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Examples of harmonic Hardy-Orlicz spaces on the plane with finitely many punctures

By

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Abstract

In [11] we showed that, given a $\mathcal{P}$-Brelot harmonic space various vector spaces of harmonic functions coincide if and only if they are finite dimensional. We give two examples which satisfy the above property by setting up appropriate differential elliptic operator on the plane with finitely many punctures.

§ 1. Introduction

Let $(\Omega, \mathcal{H})$ be a $\mathcal{P}$-Brelot harmonic space. Suppose that there exists a countable base for the open sets of Ω and that constant functions are harmonic on Ω. For an open set ω, denote by $H(\omega)$ the set of harmonic functions on ω. Set $HP_+(\Omega) = \{h \in H(\Omega) \mid h \geq 0 \text{ on } \Omega\}$, $HP(\Omega) = \{h_1 - h_2 \mid h_j \in HP_+(\Omega), j = 1, 2\}$ and $HB(\Omega) = \{h \in H(\Omega) \mid h \text{ is bounded on } \Omega\}$. Set

$$\mathcal{N} = \{\Phi : [0, +\infty) \to [0, +\infty) \mid \Phi \text{ is convex and strictly increasing}, \Phi(0) = 0, \text{ and } \lim_{t \to +\infty} \frac{\Phi(t)}{t} = +\infty\}.$$

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For $\Phi \in \mathcal{N}$, set

$$H_\Phi(\Omega) = \{h \in H(\Omega) \mid \text{there exists } \alpha > 0 \text{ such that } \Phi(\alpha|h|) \text{ has a harmonic majorant on } \Omega\}.$$

The space $H_\Phi(\Omega)$ is called the harmonic Hardy-Orlicz space associated to Φ.

In [11] we established the following two results:

Theorem 1.1. *Let Φ and Ψ be elements of $\mathcal{N}$. Suppose that*

$$\limsup_{t \to +\infty} \frac{\Phi(\alpha t)}{\Psi(t)} = +\infty$$

for all positive α. Then the following four conditions are equivalent:

(i) *$H_\Phi(\Omega)$ and $H_\Psi(\Omega)$ coincide;*

(ii) $\dim H_\Psi(\Omega) < +\infty$;

(iii) $\dim H_\Phi(\Omega) < +\infty$;

(iv) $\dim HB(\Omega) < +\infty$.

Theorem 1.2. *Let $\Phi \in \mathcal{N}$. Then the following two conditions are equivalent.*

(i) *$H_\Phi(\Omega)$ and $HP(\Omega)$ coincide;*

(ii) $\dim H_\Phi(\Omega) = \dim HP(\Omega) < +\infty$.

In this paper we give two concrete examples of harmonic spaces that clarify the difference between the above theorems. In the first example the conditions of Theorems 1.1 and 1.2 hold. In the second example we slightly modify the first example so that the claim of Theorem 1.1 holds, but that of Theorem 1.2 fails. These examples are constructed via appropriate elliptic differential operators on the plane with finitely many punctures.

§ 2. Preliminaries

Let Ω be a locally compact, non-compact, connected and locally connected Hausdorff space. Let $\mathcal{H}$ be a class of real-valued continuous functions, called harmonic functions, on open subsets of Ω satisfying the following three axioms:

Axiom 1. $\mathcal{H}$ forms a linear sheaf.

Axiom 2. There is a base for the topology of Ω such that each set in the base is a regular domain for $\mathcal{H}$.

Axiom 3. For any domain U in Ω, any ordered increasing directed family of harmonic functions defined on U has an upper envelope which is either $+\infty$ everywhere in U or harmonic in U.

A pair $(\Omega, \mathcal{H})$ satisfying the above properties is called a *Brelot harmonic space* (cf. [1], [2]). Furthermore, we assume that there exists a positive potential on Ω. Such a Brelot harmonic space is called a $\mathcal{P}$-Brelot harmonic space (cf. [2]). For an open set $U \subset \Omega$, set $H(U) = \{u \in \mathcal{H} : u \text{ is harmonic in } U\}$. Throughout this paper we further assume that there exists a countable base for the open sets of Ω and that the constant function $h(x) \equiv 1$ belongs to $H(\Omega)$.

