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Sobolev's inequality for Riesz potentials in central Lorentz-Morrey spaces of variable exponent

By

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Abstract

In the present paper we discuss the boundedness of the maximal operator in the central Lorentz-Morrey space of variable exponent defined by the symmetric decreasing rearrangement in the sense of Almut [1]. Further we establish Sobolev's inequality for Riesz potentials.

§ 1. Introduction

In this paper we use $B(x, r)$ to denote the open ball centered at x of radius $r > 0$. The volume of a measurable set $E \subset \mathbf{R}^n$ is written as $|E|$. We denote by χ_E the characteristic function of E.

Given a measurable function f on $\mathbf{R}^n$, recall the symmetric decreasing rearrangement of f defined by

$$f^\star(x) = \int_0^\infty \chi_{E_f(t)^\star}(x)dt,$$

where $E^\star = \{x : |B(0, |x|)| < |E|\}$ and $E_f(t) = \{y : |f(y)| > t\}$ (see Almut [1]). Note here that

$$f^*(|B(0, |x|)|) = f^\star(x),$$

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where f^* is the usual decreasing rearrangement of f. The fundamental fact of the symmetric decreasing rearrangement of f is that

$$|E_f(t)| = |E_{f^\star}(t)|$$

for all $t \geq 0$. This readily gives the rearrangement preserving L^p-norm property such as

$$\|f\|_{L^p(\mathbf{R}^n)} = \|f^\star\|_{L^p(\mathbf{R}^n)}$$

for $1 \leq p \leq \infty$. For fundamental properties of the symmetric decreasing rearrangement, we refer the reader to the Lecture Notes by Almut [1]; see also his papers [2, 3].

For variable exponents p, q and a constant $\mu \geq 0$, the central Lorentz-Morrey space of variable exponent $\mathcal{L}^{q,p,\mu}(\mathbf{R}^n)$ is defined as the set of all measurable functions f on $\mathbf{R}^n$ with

$$\|f\|_{\mathcal{L}^{q,p,\mu}(\mathbf{R}^n)} = \inf\left\{\lambda > 0 : \sup_{r>0} \int_{B(0,r)} r^{-\mu p(y)} |f^\star(y)/\lambda|^{p(y)} |y|^{n\left(\frac{p(y)}{q(y)}-1\right)} dy \leq 1\right\} < \infty.$$

If p and q are radial and $\mu = 0$, then we refer the reader to the paper by Ephremidze, Kokilashvili and Samko [9].

In central Lorentz-Morrey spaces of variable exponent, we establish the Sobolev inequality for the Riesz potential

$$I_\alpha f(x) = \int_{\mathbf{R}^n} |x-y|^{\alpha-n} f(y) dy$$

of order α; for fundamental properties of Riesz potentials, see e.g. [12].

§ 2. Symmetric decreasing rearrangement

Let us recall the Hardy-Littlewood inequality for the symmetric decreasing rearrangement (see Almut [1, Lemma 1.6]).

Lemma 2.1. *For all nonnegative measurable functions f and g on $\mathbf{R}^n$,*

$$\int_{\mathbf{R}^n} f(x)g(x)dx \leq \int_{\mathbf{R}^n} f^\star(x)g^\star(x)dx.$$

The (centered) maximal function Mf of a measurable function f on $\mathbf{R}^n$ is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy.$$

Lemma 2.2. *For all measurable functions f on $\mathbf{R}^n$,*

$$(Mf)^\star(x) \leq C \frac{1}{|B(0,|x|)|} \int_{B(0,|x|)} f^\star(y)dy \leq CMf^\star(x),$$

where C is a positive constant independent of f.

Proof. Recall the definition of the symmetric decreasing rearrangement and thus

$$(Mf)^\star(x) = \sup\{r > 0 : |B(0,|x|)| < |\{z : Mf(z) \ge r\}|\}.$$

Set $r_0 = (Mf)^\star(x)$. Then, using the covering property (see [12, Theorem 1.10.1]) and Lemma 2.1, we have for $0 < r < r_0$

$$\begin{aligned} |\{z : Mf(z) \ge r\}| &\le Cr^{-1} \int_{\{z: f(z) > r/2\}} f(y)dy \\ &\le Cr^{-1} \int_{\{z: f^\star(z) > r/2\}} f^\star(y)dy. \end{aligned}$$

If $\{z : f^\star(z) > r/2\} \subset B(0,|x|)$, then

$$r \le C \frac{1}{|B(0,|x|)} \int_{B(0,|x|)} f^\star(y)dy \le CM(f^\star)(x).$$

If $\{z : f^\star(z) > r/2\} \supset B(0,|x|)$, then

$$|\{z : f^\star(z) > r/2\}| \le \frac{2}{r} \int_{\{z: f^\star(z) > r/2\}} f^\star(y)dy.$$

Noting that

$$\frac{1}{|B(0,t)|} \int_{B(0,t)} f^\star(y)dy \le \frac{1}{|B(0,s)|} \int_{B(0,s)} f^\star(y)dy$$

when $0 < s < t$, we obtain

$$\begin{aligned} \frac{r}{2} &\le \frac{1}{|\{z : f^\star(z) > r/2\}|} \int_{\{z: f^\star(z) > r/2\}} f^\star(y)dy \\ &\le \frac{1}{|B(0,|x|)|} \int_{B(0,|x|)} f^\star(y)dy \le CMf^\star(x), \end{aligned}$$

as required. □

The following is known as Riesz' inequality (see Almut [1, §1.3]).

Lemma 2.3. *For all nonnegative measurable functions f, g and h on $\mathbf{R}^n$,*

$$\int_{\mathbf{R}^n} f(x)(g * h)(x)dx \le \int_{\mathbf{R}^n} f^\star(x)(g^\star * h^\star)(x)dx,$$

where

$$g * h(x) = \int_{\mathbf{R}^n} g(x-y)h(y)dy.$$

Lemma 2.4. *For all nonnegative measurable functions f on $\mathbf{R}^n$,*

$$(I_\alpha f)^\star(x) \le C \int_{\mathbf{R}^n} (|x| + |y|)^{\alpha - n} f^\star(y)dy \le C(I_\alpha f^\star)(x),$$

where C is a positive constant independent of f.

Proof. Set $r_0 = (I_\alpha f)^\star(x)$. For $0 < r < r_0$, write

$$|\{z : I_\alpha f(z) > r\}| = |B(0,t)|.$$

We have

$$\begin{aligned}
|B(0,t)| &= |\{z : I_\alpha f(z) > r\}| \\
&\le r^{-1} \int_{\{z: I_\alpha f(z) > r\}} I_\alpha f(\zeta) d\zeta \\
&\le r^{-1} \int_{\{z: (I_\alpha f)^\star(z) > r\}} I_\alpha f^\star(\zeta) d\zeta \qquad \text{(by Riesz' inequality)} \\
&= r^{-1} \int_{B(0,t)} I_\alpha f^\star(\zeta) d\zeta \\
&= r^{-1} \int_{\mathbf{R}^n} \left(\int_{B(0,t)} |\zeta - y|^{\alpha-n} d\zeta \right) f^\star(y) dy \\
&\le C r^{-1} t^n \int_{\mathbf{R}^n} (t + |y|)^{\alpha-n} f^\star(y) dy,
\end{aligned}$$

so that

$$r \le C \int_{\mathbf{R}^n} (t + |y|)^{\alpha-n} f^\star(y) dy.$$

Since $t \ge |x|$,

$$r \le C \int_{\mathbf{R}^n} (|x| + |y|)^{\alpha-n} f^\star(y) dy,$$

which gives the required inequality. □

Remark 2.5. In case $\alpha = 0$, I_α might be replaced by the singular integral operator (see [4] and [9, Theorem 3.14]).

