+),\ 0 < p \leq \infty$ and let $f \in L^p_w(\mathbb{R})$.

(I) If $\rho_p(f) < \infty$, then f has an analytic extension to an entire function of order $\lambda(f) \leq \rho_p(f)$.

(II) Let $\{a_n\}$ be the Mhaskar-Rakhmanov-Saff numbers for w. If there exist $\nu > 1$ and $C > 0$ such that for every $n \in \mathbb{N}$,

$$a_n \leq Cn^{1/\nu} \tag{5.1}$$

holds, then for every entire function f with $\lambda(f) < \nu$, we have

$$\rho_p(f) \leq \frac{\lambda(f)\nu}{\nu - \lambda(f)}. \tag{5.2}$$

(III) If w is a Freud-type weight and $\rho_p(f) < \infty$, then

$$\frac{1}{\lambda(f)} - \frac{1}{B} \geq \frac{1}{\rho_p(f)}. \tag{5.3}$$

If (I)-(III) hold, then we have Theorem 1.2. In fact, by (I) and (II), $\rho_p(f) = 0$ if and only if $\lambda(f) = 0$, so that we may assume that $\rho_p(f) \neq 0$ and $\lambda(f) \neq 0$. If w is a Freud-type weight, then for any fixed $\varepsilon > 0$, Lemma 4.7 shows that (5.1) holds for $\nu = A - \varepsilon$. Hence by (5.2), $1/\rho_p(f) \geq 1/\lambda(f) - 1/(A-\varepsilon)$ holds. Since $\varepsilon > 0$ is arbitrary, this inequality and (5.3) give us (1.4). For an Erdös-type weight w, by (3.3), we can take ν large enough as we desire, so we have $\rho_p(f) \leq \lambda(f)$ by (5.2), and with (I) we have $\rho_p(f) = \lambda(f)$.

Proof of (I). Let $a > 0$ and suppose $\rho_p(f) < \infty$. Let $P \in \mathcal{P}_n$. Since $w(x) \geq w(a)$ for $x \in I_a := [-a, a]$, we see

$$\|f - P\|_{L^p(I_a)} \leq w(a)^{-1}\|w \cdot (f - P)\|_{L^p(I_a)} \leq w(a)^{-1}\|w \cdot (f - P)\|_{L^p(\mathbb{R})},$$

so that

$$E_{p,n}(f; I_a) \leq w(a)^{-1} E_{p,n}(f; w)$$

holds. Hence

$$\rho^*_{p,a}(f) \leq \rho_p(f) < \infty.$$

Then, by Theorem 1.1, f is the restriction to I_a of an entire function of order $\lambda(f) = \rho^*_{p,a}(f)$. Since we can take $a > 0$ arbitrarily, we conclude that f is the restriction to $\mathbb{R}$ of an entire function of order $\lambda(f) \leq \rho_p(f)$.

Proof of (II). Suppose that $f(z) := \sum_{k=0}^{\infty} d_k z^k$ is an entire function with order $\lambda(f) < \infty$, and (5.1) holds. By Lemma 4.2, we see

$$\lambda(f) = \limsup_{n\to\infty} \frac{n \log n}{\log(1/|d_n|)}. \tag{5.4}$$

When $\lambda(f) < \nu$, we will show (5.2). We suppose $p \geq 1$. Then

$$E_{p,n}(f; w) \leq \|w \cdot (f - \sum_{k=0}^{n} d_k x^k)\|_{L^p(\mathbb{R})} \leq \sum_{k=n+1}^{\infty} |d_k| \|wx^k\|_{L^p(\mathbb{R})}. \tag{5.5}$$

For exactness, we denote by C_1 and C_2 the constants in Lemma 4.1 and (5.1), respectively. Then by Lemma 4.1,

$$\begin{aligned}
\|wx^k\|_{L^p(\mathbb{R})} &\leq C_1 \|wx^k\|_{L^p(|x|\leq a_k)} \leq C_1 a_k \|wx^{k-1}\|_{L^p(|x|\leq a_k)} \\
&\leq C_1^2 a_k \|wx^{k-1}\|_{L^p(|x|\leq a_{k-1})} \leq C_1^2 a_k a_{k-1} \|wx^{k-2}\|_{L^p(|x|\leq a_{k-1})} \\
&\leq C_1^3 a_k a_{k-1} \|wx^{k-2}\|_{L^p(|x|\leq a_{k-2})} \leq \cdots \leq C_1^k a_k a_{k-1} \cdots a_1 \|w\|_{L^p(|x|\leq a_1)},
\end{aligned}$$

which implies

$$E_{p,n}(f; w) \leq \|w\|_{L^p(\mathbb{R})} \sum_{k=n+1}^{\infty} |d_k| C_1^k a_k a_{k-1} \cdots a_1. \tag{5.6}$$