For a $\mathcal{P}$-Brelot harmonic space $(\Omega, \mathcal{H})$, we define the various classes of harmonic functions described in the introduction analogously, replacing harmonic functions by elements of $\mathcal{H}$.

Let S^+ be the set of non-negative superharmonic functions on Ω, defined, as usual, via a comparison principle. Denote by $\Delta_1^M = \Delta_1^{\Omega,M}$ the set of extreme harmonic functions (or minimal harmonic functions after Martin) of a compact and metrizable base Λ of the cone S^+. As usual, we call it the *minimal Martin boundary* of Ω. We refer to [2], [3], [5], [6] and [7] for a detailed discussion on the minimal Martin boundary and on the Martin boundary. The intuitive picture to have in mind is to consider points in the boundary of the unit disk both as points and Poisson kernels (associated to these points).

The following Martin representation theorem (cf. [2],[3],[7]) is the fundamental result regarding Martin boundaries. It also explains why we concentrate on the minimal Martin boundary Δ_1^M.

Theorem 2.1 (Martin representation theorem). *For each $u \in HP_+(\Omega)$, there is a unique positive measure μ_u on the Martin boundary Δ^M so that $\mu_u(\Delta^M \setminus \Delta_1^M) = 0$ and*

$$u(z) = \int_{\Delta_1^M} h(z)\ d\mu_u(h).$$

Let us introduce some further concepts that will be needed in what follows. Let ω be a subdomain of Ω such that every boundary point of ω is regular. Denote by $HP_+(\omega)$ the set of non-negative harmonic functions on ω. Set

$$HP_+(\Omega, \omega) = \{u \in HP_+(\Omega) \mid \hat{R}_u^{\Omega\setminus\omega} \text{ is a potential}\},$$

where $\hat{R}_h^{\Omega\setminus\omega}$ is the balayage of h relative to $\Omega \setminus \omega$, and

$$HP_+^0(\omega, \Omega) = \{U \in HP_+(\omega) \mid \text{there is } h \in HP_+(\Omega) \text{ with } U \le h \text{ on } \omega \text{ and } \lim_{x\to\partial\omega} U(x) = 0\}.$$

Let h be an element of $HP_+(\Omega,\omega)$. Set $S(h) = S_\omega(h) = (h - \hat{R}_h^{\Omega\setminus\omega})|_\omega$. Clearly $S(h) \in HP_+^0(\omega,\Omega)$. Let H be an element of $HP_+^0(\omega,\Omega)$. Set

$$R(H)(x) = R_\omega(H)(x) = \inf\{s(x) \mid s \in S^+ \text{ with } H \le s \text{ on } \omega\}.$$

Clearly $R(H) \in HP_+(\Omega)$.

Proposition 2.2. *Let ω be a subdomain of Ω such that every boundary point of ω is regular. $S_\omega : HP_+(\Omega,\omega) \to HP_+^0(\omega,\Omega)$ is a bijection and its inverse map is R_ω.*

Proof. In this proof, we abbreviate $R = R_\omega$ and $S = S_\omega$. We may suppose that $HP_+(\Omega,\omega) \ne \{0\}$. For, suppose that $HP_+^0(\omega,\Omega) \ne \{0\}$. From the discussion of next paragraph it follows that $HP_+(\Omega,\omega) \ne \{0\}$. Hence, if $HP_+(\Omega,\omega) = \{0\}$, then $HP_+^0(\omega,\Omega) = \{0\}$. Hence, the statement of proposition holds.

We show first that $R(F) \in HP_+(\Omega,\omega)$ if $F \in HP_+^0(\omega,\Omega)$. To this end, take a $u \in HP_+(\Omega)$ with $u \le \hat{R}_{R(F)}^{\Omega\setminus\omega}$ on Ω. Set

$$\tilde{F}(x) = \begin{cases} F(x) & \text{if } x \in \omega, \\ 0 & \text{if } x \in \Omega \setminus \omega. \end{cases}$$

Then it is easily seen that $\tilde{F}$ is subharmonic on Ω. Hence $R(F) - \tilde{F}$ is non-negative and superharmonic on Ω. By definition $R(F) - \tilde{F} = R(F)$ on $\Omega \setminus \omega$. Hence

$$R(F) - \tilde{F} \ge \hat{R}_{R(F)}^{\Omega\setminus\omega} \ge u \text{ on } \Omega$$

and $R(F) - u \ge \tilde{F} = F$ on ω. Since $R(F) - u \in HP_+(\Omega)$, $R(F) - u \ge R(F)$ on Ω. Therefore, $u \le 0$, so that $u = 0$ on Ω. Consequently, $\hat{R}_{R(F)}^{\Omega\setminus\omega}$ is a potential on Ω. Hence $R(F) \in HP_+(\Omega,\omega)$.