§ 3. Central Lorentz-Morrey spaces of variable exponent

A function p on $\mathbf{R}^n$ is said to be log-Hölder continuous if

(P1) p is locally log-Hölder continuous, namely

$$|p(x) - p(y)| \le \frac{C_0}{\log(1/|x-y|)} \quad \text{for } |x-y| \le \frac{1}{e}$$

with a constant $C_0 \ge 0$, and

(P2) p is log-Hölder continuous at infinity, namely

$$|p(x) - p(\infty)| \le \frac{C_\infty}{\log(e + |x|)}$$

with constants $C_\infty \geq 0$ and $p(\infty)$. Let $\mathcal{P}(\mathbf{R}^n)$ be the class of all log-Hölder continuous functions p on $\mathbf{R}^n$. If in addition p satisfies

(P3) $1 < p^- := \inf_{x\in\mathbf{R}^n} p(x) \leq \sup_{x\in\mathbf{R}^n} p(x) =: p^+ < \infty$,

then we write $p \in \mathcal{P}_1(\mathbf{R}^n)$.

Definition 3.1. Let $\nu \in \mathcal{P}(\mathbf{R}^n)$, $p \in \mathcal{P}_1(\mathbf{R}^n)$ and $\mu \geq 0$. For an open set $G \subset \mathbf{R}^n$, we define the $L^{p(\cdot)}$-norm by

$$\|f\|_{L^{p(\cdot)}(G)} = \inf\left\{\lambda > 0 : \int_G |f(y)/\lambda|^{p(y)} dy \leq 1\right\}$$

for measurable functions f on G. By $L^{\nu,p,\mu}(G)$ we denote the weighted $L^{p(\cdot)}$-Morrey space of all functions f on G with

$$\|f\|_{L^{\nu,p,\mu}(G)} = \sup_{r>0} r^{-\mu} \|f(x)|x|^{\nu(x)}\|_{L^{p(\cdot)}(G\cap B(0,r))} < \infty.$$

We write $L^{0,p,0}(G) = L^{p(\cdot)}(G)$ and $L^{0,p,\mu}(G) = L^{p,\mu}(G)$ for simplicity; set $\|f\|_{L^{0,p,0}(G)} = \|f\|_{L^{p(\cdot)}(G)}$ and $\|f\|_{L^{0,p,\mu}(G)} = \|f\|_{L^{p,\mu}(G)}$.

Definition 3.2. For $p, q \in \mathcal{P}_1(\mathbf{R}^n)$, set $\nu(x) = n\left(\frac{1}{q(x)} - \frac{1}{p(x)}\right)$. We denote by $\mathcal{L}^{q,p,\mu}(\mathbf{R}^n)$ the family of measurable functions f on $\mathbf{R}^n$ such that

$$\|f\|_{\mathcal{L}^{q,p,\mu}(\mathbf{R}^n)} = \|f^\star\|_{L^{\nu,p,\mu}(\mathbf{R}^n)} < \infty.$$

Throughout this paper, let C denote various constants independent of the variables in question. For functions f, g, we write $f \sim g$ if there is a constant $C > 1$ such that $C^{-1}g \leq f \leq Cg$.

Lemma 3.3. *Let $p \in \mathcal{P}_1(\mathbf{R}^n), \mu \geq 0$ and $\nu \in \mathcal{P}(\mathbf{R}^n)$. Set*

$$\omega(r) = \begin{cases} r^{\mu p(0)} & \text{for } 0 < r \leq 1, \\ r^{\mu p(\infty)} & \text{for } r > 1. \end{cases}$$

Then there exists a constant $C > 0$ such that

$$\int_{B(0,r)} |f(y)|^{p(y)} |y|^{\nu(y)p(y)} dy \leq C\omega(r)$$

for all $r > 0$, whenever $\|f\|_{L^{\nu,p,\mu}(\mathbf{R}^n)} \leq 1$. Conversely, there exists a constant $C' > 0$ such that

$$\|f\|_{L^{\nu,p,\mu}(\mathbf{R}^n)} \leq C'$$

whenever $\int_{B(0,r)} |f(y)|^{p(y)} |y|^{\nu(y)p(y)} dy \leq \omega(r)$ for all $r > 0$.

Proof. We only show the case $\mu > 0$. If $0 < r < 1$, then, noting from (P1) that $r^{p(y)} \sim r^{p(0)}$ for $y \in B(0,r)$, we find

$$\int_{B(0,r)} r^{-\mu p(y)} |f(y)|^{p(y)} |y|^{\nu(y)p(y)} dy \sim r^{-\mu p(0)} \int_{B(0,r)} |f(y)|^{p(y)} |y|^{\nu(y)p(y)} dy.$$

This proves the case when $0 < r < 1$.

Suppose $\|f\|_{L^{\nu,p,\mu}(\mathbf{R}^n)} \leq 1$. If $r > 1$, then, since $|x|^{p(x)} \sim |x|^{p(\infty)}$ when $x \in \mathbf{R}^n \setminus B(0,1/2)$ by (P2), we have

$$\begin{aligned}
&r^{-\mu p(\infty)} \int_{B(0,r)\setminus B(0,1)} |f(y)|^{p(y)} |y|^{\nu(y)p(y)} dy \\
&\leq r^{-\mu p(\infty)} \sum_{j=1}^{j_0} \int_{B(0,2^{-j+1}r)\setminus B(0,2^{-j}r)} |f(y)|^{p(y)} |y|^{\nu(y)p(y)} dy \\
&= \sum_{j=1}^{j_0} 2^{-j\mu p(\infty)} \int_{B(0,2^{-j+1}r)\setminus B(0,2^{-j}r)} (2^{-j}r)^{-\mu p(\infty)} |f(y)|^{p(y)} |y|^{\nu(y)p(y)} dy \\
&\leq C \sum_{j=1}^{j_0} 2^{-j\mu p(\infty)} \int_{B(0,2^{-j+1}r)\setminus B(0,2^{-j}r)} (2^{-j}r)^{-\mu p(y)} |f(y)|^{p(y)} |y|^{\nu(y)p(y)} dy \\
&\leq C \sum_{j=1}^{\infty} 2^{-j\mu p(\infty)} < \infty,
\end{aligned}$$

where j_0 is the positive integer such that $2^{-j_0} r \leq 1 < 2^{-j_0+1} r$. Conversely, suppose

$$\sup_{r>1} r^{-\mu p(\infty)} \int_{B(0,r)} |f(y)|^{p(y)} |y|^{\nu(y)p(y)} dy \leq 1.$$

For $r > 1$, taking j_0 as above, we obtain by (P2)

$$\begin{aligned}
&\int_{B(0,r)\setminus B(0,1)} r^{-\mu p(y)} |f(y)|^{p(y)} |y|^{\nu(y)p(y)} dy \\
&\leq \sum_{j=1}^{j_0} 2^{-j\mu} \int_{B(0,2^{-j+1}r)\setminus B(0,2^{-j}r)} (2^{-j}r)^{-\mu p(y)} |f(y)|^{p(y)} |y|^{\nu(y)p(y)} dy \\
&\leq C \sum_{j=1}^{\infty} 2^{-j\mu} (2^{-j}r)^{-\mu p(\infty)} \int_{B(0,2^{-j+1}r)\setminus B(0,2^{-j}r)} |f(y)|^{p(y)} |y|^{\nu(y)p(y)} dy \\
&\leq C \sum_{j=1}^{\infty} 2^{-j\mu} < \infty.
\end{aligned}$$

The case $r > 1$ is now established, with the aid of the first case. □

§ 4. The boundedness of maximal operator in central Lorentz-Morrey spaces of variable exponent

Theorem 4.1. *Let $\nu \in \mathcal{P}(\mathbf{R}^n)$, $p \in \mathcal{P}_1(\mathbf{R}^n)$ and $\mu \geq 0$. Suppose*

(T1) $\mu p(0) - n < \nu(0)p(0) < n\,(p(0) - 1)$ *and* $\mu p(\infty) - n < \nu(\infty)p(\infty) < n\,(p(\infty) - 1)$.