Take ρ with $\lambda(f) < \rho < \nu$. Then by (5.4) and Proposition 2.2, there exist a positive constant C_3 and a large number N_1 such that for every $n \geq N_1$,

$$n^n < (1/|d_n|)^\rho, \quad \text{that is,} \quad |d_n| \leq \left(\frac{1}{n^n}\right)^{1/\rho} \leq C_3\left(\frac{n^{1/2}}{n!e^n}\right)^{1/\rho} \leq C_3\left(\frac{1}{n!}\right)^{1/\rho}.$$

Since $a_j \leq C_2 j^{1/\nu}$ for every $j \in \mathbb{N}$ by (5.1), (5.6) implies

$$\begin{aligned}
E_{p,n}(f;w) &\leq C_4 \sum_{k=n+1}^{\infty} C_1^k (k!)^{-1/\rho} a_k a_{k-1} \cdots a_1 \leq C_4 \sum_{k=n+1}^{\infty} (C_1C_2)^k (k!)^{1/\nu - 1/\rho} \\
&= C_4 (n!)^{1/\nu-1/\rho}(C_1C_2)^n \sum_{k=n+1}^{\infty} (C_1C_2)^{k-n}\left(\frac{1}{k(k-1)\cdots(n+1)}\right)^{1/\rho-1/\nu},
\end{aligned}$$

where $C_4 = C_3\|w\|_{L^p(\mathbb{R})}$. Let $C_5 = (C_1C_2)^{\rho\nu/(\nu-\rho)}$ and take a number N_2 such that $N_2/C_5 > 2$. Then for every $n \geq N_2$, we have

$$\begin{aligned}
&\sum_{k=n+1}^{\infty} (C_1C_2)^{k-n}\left(\frac{1}{k(k-1)\cdots(n+1)}\right)^{1/\rho-1/\nu} \\
&\quad = \sum_{k=n+1}^{\infty}\left(\frac{1}{(k/C_5)((k-1)/C_5)\cdots((n+1)/C_5)}\right)^{1/\rho-1/\nu} \\
&\quad \leq \sum_{m=1}^{\infty} (1/2)^{m(1/\rho-1/\nu)} < \infty.
\end{aligned}$$

Hence the above estimates, together with Proposition 2.2, imply

$$E_{p,n}(f;w) \leq C^{n+1}(n!)^{1/\nu-1/\rho} \leq C^{n+1}(n^n)^{1/\nu-1/\rho}e^{-n(1/\nu-1/\rho)} \leq C^{n+1}(n^n)^{1/\nu-1/\rho}.$$

This gives us

$$\log E_{p,n}(f;w) \leq (n+1)\log C + (1/\nu - 1/\rho)n\log n,$$

and hence

$$\begin{aligned}
\rho_p(f) &= \limsup_{n\to\infty} \frac{n\log n}{\log(1/E_{p,n}(f;w)} \\
&\leq \limsup_{n\to\infty} \frac{n\log n}{-(n+1)\log C - (1/\nu - 1/\rho)n\log n} = \frac{\rho\nu}{\nu-\rho}.
\end{aligned}$$

Letting ρ to $\lambda(f)$, we have (5.2).

When $0 < p < 1$, in stead of (5.5), we obtain

$$E_{p,n}(f;w)^p \leq \sum_{k=n+1}^{\infty} |d_k|^p \|wx^k\|_{L^p(\mathbb{R})}^p,$$

and hence, we see

$$E_{p,n}(f;w)^p \le C^{p(n+1)}(n^n)^{p(1/\nu-1/\rho)}$$

so that as in the above argument, we have (5.2).

Proof of (III). We first show that if $\rho_p(f) < \infty$, then $\rho_p(f) = \rho_q(f)$ holds for every $0 < p \le \infty$ and $0 < q \le \infty$. We may repeat the method in the proof of Theorem 1.1. In fact, let

$$M_n^* := \max\left\{ a_n^{|1/p-1/q|}, \left(\frac{n\sqrt{T(a_n)}}{a_n}\right)^{|1/p-1/q|} \right\}.$$

Since $T(a_n) \le Cn^{2-\varepsilon}$ by Lemma 4.6 and since $a_n \le Cn^{1/\Lambda}$ by (3.2), we see $M_n^* \le n^{2|1/p-1/q|}$ for n large enough. Hence (4.3) and (4.4) show

$$\|wP\|_{L^q(\mathbb{R})} \le Cn^{2|1/p-1/q|}\|wP\|_{L^p(\mathbb{R})}$$

for every $P \in \mathcal{P}_n$. If we exchange $\|f\|_{L^q(I)}$ and $\|P\|_{L^q(I)}$ for $\|wf\|_{L^q(\mathbb{R})}$ and $\|wP\|_{L^q(\mathbb{R})}$ in the proof of Theorem 1.1, respectively, we obtain the desired result.