To prove the statement of the proposition it suffices to show that $R \circ S = id|_{HP_+(\Omega,\omega)}$ and $S \circ R = id|_{HP_+^0(\omega,\Omega)}$.

First we prove that $R \circ S = id|_{HP_+(\Omega,\omega)}$. Take an element $f \in HP_+(\Omega,\omega)$. Then $\hat{R}_f^{\Omega\setminus\omega}$ is a potential on Ω. Now $R(S(f)) \ge S(f) = f - \hat{R}_f^{\Omega\setminus\omega}$ on ω. Since $\partial\omega$ consists of only regular boundary points, $\hat{R}_f^{\Omega\setminus\omega} = f$ on $\Omega \setminus \omega$. Hence $R(S(f)) + \hat{R}_f^{\Omega\setminus\omega} \ge f$ on Ω. Since $R(S(f))$ and f are harmonic on Ω, and $\hat{R}_f^{\Omega\setminus\omega}$ is a potential on Ω, by the Riesz decomposition theorem, we find that $R(S(f)) \ge f$ on Ω. The converse inequality holds in general. Hence $R(S(f)) = f$ on Ω.

Next we prove that $S \circ R = id|_{HP_+^0(\omega,\Omega)}$. By the discussion in the first paragraph of this proof we infer that $R(F) - \tilde{F} \ge \hat{R}_{R(F)}^{\Omega\setminus\omega}$ on Ω whenever $F \in HP_+^0(\omega,\Omega)$. For the reverse inequality we only need to check that $\tilde{F} + \hat{R}_{R(F)}^{\Omega\setminus\omega}$ is non-negative and superharmonic on Ω because $\tilde{F} + \hat{R}_{R(F)}^{\Omega\setminus\omega} \ge F$ on ω. We easily find that $\tilde{F} + \hat{R}_{R(F)}^{\Omega\setminus\omega}$ is lower semi-continuous on Ω and that it is harmonic on $\Omega \setminus \partial\omega$. Take any point $x_0 \in \partial\omega$ and regular domain U with $x_0 \in U$. Then, we have

$$(\tilde{F} + \hat{R}^{\Omega\setminus\omega}_{R(F)})(x_0) = R(F)(x_0) = \int_{\partial U} R(F)(x) d\omega^U_{x_0} \geq \int_{\partial U} (\tilde{F} + \hat{R}^{\Omega\setminus\omega}_{R(F)})(x) d\omega^U_{x_0},$$

where $\omega^U_{x_0}$ is the harmonic measure relative to U and x_0.

This implies that $\tilde{F} + \hat{R}^{\Omega\setminus\omega}_{R(F)}$ is non-negative and superharmonic on Ω. Hence we find that $R(F) - \tilde{F} = \hat{R}^{\Omega\setminus\omega}_{R(F)}$ on Ω. Thus $S \circ R = id|_{HP^0_+(\omega,\Omega)}$. □

Definition 2.3. Let ω be subdomain of Ω such that every boundary point of ω is regular, and let $H \in HP^0_+(\omega, \Omega)$. Then, if any non-negative harmonic minorant of H on ω is proportional to H, H is called a *minimal* element of $HP^0_+(\omega, \Omega)$.

Proposition 2.4. *Let ω be a subdomain of Ω such that every boundary point of ω is regular. Suppose that $HP_+(\Omega, \omega) \neq \{0\}$ and let $h \in HP_+(\Omega, \omega)$. Then h is a minimal element of $HP_+(\Omega)$ if and only if $S(h)$ is a minimal element of $HP^0_+(\omega, \Omega)$.*

Proof. Write $R = R_\omega$ and $S = S_\omega$. Suppose first that h is a minimal element of $HP_+(\Omega)$. Let F be a non-negative harmonic minorant of $S(h)$ on ω. Then $0 \leq R(F) \leq h$ on Ω and, by minimality, there exists a constant $\alpha \geq 0$ such that $R(F) = \alpha h$ on Ω. By Proposition 2.2

$$F = S(R(F)) = S(\alpha h) = \alpha S(h).$$

Hence $S(h)$ is a minimal element of $HP^0_+(\omega, \Omega)$.