Then the maximal operator $\mathcal{M} : f \longrightarrow Mf$ is bounded from $L^{\nu,p,\mu}(\mathbf{R}^n)$ into itself, namely, there is a constant $C > 0$ such that

$$\|Mf\|_{L^{\nu,p,\mu}(\mathbf{R}^n)} \leq C\|f\|_{L^{\nu,p,\mu}(\mathbf{R}^n)}$$

for all $f \in L^{\nu,p,\mu}(\mathbf{R}^n)$.

Our Theorem 4.1 includes the boundedness of the maximal operator in the weighted $L^{p(\cdot)}$-Morrey space as in [13], which is an extension of Diening [8] and Cruz-Uribe, Fiorenza and Neugebauer [6] . When $\mu = 0$, Theorem 4.1 is a special case of Theorem 1.1 in Hästö and Diening [10] (see also Cruz-Uribe, Diening and Hästö [5] and Cruz-Uribe, Fiorenza and Neugebauer [7]).

With the aid of Lemma 2.2, we find the following result.

Corollary 4.2. *Let $p, q \in \mathcal{P}_1(\mathbf{R}^n)$. Then the maximal operator $\mathcal{M} : f \longrightarrow Mf$ is bounded in $\mathcal{L}^{q,p,\mu}(\mathbf{R}^n)$.*

In what follows, we prepare lemmas required for a proof of Theorem 4.1. For a measurable function f on an open set $G \subset \mathbf{R}^n$, we define

$$M_G f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)\cap G} |f(y)|dy.$$

Let us begin with the following result due to Cruz-Uribe, Fiorenza and Neugebauer [6].

Lemma 4.3. *Let $p \in \mathcal{P}(\mathbf{R}^n)$ and G be an open set in $\mathbf{R}^n$. If*

$$p^-(G) = \inf_{x\in G} p(x) > 1,$$

then there exists a constant $C > 0$ such that

$$\|M_G f\|_{L^{p(\cdot)}(G)} \leq C\|f\|_{L^{p(\cdot)}(G)}.$$

Lemma 4.4. *Let $p \in \mathcal{P}_1(\mathbf{R}^n), \mu \geq 0, \nu \in \mathcal{P}(\mathbf{R}^n)$ and $0 < r_0 < 1 < R_0 < \infty$. If $\mu p(0) - n < \nu(0)p(0)$ and $\nu(\infty)p(\infty) < n(p(\infty) - 1)$, then there exists a constant $C > 0$ such that*

$$\|Mf_{r_0,R_0}\|_{L^{\nu,p,\mu}(\mathbf{R}^n)} \leq C$$

for all $f \geq 0$ satisfying $\|f\|_{L^{\nu,p,\mu}(\mathbf{R}^n)} \leq 1$, where $f_{r_0,R_0}(y) = f(y)\chi_{B(0,R_0)\setminus B(0,r_0)}(y)$ with χ_E denoting the characteristic function of a measurable set $E \subset \mathbf{R}^n$.

Proof. Let f be a nonnegative measurable function on $\mathbf{R}^n$ satisfying

$$\|f\|_{L^{\nu,p,\mu}(\mathbf{R}^n)} \le 1,$$

and $0 < r_0 < 1 < R_0 < \infty$. Then note that $\|f_{r_0,R_0}\|_{L^{p(\cdot)}(\mathbf{R}^n)} \le C$.

For $x \in B(0, r_0/2)$, we see that

$$\begin{aligned} Mf_{r_0,R_0}(x) &\le \frac{1}{|B(0,r_0/2)|} \int_{\mathbf{R}^n} f_{r_0,R_0}(y)dy \\ &\le \frac{1}{|B(0,r_0/2)|} \left(|B(0,R_0)| + \int_{\mathbf{R}^n} \{f_{r_0,R_0}(y)\}^{p(y)} dy \right) \le C, \end{aligned}$$

so that

$$\int_{B(0,t)} \{Mf_{r_0,R_0}(x)\}^{p(x)} |x|^{\nu(x)p(x)} dx \le C \int_{B(0,t)} |x|^{\nu(x)p(x)} dx \le Ct^{\nu(0)p(0)+n} \le Ct^{\mu p(0)}$$

for all $0 < t < r_0/2$, since $\mu p(0) - n < \nu(0)p(0)$.

Moreover, for $r_0/2 < t < 2R_0$ we have by Lemma 4.3

$$\begin{aligned} \int_{B(0,t)\setminus B(0,r_0/2)} \{Mf_{r_0,R_0}(x)\}^{p(x)} |x|^{\nu(x)p(x)} dx &\le C \int_{\mathbf{R}^n} \{Mf_{r_0,R_0}(x)\}^{p(x)} dx \\ &\le C \le C \min\{t^{\mu p(0)}, t^{\mu p(\infty)}\}, \end{aligned}$$

since $\|f_{r_0,R_0}\|_{L^{p(\cdot)}(\mathbf{R}^n)} \le C$.

Finally, we find for $x \in \mathbf{R}^n \setminus B(0, 2R_0)$

$$Mf_{r_0,R_0}(x) \le C \frac{1}{|B(0,|x|)|} \int_{\mathbf{R}^n} f_{r_0,R_0}(y)dy \le C|x|^{-n}.$$

Hence we obtain for $t > 2R_0$

$$\begin{aligned} \int_{B(0,t)\setminus B(0,2R_0)} \{Mf_{r_0,R_0}(x)\}^{p(x)} |x|^{\nu(x)p(x)} dx &\le C \int_{\mathbf{R}^n\setminus B(0,2R_0)} |x|^{\nu(x)p(x)-np(x)} dx \\ &\le C \le Ct^{\mu p(\infty)}, \end{aligned}$$

since $\nu(\infty)p(\infty) < n(p(\infty) - 1)$, which completes the proof with the aid of Lemma 3.3. □

Lemma 4.5. *Let $p \in \mathcal{P}_1(\mathbf{R}^n)$ and $\nu \in \mathcal{P}(\mathbf{R}^n)$. Let $p_0 > 1$. Suppose*

$$\nu(0) < n\left(1 - \frac{1}{p_0}\right).$$

For $0 < t \le 2$ and a measurable function f on $\mathbf{R}^n$, set

$$I = \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy$$

and

$$J_t = \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} g_{0,t}(y) dy \right)^{1/p_0},$$

where $g_{0,t}(y) = \{|f(y)||y|^{\nu(y)}\}^{p_0} \chi_{B(0,3t)}(y)$. *Then there exists a constant* $C > 0$ *such that*

$$I \le C \left\{ |x|^{-\nu(x)} J_t + H(x) \right\}$$

for all $x \in B(0,t)$, $r > 0$ *and* $0 < t \le 2$, *where*

$$H(x) = Hf(x) = \int_{\mathbf{R}^n \setminus B(0,|x|)} |f(y)||y|^{-n} dy.$$

Proof. Let f be a nonnegative measurable function on $\mathbf{R}^n$ and let $x \in B(0,t)$ be fixed.