We assume that $p = 2$ and $f \in L_w^2(\mathbb{R})$. Let $\{p_k\}_{k=0}^\infty$ be the sequence of orthonormal polynomials for the weight w^2 and $\{b_k\}_{k=0}^\infty$ be the Fourier-type coefficients for f defined in (4.8). Take $\beta > \rho_2(f)$ and let $\varepsilon > 0$ arbitrarily. By Lemma 4.8, the series

$$\sum_{n=0}^\infty \frac{1}{n!}\sum_{k=n}^\infty |b_k||p_k^{(n)}(0)||z|^n \le Cw^{-1}(0)\sum_{n=0}^\infty C^n(n!)^{-1/(B+\varepsilon)-1/\beta}|z|^n$$

converges uniformly on compact subsets of the complex plane. Interchanging the order of summation, we get

$$\sum_{n=0}^\infty \frac{1}{n!}\sum_{k=n}^\infty b_k p_k^{(n)}(0)z^n = \sum_{k=0}^\infty b_k \sum_{n=0}^k \frac{p_k^{(n)}(0)}{n!}z^n = \sum_{k=0}^\infty b_k p_k(z).$$

This shows that the above entire function is the analytic extension of f, because $f = \sum_{k=0}^\infty b_k p_k$ in $L_w^2(\mathbb{R})$. We describe this entire function as f again, then

$$f(z) = \sum_{n=0}^\infty \left(\frac{1}{n!}\sum_{k=n}^\infty b_k p_k^{(n)}(0)\right) z^n =: \sum_{n=0}^\infty d_n z^n.$$

Hence, from Lemma 4.8,

$$|d_n| \le \frac{1}{n!}\sum_{k=n}^\infty |b_k||p_k^{(n)}(0)| \le C^{n+1}(n!)^{-1/(B+\varepsilon)-1/\beta},$$

so that

$$\log(1/|d_n|) \ge (1/(B+\varepsilon)+1/\beta)\log n! - (n+1)\log C. \tag{5.7}$$

By Lemma 4.2 and Proposition 2.2, the order $\lambda(f)$ of f is given by

$$\lambda(f) = \limsup_{n\to\infty} \frac{n\log n}{\log 1/|d_n|} = \limsup_{n\to\infty} \frac{\log n!}{\log 1/|d_n|}.$$

So, (5.7) gives us

$$\frac{1}{\lambda(f)} \geq \frac{1}{B+\varepsilon} + \frac{1}{\beta}.$$

Since $\beta > \rho_2(f)$ and $\varepsilon > 0$ are arbitrary, we have

$$\frac{1}{\lambda(f)} \geq \frac{1}{B} + \frac{1}{\rho_2(f)},$$

and (5.3) follows.

References

[1] S. N. Bernstein, *Leçon sur les propriétés extrémales et la meilleure approximation des fonction analytiques d'une variable réell*, Gauthier-Villars, Paris, 1926.
[2] R. P. Boas, *Entire functions*, Academic Press, New York, 1954.
[3] H. S. Jung and R. Sakai, *Specific examples of exponential weights*, Commun. Korean Math. Soc. **24** (2009), no.2, 303-319.
[4] A.L. Levin and D.S. Lubinsky, *Orthogonal polynomials for exponential weights*, Springer, New York, 2001.
[5] H. N. Mhaskar, *Weighed polynomial approximation of entire functions, II*, J. Approx. Theory **33** (1981), 59-68.
[6] ———, *Introduction to the theory of weighted polynomial approximation*, World Scientific, Singapore, 1996.
[7] E. B. Saff and V. Totik, *Logarithmic potentials with external fields*, Springer, New York, 1997.
[8] R. Sakai and N. Suzuki, *Favard-type inequalities for exponential weights*, Pioneer J. of Math. **3** (2011), no.1 1-16.
[9] R. S. Varga, *On an extension of a result of S. N. Bernstein*, J. Approx. Theory **1** (1968), 176-179.