To prove the converse, suppose that $S(h)$ is a minimal element of $HP^0_+(\omega, \Omega)$. Let f be a non-negative harmonic minorant of h on Ω. Since $S(f) \leq S(h)$ on ω, by minimality, there exists a non-negative constant α such that $S(f) = \alpha S(h)$ on ω. By Proposition 2.2,

$$f = R(S(f)) = R(\alpha S(h)) = \alpha R(S(h)) = \alpha h.$$

Hence h is a minimal element of $HP_+(\Omega)$. □

Proposition 2.5. *Let ω_j, $j = 1, 2$, be subdomains of Ω such that all boundary points of ω_j are regular and that $HP_+(\Omega, \omega_j) \neq \{0\}$. If $\omega_1 \cap \omega_2 = \emptyset$, then*

$$HP_+(\Omega, \omega_1) \cap HP_+(\Omega, \omega_2) = \{0\}.$$

Proof. Suppose that $HP_+(\Omega, \omega_1) \cap HP_+(\Omega, \omega_2) \neq \{0\}$. Take any minimal element ν of $HP_+(\Omega, \omega_1) \cap HP_+(\Omega, \omega_2)$. Then, $\hat{R}^{\Omega\setminus\omega_j}_\nu$ $(j = 1, 2)$ is a potential on Ω. Hence, by [3, Hilfssatz 11.2](cf. [14]),

$$\hat{R}^{\Omega\setminus\omega_j}_\nu \neq \nu \quad (j = 1, 2).$$

Since $\omega_1 \cap \omega_2 = \emptyset$ and all boundary points of ω_1 are regular, $\hat{R}_\nu^{\Omega\setminus\omega_1} = \nu \geq S_{\omega_2}(\nu)$ on ω_2. By the definition of R_{ω_2},

$$\hat{R}_\nu^{\Omega\setminus\omega_1} \geq R_{\omega_2}(S_{\omega_2}(\nu)) = \nu$$

on Ω. Hence $\hat{R}_\nu^{\Omega\setminus\omega_1} = \nu$ on Ω. This contradicts our assumption. □

§ 3. Examples

Example 3.1. Given an integer $q \geq 2$, let

$$\Omega_q = \mathbb{R}^2 \setminus \bigcup_{j=1}^{q-1} \{(3j, 0)\}.$$

We construct such a harmonic space on Ω_q that the minimal Martin boundary consists of exactly q elements and that the conditions (i) - (iv) of Theorem 1.1 and (i)-(ii) of Theorem 1.2 hold. Towards this end, denote by $B_r(x)$ the disk with center x and radius r. Set $D_0 = \mathbb{R}^2 \setminus \overline{B}_{3q+2}(0)$ and

$$D_j = B_1((3j,0)) \setminus \{(3j,0)\} = \{(x,y) \in \mathbb{R}^2 \mid 0 < (x-3j)^2 + y^2 < 1\}$$

for each $1 \leq j \leq q-1$. We endow Ω_q with the usual Euclidean topology.

Fix $\epsilon > 0$ and set

$$d\mu(x) = \begin{cases} (2/(1+3q+2))^\epsilon (1+|x|)^\epsilon dx & \text{if } x \in D_0, \\ (1+|x-(3j,0)|^{-1})^\epsilon dx & \text{if } x \in D_j,\ j = 1, \cdots, q-1, \\ 2^\epsilon dx & \text{if } x \notin \bigcup_{j=0}^{q-1} D_j. \end{cases} \tag{3.1}$$

For $z \in \Omega_q$, we define

$$L_\mu = \begin{cases} (2/(1+3q+2))^\epsilon div((1+|x|)^\epsilon \nabla) & \text{if } x \in D_0, \\ div((1+|x-(3j,0)|^{-1})^\epsilon \nabla) & \text{if } x \in D_j,\ j = 1, \cdots, q-1, \\ 2^\epsilon \Delta & \text{if } x \notin \bigcup_{j=0}^{q-1} D_j. \end{cases} \tag{3.2}$$

Then L_μ is a second order elliptic differential operator of divergence form on Ω_q.