First suppose $r < 2|x|$. Then $B(x,r) \subset B(0,3t)$, and we have by Hölder's inequality

$$I \le J_t \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |y|^{-\nu(y)p_0'} dy \right)^{1/p_0'}.$$

If $r \le |x|/2 < t/2$, then $|y| \sim |x|$ and

$$|y|^{\nu(y)} \sim |y|^{\nu(0)} \sim |x|^{\nu(0)} \sim |x|^{\nu(x)}$$

by (P1). Hence, in this case,

$$\begin{aligned} \frac{1}{|B(x,r)|} \int_{B(x,r)} |y|^{-\nu(y)p_0'} dy &\le C \frac{1}{|B(x,r)|} \int_{B(x,r)} |x|^{-\nu(x)p_0'} dy \\ &\le C|x|^{-\nu(x)p_0'}. \end{aligned}$$

If $|x|/2 < r \le 2|x| < 2t$, then, since $|y|^{\nu(y)} \sim |y|^{\nu(0)}$ for $y \in B(0,6)$, we find

$$\begin{aligned} \frac{1}{|B(x,r)|} \int_{B(x,r)} |y|^{-\nu(y)p_0'} dy &\le C \frac{1}{|B(0,3|x|)|} \int_{B(0,3|x|)} |y|^{-\nu(0)p_0'} dy \\ &\le C|x|^{-\nu(0)p_0'} \le C|x|^{-\nu(x)p_0'} \end{aligned}$$

since $\nu(0) < n(1 - 1/p_0)$. Hence

$$I \le C|x|^{-\nu(x)} J_t$$

when $r \le 2|x| < 2t$.

Finally, if $r > 2|x|$, then

$$\begin{aligned} I &\le \frac{1}{|B(x,r)|} \int_{B(x,r) \cap B(0,|x|)} |f(y)| dy + \frac{1}{|B(x,r)|} \int_{B(x,r) \setminus B(0,|x|)} |f(y)| dy \\ &\le \frac{1}{|B(x,2|x|)|} \int_{B(x,2|x|)} |f(y)| dy + CH(x) \\ &\le C|x|^{-\nu(x)} J_t + CH(x) \end{aligned}$$

by the above discussions in the case $r = 2|x|$.

Now the present lemma is obtained. $\square$

Next we treat the Hardy type operator H along the same manner as in [13].

Lemma 4.6. *Let $p \in \mathcal{P}_1(\mathbf{R}^n)$, $\mu \geq 0$, $\nu \in \mathcal{P}(\mathbf{R}^n)$ and $\varepsilon > 0$. For a measurable function f on $\mathbf{R}^n$ and $\beta \geq 0$, set*

$$H_{\beta,1} = H_{\beta,1}(x) = H_{\beta,1}f(x) = \int_{B(0,1)\setminus B(0,|x|)} |f(y)||y|^{\beta-n}dy$$

and

$$K_{\varepsilon,1} = K_{\varepsilon,1}(x) = \left(|x|^{\varepsilon-\mu p(x)} \int_{B(0,1)\setminus B(0,|x|)} g(y)|y|^{-\varepsilon}dy\right)^{1/p(x)},$$

where $g(y) = |f(y)|^{p(y)}|y|^{\nu(y)p(y)}$. If $0 < \delta < \varepsilon - \mu p(0) < (n + \nu(0)p(0))/p(0) - \mu - \beta$, then there exists a constant $C > 0$ such that

$$H_{\beta,1} \leq C|x|^{\beta-(\nu(x)p(x)+n)/p(x)+\mu}K_{\varepsilon,1} + C|x|^{\beta-(\nu(x)p(x)-\delta+n)/p(x)+\mu}$$

for all $x \in B(0,1)$ and f with $\|f\|_{L^{\nu,p,\mu}(\mathbf{R}^n)} \leq 1$.

Proof. Let f be a nonnegative measurable function on $\mathbf{R}^n$ with $\|f\|_{L^{\nu,p,\mu}(\mathbf{R}^n)} \leq 1$, and $x \in B(0,1)$. Set $E = \{y : f(y) \geq |y|^{-(\nu(y)p(y)+n)/p(y)+\mu}\}$. Noting that $|y|^{\tau(y)} \sim |y|^{\tau(0)}$ when $|y| < 1$ and $\tau \in \mathcal{P}(\mathbf{R}^n)$ by (P1), we have

$$\begin{aligned}
H_{\beta,1,1} &\equiv \int_{E\cap(B(0,1)\setminus B(0,|x|))} f(y)|y|^{\beta-n}dy \\
&\leq \int_{E\cap(B(0,1)\setminus B(0,|x|))} f(y)|y|^{\beta-n}\left(\frac{f(y)}{|y|^{-(\nu(y)p(y)+n)/p(y)+\mu}}\right)^{p(y)-1}dy \\
&\leq \int_{B(0,1)\setminus B(0,|x|)} f(y)^{p(y)}|y|^{\nu(y)p(y)-\varepsilon}|y|^{\beta+\varepsilon-\mu p(y)-(\nu(y)p(y)+n)/p(y)+\mu}dy \\
&\leq C\int_{B(0,1)\setminus B(0,|x|)} g(y)|y|^{-\varepsilon}|y|^{\beta+\varepsilon-\mu p(0)-(\nu(0)p(0)+n)/p(0)+\mu}dy \\
&\leq C|x|^{\beta-(\nu(0)p(0)+n)/p(0)+\mu}|x|^{\varepsilon-\mu p(0)}\int_{B(0,1)\setminus B(0,|x|)} g(y)|y|^{-\varepsilon}dy \\
&\leq C|x|^{\beta-(\nu(x)p(x)+n)/p(x)+\mu}K_{\varepsilon,1}
\end{aligned}$$

since

$$\begin{aligned}
K_{\varepsilon,1}^{p(x)} &\leq |x|^{\varepsilon-\mu p(x)}\sum_{j=1}^{\infty}\int_{B(0,1)\cap(B(0,2^j|x|)\setminus B(0,2^{j-1}|x|))} g(y)|y|^{-\varepsilon}dy \\
&\leq C|x|^{\varepsilon-\mu p(x)}\sum_{j=1}^{\infty}(2^j|x|)^{\mu p(0)-\varepsilon} \leq C\sum_{j=1}^{\infty}2^{j(\mu p(0)-\varepsilon)} \leq C \qquad (4.1)
\end{aligned}$$

and $0 < \varepsilon - \mu p(0) < (n + \nu(0)p(0))/p(0) - \mu - \beta$.

We next obtain by Hölder's inequality and the fact that $|y|^{\tau(x)} \sim |y|^{\tau(y)} \sim |y|^{\tau(0)}$ when $|x| \le |y| < 1$ and $\tau \in \mathcal{P}(\mathbf{R}^n)$ by (P1)

$$
\begin{aligned}
H_{\beta,1,2} &\equiv \int_F f(y)|y|^{\beta-n} dy \\
&\le \left(\int_F |y|^{(\beta-(\nu(y)p(y)-\varepsilon)/p(y)-n)p'(x)} dy\right)^{1/p'(x)} \left(\int_F f(y)^{p(x)}|y|^{\nu(y)p(y)-\varepsilon} dy\right)^{1/p(x)} \\
&\le C\left(\int_F |y|^{(\beta-(\nu(0)p(0)-\varepsilon)/p(0)-n)p'(0)} dy\right)^{1/p'(x)} \left(\int_F f(y)^{p(x)}|y|^{\nu(y)p(y)-\varepsilon} dy\right)^{1/p(x)} \\
&\le C|x|^{(\beta-(\nu(0)p(0)-\varepsilon+n)/p(0))p'(0)/p'(x)} \left(\int_F f(y)^{p(x)}|y|^{\nu(y)p(y)-\varepsilon} dy\right)^{1/p(x)} \\
&\le C|x|^{\beta-(\nu(x)p(x)-\varepsilon+n)/p(x)} \left(\int_F f(y)^{p(x)}|y|^{\nu(x)p(x)-\varepsilon} dy\right)^{1/p(x)}
\end{aligned}
$$