We choose $\mathcal{H}$ to consist of (weak) solutions to $L_\mu u = 0$ on open subsets of Ω_q, and claim that $(\Omega_q, \mathcal{H})$ is a $\mathcal{P}$-Brelot harmonic space with a countable base, and that the minimal Martin boundary $\Delta_{1,\mu}^M$ consists of exactly q points.

First we check that $(\Omega_q, \mathcal{H})$ is a $\mathcal{P}$-Brelot harmonic space.

Axiom 1 is clear. Then we check Axiom 2. Clearly, L_μ is a second order differential operator of divergence form and locally uniformly elliptic on Ω_q. Take any $z \in \Omega_q$ and $B_r(z)$ with $\overline{B_r(z)} \subset \Omega_q$.

By [12] (see also [4], [7], [10]), for any Lipschitz continuous function $f \in \partial B_r(z)$, there exists a weak solution $u \in W^{1,2}(B_r(z))$ to $L_\mu u = 0$ on $B_r(z)$, so that $u|_{\partial B_r(z)} = f$. Moreover, if $f \geq 0$ on $\partial B_r(z)$, then $u \geq 0$ on $B_r(z)$. The discussion in [10, pp. 5–6] gives us the existence of a harmonic measure $\omega_x^{B_r(z)}$ $(x \in B_r(z))$ such that, for $f \in C(\partial B_r(z))$, $v(x) := \int_{\partial B_r(z)} f d\omega_x^{B_r(z)}$ is a weak solution on $B_r(z)$ to $L_\mu u = 0$, continuous up to boundary of $B_r(z)$, and $v|_{\partial B_r(z)} = f$ on $\partial B_r(z)$. We conclude that Axiom 2 is satisfied.

By Moser's theorem ([13], [4], [7]), for any relatively compact subdomain $G \subset \Omega_q$, the Harnack inequality holds on G with respect to L_μ. Axiom 3 follows from this fact when combined with Axiom 1.

Next we prove that there exists a potential f on Ω_q. For this, we employ (weighted) nonlinear potential theory [7]. Set $w(x) = (1 + |x|)^\epsilon$ and $\mu(E) = \int_E w(x)dx$. Fix $1 < p < 2$, and set

$$f(x) = \begin{cases} x & \text{for } |x| \leq 1, \\ x|x|^\gamma & \text{for } |x| > 1, \end{cases} \tag{3.3}$$

where $\gamma = \epsilon/(2 - p)$.

Then, $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a quasiconformal mapping and the Jacobian determinant $J_f(x)$ of f at $x (\in \mathbb{R}^2)$ satisfies

$$J_f^{1-p/2}(x) \approx \begin{cases} 1 & \text{for } |x| \leq 1 \\ |x|^\epsilon & \text{for } |x| > 1 \end{cases} = \max\{1, |x|^\epsilon\}. \tag{3.4}$$

By [7], $J_f^{1-p/2}$ generates a doubling measure that supports a p-Poincaré inequality and, in particular a 2-Poincaré inequality. Since w is comparable to $J_f^{1-p/2}$, it easily follows that the doubling property and the 2-Poincaré inequality hold for w as well. In the terminology of [7], this means that w is 2-admissible and hence the full theory of [7] is at our disposal.

Let $B(r) = B_r(0)$ be the disk with center 0 and radius $r > 0$. In order to prove the existence of a nonconstant positive potential, we first prove that there exists a nonnegative harmonic function f_0 on $\mathbb{R}^2 \setminus \overline{B(3q + 2)}$ with respect to L_μ such that $f_0 = 1$ on $\partial B(3q + 2)$ and $\lim_{x\to\infty} f_0(x) = 0$. To see this, by the statement and proof of [7, Theorem 9.22] (see also [9]), it suffices to prove that $\mathrm{cap}_{2,\mu}(\overline{B(3q + 2)}, \mathbb{R}^2) > 0$. Here, given an open set G and a compact set $E \subset G$,

$$\mathrm{cap}_{2,\mu}(E, G) = \inf_{v\in W(E,G)} \int_G |\nabla v|^2 d\mu, \tag{3.5}$$

where $W(E,G)) = \{v \in C_0^\infty(G) : v \geq 1 \text{ on } E\}$.

Set $A(r,R) = B(R) \setminus \overline{B(r)}$. By [7, Theorem 2.18], for $3q+2 < R$,

$$\begin{aligned}
\operatorname{cap}_{2,\mu}(\overline{B(3q+2)}, B(R)) &\geq (2\pi)^2 \Big(\int_{A(3q+2,R)} |x|^{-2}(1+|x|)^{-\epsilon} dx\Big)^{-1} \\
&= 2\pi\Big(\int_{3q+2}^{R} r^{-2}(1+r)^{-\epsilon} r dr\Big)^{-1} \\
&> 2\pi\Big(\int_{3q+2}^{\infty} r^{-1-\epsilon} dr\Big)^{-1} \\
&= 2\pi\epsilon(3q+2)^{\epsilon}.
\end{aligned}$$

Since the lower bound is independent of R, we conclude that

$$\operatorname{cap}_{2,\mu}(\overline{B(3q+2)}, \mathbb{R}^2) \geq 2\pi\epsilon(3q+2)^{\epsilon} > 0, \tag{3.6}$$

and the existence of f_0 is shown.

By the reflection and a discussion similar to that above we obtain, for each $1 \leq j \leq q-1$, a non-negative harmonic function f_j on D_j with respect to L_μ so that $f_j = 1$ on ∂D_j and $\lim_{x\to(3j,0)} f_j(x) = 0$.

Set

$$\varphi(x) = \begin{cases} 1 & \text{if } x \in \overline{B(3q+2)} \setminus \bigcup_{j=1}^{q-1} D_j, \\ f_j(x) & \text{if } x \in D_j,\ 0 \leq j \leq q-1. \end{cases} \tag{3.7}$$

Then φ is positive superharmonic function on Ω_q with respect to L_μ. Moreover, φ is a potential on Ω_q. Indeed, let u be a non-negative L_μ-harmonic function on Ω_q with $u \leq \varphi$ on Ω_q. Since

$$\lim_{x\to\infty} \varphi(x) = 0$$

and

$$\lim_{x\to(3j,0)} \varphi(x) = 0$$

for all j, the maximum principle yields that $u = 0$.

Consequently, $(\Omega_q, \mathcal{H})$ is a $\mathcal{P}$-Brelot harmonic space with a countable base.

Next we prove that $\Delta_{1,\mu}^{\Omega_q,M}$ consists of exactly q points. Set

$$HP_+^0(D_0) = \{u \in HP_+(D_0) \mid \lim_{x\to\partial D_0} u(x) = 0\}.$$

First we prove that $HP_+^0(D_0,\Omega_q) = HP_+^0(D_0)$ and that $HP_+^0(D_0,\Omega_q)$ has only one minimal element up to proportionality. To show that $HP_+^0(D_0,\Omega_q) = HP_+^0(D_0)$ it suffices to prove that each element of $HP_+^0(D_0)$ is bounded.

Take an element u of $HP_+^0(D_0)$. For $r > 0$, write C_r for the circle of radius r centered at the origin 0. By the Harnack inequality we find that there exists a positive γ independent of u and $r > 3q + 6$ so that

$$\frac{1}{\gamma}u(x) \le u(y) \le \gamma u(x) \quad \text{for all } x, y \in C_r. \tag{3.8}$$

Set

$$A_r = A(r, 3q + 2) = B(r) \setminus \overline{B(3q + 2)}$$

and $x_0 = (3q+4, 0)$. Let $\omega_{x_0}^{A_r}$ be the L_μ-harmonic measure relative to A_r and x_0. Then

$$u(x_0) = \int_{C_r} u d\omega_{x_0}^{A_r}. \tag{3.9}$$

Recall that there exists a non-negative harmonic function f_0 on $\mathbb{R}^2 \setminus \overline{B(3q + 2)}$ with respect to L_μ such that $f_0 = 1$ on $\partial B(3q + 2)$ and $\lim_{x\to\infty} f_0(x) = 0$. Set $g_0 = 1 - f_0$. By the maximum principle and symmetry of L_μ, we find that $g_0(x) = g_0(y)$, for every $x, y \in C_r$. Replacing u by g_0 in (3.9) we obtain

$$g_0(x_0) = \int_{C_r} g_0 d\omega_{x_0}^{A_r} = g_0((r, 0))\omega_{x_0}^{A_r}(C_r). \tag{3.10}$$