since $\beta - (\nu(0)p(0) - \varepsilon + n)/p(0) < (\mu - \varepsilon/p(0))(p(0) - 1)$, where

$$F = (B(0,1) \setminus B(0,|x|)) \setminus E.$$

Here, since $0 < \delta < \varepsilon - \mu p(0)$, we see that

$$
\begin{aligned}
&\left(\int_F f(y)^{p(x)}|y|^{\nu(x)p(x)-\varepsilon} dy\right)^{1/p(x)} \\
&= \left(\int_F (f(y)|y|^{(\nu(x)p(x)+n-\mu p(x))/p(x)})^{p(x)}|y|^{-(\varepsilon-\mu p(x))-n} dy\right)^{1/p(x)} \\
&\le C\left(\int_F |y|^{\delta}|y|^{-(\varepsilon-\mu p(y))-n} dy\right)^{1/p(x)} \\
&\quad + C\left(\int_F (f(y)|y|^{(\nu(x)p(x)+n-\mu p(x))/p(x)})^{p(y)}|y|^{-(\varepsilon-\mu p(x))-n} dy\right)^{1/p(x)} \\
&\le C|x|^{(\delta-\varepsilon)/p(x)+\mu} + C\left(\int_F f(y)^{p(y)}|y|^{\nu(y)p(y)-\varepsilon} dy\right)^{1/p(x)},
\end{aligned}
$$

since $(f(y)|y|^{(\nu(x)p(x)+n-\mu p(x))/p(x)})^{p(x)} \le C(f(y)|y|^{(\nu(x)p(x)+n-\mu p(x))/p(x)})^{p(y)}$ when $|x| \le |y| < 1$ and $|y|^{\delta/p(x)} < f(y)|y|^{(\nu(x)p(x)+n-\mu p(x))/p(x)} \le 1$, so that

$$H_{\beta,1,2} \le C|x|^{\beta-(\nu(x)p(x)-\delta+n)/p(x)+\mu} + C|x|^{\beta-(\nu(x)p(x)+n)/p(x)+\mu} K_{\varepsilon,1},$$

which completes the proof. □

For $p \in \mathcal{P}_1(\mathbf{R}^n)$ and $\beta \ge 0$, set

$$\frac{1}{p_\beta} = \frac{1}{p} - \frac{\beta}{n}.$$

Corollary 4.7. *Let $p \in \mathcal{P}_1(\mathbf{R}^n)$, $\mu \geq 0$ and $\nu \in \mathcal{P}(\mathbf{R}^n)$ satisfy $\beta p(0) + \mu p(0) - n < \nu(0)p(0)$. If $0 \leq \beta < n/p^+$, then there exists a constant $C > 0$ such that*

$$\int_{B(0,t)} \left\{ H_{\beta,1} f(x) |x|^{\nu(x)} \right\}^{p_\beta(x)} dx \leq C t^{\mu p_\beta(0)}$$

for all $t > 0$ and $f \geq 0$ satisfying $\|f\|_{L^{\nu,p,\mu}(\mathbf{R}^n)} \leq 1$.

Proof. Let f be a nonnegative measurable function on $\mathbf{R}^n$ satisfying

$$\|f\|_{L^{\nu,p,\mu}(\mathbf{R}^n)} \leq 1.$$

Take δ and ε such that

$$0 < \delta < \varepsilon - \mu p(0) < (n + \nu(0)p(0))/p(0) - \mu - \beta.$$

In view of Lemma 4.6, we find

$$H_{\beta,1}(x)|x|^{\nu(x)} \leq C|x|^{\mu - n/p_\beta(x)} K_{\varepsilon,1}(x) + C|x|^{\delta/p(x) + \mu - n/p_\beta(x)} \tag{4.2}$$

for $x \in B(0,1)$; and note that $H_{\beta,1} = 0$ outside $B(0,1)$.

First we consider the case $\mu = 0$. Then note that

$$\int_{\mathbf{R}^n} g(y) dy \leq C.$$

Since $K_{\varepsilon,1}(x) < C$ by (4.1), we obtain

$$\begin{aligned}
&\int_{B(0,t)} \{H_{\beta,1}(x)|x|^{\nu(x)}\}^{p_\beta(x)} dx \\
&\leq C \int_{B(0,t)} \{K_{\varepsilon,1}(x)\}^{p(x)} |x|^{-n} dx + C \int_{B(0,t)} |x|^{\delta p_\beta(x)/p(x) - n} dx \\
&\leq C \int_{B(0,t)} \left(|x|^{\varepsilon} \int_{B(0,1) \setminus B(0,|x|)} g(y)|y|^{-\varepsilon} dy \right) |x|^{-n} dx + C t^{\delta p_\beta(0)/p(0)} \\
&\leq C \int_{\mathbf{R}^n} g(y)|y|^{-\varepsilon} \left(\int_{B(0,|y|)} |x|^{\varepsilon - n} dx \right) dy + C \\
&\leq C \int_{\mathbf{R}^n} g(y) dy + C \leq C
\end{aligned}$$

for $0 < t < 1$.

Next we consider the case $\mu > 0$. By (4.1) and (4.2), we have

$$H_{\beta,1}(x)|x|^{\nu(x)} \leq C|x|^{\mu - n/p_\beta(x)}$$

for $x \in B(0,1)$. Hence we obtain

$$\int_{B(0,t)} \{|x|^{\nu(x)} H_{\beta,1}(x)\}^{p_\beta(x)} dx \le C \int_{B(0,t)} |x|^{\mu p_\beta(x)-n} dx \le C t^{\mu p_\beta(0)}$$

for $0 < t < 1$, as required. □

In the same manner as Lemma 4.5, we can prove the following result.

Lemma 4.8. *Let $p \in \mathcal{P}_1(\mathbf{R}^n)$ and $\nu \in \mathcal{P}(\mathbf{R}^n)$. Let $p_\infty > 1$. Suppose*

$$\nu(\infty) < n\left(1 - \frac{1}{p_\infty}\right).$$

For $t > 2$ and a measurable function f on $\mathbf{R}^n$, set

$$J_{t,\infty} = \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} g_{\infty,t}(y) dy\right)^{1/p_\infty},$$

where $g_{\infty,t}(y) = \{|f(y)||y|^{\nu(y)}\}^{p_\infty} \chi_{B(0,3t)}(y)$. Then there exists a constant $C > 0$ such that

$$I \le C|x|^{-\nu(x)} J_{t,\infty} + CH(x)$$

for all $x \in B(0,t) \setminus B(0,1)$, $r > 0$ and f such that $f = 0$ on $B(0,1)$.

Lemma 4.9. *Let $p \in \mathcal{P}_1(\mathbf{R}^n)$, $\mu \ge 0$ and $\nu \in \mathcal{P}(\mathbf{R}^n)$. For a measurable function f on $\mathbf{R}^n$, $\beta \ge 0$ and $\eta > \mu p(\infty)$, set*

$$H_\beta = H_\beta f(x) = \int_{\mathbf{R}^n \setminus B(0,|x|)} |f(y)||y|^{\beta-n} dy$$

and

$$K_\eta = K_\eta(x) = \left(|x|^{\eta - \mu p(\infty)} \int_{\mathbf{R}^n \setminus B(0,|x|)} g(y)|y|^{-\eta} dy\right)^{1/p(x)},$$

where $g(y) = |f(y)|^{p(y)}|y|^{\nu(y)p(y)}$. If $\varepsilon > 0$ and $0 < \varepsilon(p(\infty)-1) + \eta - \mu p(\infty) < (n + \nu(\infty)p(\infty))/p(\infty) - \mu - \beta$, then there exists a constant $C > 0$ such that

$$H_\beta \le C|x|^{\beta-(\nu(x)p(x)+n)/p(x)+\mu} K_\eta + C|x|^{\beta-\varepsilon-(\nu(x)p(x)+n)/p(x)+\mu}$$

for all $x \in \mathbf{R}^n \setminus B(0,1)$ and $f \ge 0$ satisfying $\|f\|_{L^{\nu,p,\mu}(\mathbf{R}^n)} \le 1$.