By (3.8), (3.9) and (3.10) we have, for any $x \in C_r$,

$$\frac{g_0(x_0)}{\gamma g_0((r, 0))}u(x) \le u(x_0) = \int_{\partial A_r} u d\omega_{x_0}^{A_r} \le \frac{\gamma g_0(x_0)}{g_0((r, 0))}u(x). \tag{3.11}$$

Since $\lim_{x\to\infty} g_0(x) = 1$, there exists positive number $r_0 > 3q + 6$ with

$$1/2 < g_0((r, 0)) \le 1 \text{ for } r \ge r_0.$$

Hence we have

$$\frac{u(x_0)}{2\gamma g_0(x_0)} \le u(x) \le \frac{\gamma u(x_0)}{g_0(x_0)},$$

for any $x \in \mathbb{R}^2 \setminus \overline{B(r_0)}$. Thus, by the maximum principle, u is bounded on D_0 and hence

$$HP_+^0(D_0) \subset HP_+^0(D_0, \Omega_q).$$

Since the converse inclusion holds in general, we have that $HP_+^0(D_0, \Omega_q) = HP_+^0(D_0)$.

Next, let h be a minimal element of $HP_+^0(D_0, \Omega_q)$. Take any element u of $HP_+^0(D_0, \Omega_q)$. By the above discussion, we have, for each $x \in \mathbb{R}^2 \setminus \overline{B(r_0)}$, that

$$\frac{u(x_0)}{2\gamma g_0(x_0)} \le u(x) \le \frac{\gamma u(x_0)}{g_0(x_0)}, \tag{3.12}$$

and

$$\frac{h(x_0)}{2\gamma g_0(x_0)} \le h(x) \le \frac{\gamma h(x_0)}{g_0(x_0)}. \tag{3.13}$$

Hence, for all $x \in \mathbb{R}^2 \setminus \overline{B(r_0)}$

$$u(x) \le \frac{2\gamma^2 u(x_0)}{h(x_0)} h(x). \tag{3.14}$$

By maximum principle we have, for every $x \in D_0$,

$$u(x) \le \frac{2\gamma^2 u(x_0)}{h(x_0)} h(x). \tag{3.15}$$

By minimality, there exists a positive c with $u = ch$ on D_0. Hence, every element of $HP_+^0(D_0, \Omega_q)$ is proportional to h. This means that $HP_+^0(D_0, \Omega_q)$ has only one minimal element up to proportionality.

Similarly, we arrive at the conclusion that each $HP_+^0(D_i, \Omega_q)$ has only one minimal point up to proportionality.

Next we prove that $\sharp\Delta_{1,\mu}^{\Omega_q,M} \le q$. To prove this it suffices to show that, for each minimal element ν of $HP_+(\Omega_q)$, there exists a unique integer $0 \le j \le q-1$ such that $\nu \in HP_+(\Omega_q, D_j)$. For, if another minimal element ν' of $HP_+(\Omega_q)$ satisfies $\nu' \in HP_+(\Omega_q, D_j)$, both $S_{D_j}(\nu)$ and $S_{D_j}(\nu')$ are minimal elements of $HP_+(\Omega_q, D_j)$ by Proposition 2.4. Since $HP_+(\Omega_q, D_j)$ has only one minimal element up to proportionality, there exists a positive constant α with $S_{D_j}(\nu') = \alpha S_{D_j}(\nu)$, and hence, by Proposition 2.2, $\nu' = \alpha\nu$ and we arrive at the desired inequality $\sharp\Delta_{1,\mu}^{\Omega_q,M} \le q$.