Proof. Let f be a nonnegative measurable function on $\mathbf{R}^n$ satisfying

$$\|f\|_{L^{\nu,p,\mu}(\mathbf{R}^n)} \le 1,$$

and $x \in \mathbf{R}^n \setminus B(0,1)$. Then, as in (4.1), we see that

$$|x|^{\eta-\mu p(\infty)} \int_{\mathbf{R}^n \setminus B(0,|x|)} g(y)|y|^{-\eta} dy \le C$$

since $\eta > \mu p(\infty)$.

If $|x|^{-\varepsilon} < K_\eta$ ($< C$), then $K_\eta^{-p(y)} \le C K_\eta^{-p(x)}$ and $|y|^{\tau(y)} \sim |y|^{\tau(\infty)}$ when $1 < |x| \le |y|$ and $\tau \in \mathcal{P}(\mathbf{R}^n)$ by (P2), so that

$$\begin{aligned} H_\beta \le & \int_{\mathbf{R}^n \setminus B(0,|x|)} K_\eta |y|^{-(\nu(y)p(y)+n)/p(y)+\mu} |y|^{\beta-n} dy \\ & + C \int_{\mathbf{R}^n \setminus B(0,|x|)} f(y)|y|^{\beta-n} \left(\frac{f(y)}{K_\eta |y|^{-(\nu(y)p(y)+n)/p(y)+\mu}} \right)^{p(y)-1} dy \\ \le & \, C K_\eta |x|^{\beta-(\nu(\infty)p(\infty)+n)/p(\infty)+\mu} \\ & + C K_\eta^{1-p(x)} \int_{\mathbf{R}^n \setminus B(0,|x|)} g(y)|y|^{\beta-\mu p(y)-(\nu(y)p(y)+n)/p(y)+\mu} dy \\ \le & \, C K_\eta |x|^{\beta-(\nu(x)p(x)+n)/p(x)+\mu} \\ & + C K_\eta^{1-p(x)} |x|^{\beta-(\nu(\infty)p(\infty)+n)/p(\infty)+\mu} |x|^{\eta-\mu p(x)} \int_{\mathbf{R}^n \setminus B(0,|x|)} g(y)|y|^{-\eta} dy \\ \le & \, C K_\eta |x|^{\beta-(\nu(x)p(x)+n)/p(x)+\mu} \end{aligned}$$

since $0 < \eta - \mu p(\infty) < (n + \nu(\infty)p(\infty))/p(\infty) - \mu - \beta$.

Next consider the case $K_\eta \le |x|^{-\varepsilon}$. Since $|y|^{\tau(y)} \sim |y|^{\tau(\infty)}$ when $|y| > 1$ and $\tau \in \mathcal{P}(\mathbf{R}^n)$, we find

$$\begin{aligned} H_\beta \le & \int_{\mathbf{R}^n \setminus B(0,|x|)} |y|^{-\varepsilon-(\nu(y)p(y)+n)/p(y)+\mu} |y|^{\beta-n} dy \\ & + \int_{\mathbf{R}^n \setminus B(0,|x|)} f(y)|y|^{\beta-n} \left(\frac{f(y)}{|y|^{-\varepsilon-(\nu(y)p(y)+n)/p(y)+\mu}} \right)^{p(y)-1} dy \\ \le & \, C|x|^{\beta-\varepsilon-(\nu(\infty)p(\infty)+n)/p(\infty)+\mu} \\ & + C \int_{\mathbf{R}^n \setminus B(0,|x|)} g(y)|y|^{\beta+\varepsilon(p(y)-1)-\mu p(y)-(\nu(y)p(y)+n)/p(y)+\mu} dy \\ \le & \, C|x|^{\beta-\varepsilon-(\nu(x)p(x)+n)/p(x)+\mu} \\ & + C|x|^{\beta+\varepsilon(p(\infty)-1)-(\nu(\infty)p(\infty)+n)/p(\infty)+\mu} |x|^{\eta-\mu p(x)} \int_{\mathbf{R}^n \setminus B(0,|x|)} g(y)|y|^{-\eta} dy \\ \le & \, C|x|^{\beta-\varepsilon-(\nu(x)p(x)+n)/p(x)+\mu} \end{aligned}$$

since $\varepsilon(p(\infty)-1) + \eta - \mu p(\infty) < (n + \nu(\infty)p(\infty))/p(\infty) - \mu - \beta$. Thus the proof is completed. □

In the same manner as Corollary 4.7, we can prove the following result.

Corollary 4.10. *Let $p \in \mathcal{P}_1(\mathbf{R}^n)$, $\mu \geq 0$ and $\nu \in \mathcal{P}(\mathbf{R}^n)$ satisfy $\beta p(\infty) + \mu p(\infty) - n < \nu(\infty)p(\infty)$. If $0 \leq \beta < n/p^+$, then there exists a constant $C > 0$ such that*

$$\int_{B(0,t)\setminus B(0,1)} \left\{ H_\beta f(x)|x|^{\nu(x)} \right\}^{p_\beta(x)} dx \leq C t^{\mu p_\beta(\infty)}$$

for all $t > 1$ and $f \geq 0$ satisfying $\|f\|_{L^{\nu,p,\mu}(\mathbf{R}^n)} \leq 1$.

Proof of Theorem 4.1. We now show the boundedness of the maximal operator. For this purpose, take r_0, R_0 and p_0, p_∞ such that $0 < 2r_0 < 1 < R_0 < \infty$, $1 < p_0 < p(0)$, $1 < p_\infty < p(\infty)$, $\nu(0) < n(1 - 1/p_0)$ and $\nu(\infty) < n(1 - 1/p_\infty)$. Let f be a nonnegative measurable function on $\mathbf{R}^n$ satisfying $\|f\|_{L^{\nu,p,\mu}(\mathbf{R}^n)} \leq 1$, and write

$$\begin{aligned} f(y) &= f(y)\chi_{B(0,r_0)}(y) + f(y)\chi_{B(0,2R_0)\setminus B(0,r_0)}(y) + f(y)\chi_{\mathbf{R}^n\setminus B(0,2R_0)}(y) \\ &= f_{0,r_0}(y) + f_{r_0,2R_0}(y) + f_{2R_0,\infty}(y). \end{aligned}$$

Case 1: $0 < t \leq 2r_0$. If $0 < t \leq 2r_0$ and $x \in B(0,t)$, then we have by Lemma 4.5

$$|x|^{\nu(x)} M f_{0,r_0}(x) \leq C\{M(g_{0,t})(x)\}^{1/p_0} + C|x|^{\nu(x)} H f(x),$$

where $g_{0,t}(y) = \{f_{0,r_0}(y)|y|^{\nu(y)}\}^{p_0}\chi_{B(0,3t)}(y)$. Here note from Lemma 4.3 that

$$\begin{aligned} \|\{M(g_{0,t})\}^{1/p_0}\|_{L^{p(\cdot)}(B(0,t))} &\leq \|\{M(g_{0,t})\}^{1/p_0}\|_{L^{p(\cdot)}(B(0,6r_0))} \\ &\leq C\|(g_{0,t})^{1/p_0}\|_{L^{p(\cdot)}(B(0,6r_0))} \leq C t^\mu. \end{aligned}$$

Therefore we obtain by Corollary 4.7

$$\begin{aligned} \||x|^{\nu(x)} M f_{0,r_0}(x)\|_{L^{p(\cdot)}(B(0,t))} &\leq C\|\{M(g_{0,t})\}^{1/p_0}\|_{L^{p(\cdot)}(B(0,t))} \\ &\quad + C\||x|^{\nu(x)} H f(x)\|_{L^{p(\cdot)}(B(0,t))} \\ &\leq C t^\mu. \end{aligned}$$

Now it follows from Lemma 4.4 and Corollary 4.7 that

$$\||x|^{\nu(x)} M f(x)\|_{L^{p(\cdot)}(B(0,t))} \leq C t^\mu,$$

since $M f_{2R_0,\infty}(x) \leq C(Hf(x))$ for $x \in B(0, R_0/2)$.

Case 2: $2r_0 < t \leq R_0$. For $x \in \mathbf{R}^n \setminus B(0, 2r_0)$ note that

$$\begin{aligned} M f_{0,r_0}(x) &\leq C|x|^{-n} \int_{\mathbf{R}^n} f_{0,r_0}(y)dy \\ &\leq C|x|^{-n} \left(\int_{B(0,r_0)} |y|^{-\nu(y)p'(y)} dy + \int_{B(0,r_0)} f(y)^{p(y)} |y|^{\nu(y)p(y)} dy \right) \\ &\leq C|x|^{-n} \end{aligned}$$

and

$$Mf_{2R_0,\infty}(x) \le CHf(x)$$

for $x \in B(0,R_0)$. Hence we find by Lemma 4.4 and Corollaries 4.7 and 4.10

$$\begin{aligned} &\||x|^{\nu(x)}Mf(x)\|_{L^{p(\cdot)}(B(0,t)\setminus B(0,2r_0))} \\ &\le C\||x|^{\nu(x)-n}\|_{L^{p(\cdot)}(B(0,t)\setminus B(0,2r_0))} + \||x|^{\nu(x)}Mf_{r_0,2R_0}(x)\|_{L^{p(\cdot)}(B(0,t)\setminus B(0,2r_0))} \\ &\quad + C\||x|^{\nu(x)}Hf(x)\|_{L^{p(\cdot)}(B(0,1)\setminus B(0,2r_0))} + C\||x|^{\nu(x)}Hf(x)\|_{L^{p(\cdot)}(B(0,t)\setminus B(0,1))} \\ &\le Ct^{\mu}, \end{aligned}$$

so that the first case gives

$$\||x|^{\nu(x)}Mf(x)\|_{L^{p(\cdot)}(B(0,t))} \le Ct^{\mu}.$$

Case 3: $t > R_0$. If $t > R_0$ and $x \in \mathbf{R}^n \setminus B(0,R_0)$, then Lemma 4.8 gives

$$|x|^{\nu(x)}Mf_{2R_0,\infty}(x) \le C\{M(g_{\infty,t})(x)\}^{1/p_\infty} + C|x|^{\nu(x)}H(x),$$

where $g_{\infty,t}(y) = \{f_{2R_0,\infty}(y)|y|^{\nu(y)}\}^{p_\infty}\chi_{B(0,3t)}(y)$. By Lemma 4.3, we find

$$\begin{aligned} \|\{M(g_{\infty,t})\}^{1/p_\infty}\|_{L^{p(\cdot)}(B(0,t)\setminus B(0,R_0))} &\le \|\{M(g_{\infty,t})\}^{1/p_\infty}\|_{L^{p(\cdot)}(\mathbf{R}^n\setminus B(0,R_0))} \\ &\le C\|(g_{\infty,t})^{1/p_\infty}\|_{L^{p(\cdot)}(\mathbf{R}^n\setminus B(0,R_0))} \le Ct^{\mu} \end{aligned}$$

and hence

$$\begin{aligned} &\||x|^{\nu(x)}Mf_{2R_0,\infty}(x)\|_{L^{p(\cdot)}(B(0,t)\setminus B(0,R_0))} \\ &\le C\|\{M(g_{\infty,t})\}^{1/p_\infty}\|_{L^{p(\cdot)}(B(0,t)\setminus B(0,R_0))} + C\||x|^{\nu(x)}H(x)\|_{L^{p(\cdot)}(B(0,t)\setminus B(0,R_0))} \\ &\le Ct^{\mu} \end{aligned}$$

by Corollary 4.10. Now it follows from Lemma 4.4 that

$$\begin{aligned} &\||x|^{\nu(x)}Mf(x)\|_{L^{p(\cdot)}(B(0,t)\setminus B(0,R_0))} \\ &\le C\||x|^{\nu(x)-n}\|_{L^{p(\cdot)}(B(0,t)\setminus B(0,R_0))} + \||x|^{\nu(x)}Mf_{r_0,2R_0}(x)\|_{L^{p(\cdot)}(B(0,t)\setminus B(0,R_0))} \\ &\quad + C\||x|^{\nu(x)}Mf_{2R_0,\infty}(x)\|_{L^{p(\cdot)}(B(0,t)\setminus B(0,R_0))} \\ &\le Ct^{\mu}, \end{aligned}$$

and the proof is completed, with the aid of the second case. □

§ 5. Sobolev's inequality in Lorentz spaces

Our next aim in this paper is to establish the Sobolev type inequality for Riesz potentials.

For $p \in \mathcal{P}_1(\mathbf{R}^n)$, set

$$\frac{1}{p^\sharp} = \frac{1}{p} - \frac{\alpha}{n}.$$

Theorem 5.1. *Let $\nu \in \mathcal{P}(\mathbf{R}^n)$, $p \in \mathcal{P}_1(\mathbf{R}^n)$ and $\mu \geq 0$. Suppose $\alpha < n/p^+$ and*

(T2) $\alpha p(0) - n + \mu p(0) < \nu(0)p(0) < n\,(p(0) - 1)$ *and*
$\alpha p(\infty) - n + \mu p(\infty) < \nu(\infty)p(\infty) < n\,(p(\infty) - 1)$.

Then there is a constant $C > 0$ such that

$$\|I_\alpha f\|_{L^{\nu,p^\sharp,\mu}(\mathbf{R}^n)} \leq C\|f\|_{L^{\nu,p,\mu}(\mathbf{R}^n)}$$

for all $f \in L^{\nu,p,\mu}(\mathbf{R}^n)$.

With the aid of Lemma 2.4, we find the following result.

Corollary 5.2. *Let $\nu \in \mathcal{P}(\mathbf{R}^n)$, $p \in \mathcal{P}_1(\mathbf{R}^n)$, $q \in \mathcal{P}_1(\mathbf{R}^n)$ and $\mu \geq 0$. If $\alpha < n/p^+, \alpha + \mu < n/q(0)$ and $\alpha + \mu < n/q(\infty)$, then there is a constant $C > 0$ such that*

$$\|I_\alpha f\|_{\mathcal{L}^{q^\sharp,p^\sharp,\mu}(\mathbf{R}^n)} \leq C\|f\|_{\mathcal{L}^{q,p,\mu}(\mathbf{R}^n)}$$

for all $f \in \mathcal{L}^{q,p,\mu}(\mathbf{R}^n)$.

§ 6. Proof of Theorem 5.1

For a measurable function f on an open set $G \subset \mathbf{R}^n$, we define

$$I_{\alpha,G}f(x) = \int_G |x-y|^{\alpha-n} f(y)dy.$$

First we note the Sobolev's inequality for Riesz potentials of functions in $L^{p(\cdot)}(\mathbf{R}^n)$.

Lemma 6.1 ([11, Theorem 6.4])**.** *Let $p \in \mathcal{P}_1(\mathbf{R}^n)$ and G be an open set in $\mathbf{R}^n$. If $1 < p^-(G) \leq p^+(G) < n/\alpha$, then there exists a constant $C > 0$ such that*

$$\|I_{\alpha,G}f\|_{L^{p^\sharp(\cdot)}(G)} \leq C\|f\|_{L^{p(\cdot)}(G)}.$$

For a nonnegative measurable function f on $\mathbf{R}^n$, we write

$$\begin{aligned} I_\alpha f(x) &= \int_{B(x,2|x|)} |x-y|^{\alpha-n} f(y)dy + \int_{\mathbf{R}^n \setminus B(x,2|x|)} |x-y|^{\alpha-n} f(y)dy \\ &= U_\alpha f(x) + I_2(x). \end{aligned}$$

Then note that

$$I_2(x) \leq CH_\alpha f(x). \tag{6.1}$$

For $U_\alpha f$ we have the following result in the same manner as Lemmas 4.5 and 4.8.

Lemma 6.2. *Let $p \in \mathcal{P}_1(\mathbf{R}^n)$ and $\nu \in \mathcal{P}(\mathbf{R}^n)$. Let $p_0 > 1$. Suppose*

$$\nu(0) < n\left(1 - \frac{1}{p_0}\right).$$

Then there exists a constant $C > 0$ such that

$$U_\alpha f(x) \le C|x|^{-\nu(x)}\left[I_\alpha g_t(x) + \{I_{\alpha p_0}(g_t)^{p_0}(x)\}^{1/p_0}\right]$$

for all $x \in B(0,t)$ with $0 < t \le 2$ and nonnegative measurable functions f on $\mathbf{R}^n$, where $g_t(y) = f(y)|y|^{\nu(y)}\chi_{B(0,3t)}(y)$.

Proof. Let f be a nonnegative measurable function on $\mathbf{R}^n$. First note that $B(x, 2|x|) \subset B(0, 3t)$ for $x \in B(0,t)$. Set $f_t(y) = f(y)\chi_{B(0,3t)}(y)$. We have by Hölder's inequality

$$\begin{aligned}
&\int_{B(0,|x|/2)} |x-y|^{\alpha-n} f_t(y)dy \le \left(\int_{B(0,|x|/2)} |x-y|^{-n}\{|y|^{-\nu(y)}\}^{p_0'}dy\right)^{1/p_0'} \\
&\times \left(\int_{B(0,|x|/2)} |x-y|^{\alpha p_0 - n}\{f_t(y)|y|^{\nu(y)}\}^{p_0}dy\right)^{1/p_0} \\
&\le C\left(|x|^{-n}\int_{B(0,|x|/2)} |y|^{-\nu(y)p_0'}dy\right)^{1/p_0'} \{I_{\alpha p_0}(g_t)^{p_0}(x)\}^{1/p_0} \\
&\le C|x|^{-\nu(x)}\{I_{\alpha p_0}(g_t)^{p_0}(x)\}^{1/p_0},
\end{aligned}$$

as in the proof of Lemma 4.5. Moreover, if $y \in B(x, 2|x|) \setminus B(0, |x|/2)$, then $|y| \sim |x|$ and $|y|^{\tau(y)} \sim |x|^{\tau(x)}$ for $\tau \in \mathcal{P}(\mathbf{R}^n)$ by (P1), so that

$$\begin{aligned}
&\int_{B(x,2|x|)\setminus B(0,|x|/2)} |x-y|^{\alpha-n} f_t(y)dy \\
&\le C|x|^{-\nu(x)}\int_{B(x,2|x|)\setminus B(0,|x|/2)} |x-y|^{\alpha-n} f_t(y)|y|^{\nu(y)}dy \\
&\le C|x|^{-\nu(x)} I_\alpha g_t(x),
\end{aligned}$$

as required. □

In the same manner as Lemma 6.2, we can prove the following result.

Lemma 6.3. *Let $p \in \mathcal{P}_1(\mathbf{R}^n)$ and $\nu \in \mathcal{P}(\mathbf{R}^n)$. Let $p_\infty > 1$. Suppose*

$$\nu(\infty) < n\left(1 - \frac{1}{p_\infty}\right).$$

Then there exists a constant $C > 0$ such that

$$U_\alpha f(x) \le C|x|^{-\nu(x)}\left[I_\alpha g_t(x) + \{I_{\alpha p_\infty}(g_t)^{p_\infty}(x)\}^{1/p_\infty}\right]$$

all $x \in B(0,t) \setminus B(0,1)$, $t > 2$ *and nonnegative measurable functions* f *such that* $f = 0$ *on* $B(0,1)$, *where* $g_t(y) = f(y)|y|^{\nu(y)}\chi_{B(0,3t)}(y)$.

Proof of Theorem 5.1. Take r_0, R_0 and p_0, p_∞ such that $0 < 2r_0 < 1 < R_0 < \infty$, $1 < p_0 < p(0)$, $1 < p_\infty < p(\infty)$, $\nu(0) < n(1 - 1/p_0)$ and $\nu(\infty) < n(1 - 1/p_\infty)$. Let f be a nonnegative measurable function on $\mathbf{R}^n$ satisfying $\|f\|_{L^{\nu,p,\mu}(\mathbf{R}^n)} \le 1$, and write

$$\begin{aligned} f(y) &= f(y)\chi_{B(0,r_0)}(y) + f(y)\chi_{B(0,2R_0)\setminus B(0,r_0)}(y) + f(y)\chi_{\mathbf{R}^n\setminus B(0,2R_0)}(y) \\ &= f_{0,r_0}(y) + f_{r_0,2R_0}(y) + f_{2R_0,\infty}(y). \end{aligned}$$

As in the proof of Theorem 4.1, it suffices to show that

$$\||x|^{\nu(x)} I_\alpha f_{0,r_0}(x)\|_{L^{p^\sharp(\cdot)}(B(0,t))} \le Ct^\mu \qquad \text{for } 0 < t \le 2r_0$$

and

$$\||x|^{\nu(x)} I_\alpha f_{2R_0,\infty}(x)\|_{L^{p^\sharp(\cdot)}(B(0,t)\setminus B(0,R_0))} \le Ct^\mu \qquad \text{for } t \ge R_0.$$

In the rest of the present proof, we are only concerned with f_{0,r_0}. For this note from Lemmas 6.1, 6.2 and Corollary 4.7 that

$$\begin{aligned} &\||x|^{\nu(x)} I_\alpha f_{0,r_0}(x)\|_{L^{p^\sharp(\cdot)}(B(0,t))} \le C\|I_\alpha g_t\|_{L^{p^\sharp(\cdot)}(B(0,6r_0))} \\ &\quad + C\|\{I_{\alpha p_0}(g_t)^{p_0}\}^{1/p_0}\|_{L^{p^\sharp(\cdot)}(B(0,6r_0))} + C\||x|^{\nu(x)} Hf(x)\|_{L^{p^\sharp(\cdot)}(B(0,t))} \\ &\le C\|g_t\|_{L^{p(\cdot)}(B(0,6r_0))} + Ct^\mu \le Ct^\mu \end{aligned}$$

when $0 < t < 2r_0$, where $g_t(y) = f(y)|y|^{\nu(y)}\chi_{B(0,3t)}(y)$.

The remaining part of the proof is easily completed. □

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