It remains to prove the claim from the preceding paragraph. To this end, we only have to prove that, for any minimal element ν of $HP_+(\Omega_q)$, there exists a unique j such that

$$\hat{R}_\nu^{\Omega_q \setminus D_j} \ne \nu,$$

since $\hat{R}_\nu^{\Omega_q \setminus D_j}$ is a potential if and only if $\hat{R}_\nu^{\Omega_q \setminus D_j} \ne \nu$ (cf. [3, Hilfssatz 11.2], [14]). For this, take any minimal element ν of $HP_+(\Omega_q)$. Set $K = \overline{B(3q+2)} \setminus \cup_{i=1}^{q-1} D_i$. Since K is compact, $\hat{R}_\nu^K$ is a potential on Ω_q (cf. [2, Proposition 5.3.5]), and hence $\hat{R}_\nu^K \ne \nu$. Thus, there exists a j with $\hat{R}_\nu^K \ne \nu$ on D_j, that is, $\hat{R}_\nu^{\Omega_q \setminus D_j} \ne \nu$ on D_j (cf. [2, Proposition 5.3.3]). Suppose that there exists another j' with $\hat{R}_\nu^K \ne \nu$ on $D_{j'}$. We remark that every boundary point of D_i is regular. Then, on $D_j (\subset \Omega_q \setminus D_{j'})$, $\hat{R}_\nu^{\Omega_q \setminus D_{j'}} = \nu \ge S_{D_j}(\nu)$. By the definition,

$$\hat{R}_\nu^{\Omega_q \setminus D_{j'}} \ge R_{D_j}(S_{D_j}(\nu)) = \nu$$

on Ω_q, and hence, $\hat{R}_\nu^{\Omega_q \setminus D_{j'}} = \nu$ on Ω_q. Since $\hat{R}_\nu^K = \hat{R}_\nu^{\Omega_q \setminus D_{j'}}$ on $D_{j'}$, this contradicts our assumption.

Finally we prove that $\sharp\Delta_{1,\mu}^{\Omega_q,M} \geq q$. Take a minimal element h_j of $HP_+^0(D_j, \Omega_q)$. By Proposition 2.4 there exists a minimal element $R_{D_j}(h_j)$ of $HP_+(\Omega_q, D_j)$. Since $D_i \cap D_j = \emptyset$ $(i \neq j)$, by Proposition 2.5

$$HP_+(\Omega_q, D_i) \cap HP_+(\Omega_q, D_j) = \{0\},$$

that is, $R_{D_i}(h_i)$ and $R_{D_j}(h_j)$ are minimal elements of $HP_+(\Omega_q)$, but they are not proportional each other. Hence $\sharp\Delta_{1,\mu}^{\Omega_q,M} \geq q$.

In conclusion, $\sharp\Delta_{1,\mu}^{\Omega_q,M} = q$.

By the above discussion in the proof of that $\sharp\Delta_{1,\mu}^{\Omega_q,M} = q$, we find that $HP_+(\Omega_q)$ has exactly q minimal elements $h_0, \cdots, h_{q-1}$ such that each $S_{D_j}(h_j)$ is a minimal element of $HP_+^0(D_j, \Omega_q)$. By the Martin representation theorem, we find that, if $u \in HP_+(\Omega_q)$, there exist non-negative constants α_j with

$$u = \sum_{j=0}^{q-1} \alpha_j h_j.$$

Since each h_j is bounded on Ω_q, $u \in HB(\Omega_q)$ whence $HP(\Omega_q) \subset HB(\Omega_q)$. It is well-known that $HB(\Omega_q) \subset H_\Phi(\Omega_q) \subset HP(\Omega_q)$. Thus

$$HB(\Omega_q) = H_\Phi(\Omega_q) = HP(\Omega_q).$$

Therefore this $\mathcal{P}$-Brelot harmonic space $(\Omega_q, \mathcal{H})$ gives us an example of a setting where the conditions both of Theorems 1.1 and 1.2 hold.

Example 3.2. Let Ω_q, $q \geq 2$, be as in Example 3.1. Pick $y_0 = (3q, 0)$ and write $\Omega = \Omega_q \setminus \{y_0\}$. In this example we consider harmonic space $(\Omega, \mathcal{H})$, where the harmonic sheaf $\mathcal{H}$ is that inherited from $(\Omega_q, \mathcal{H})$ in Example 3.1. We show that in this setup the conditions of Theorem 1.1 hold, but those of Theorem 1.2 fail.

Let D_j, $0 \leq j \leq q-1$, be as in Example 3.1 and let

$$D_q = B_1(y_0) \setminus \{y_0\}.$$

It is well-known that

$$H_q(x) = \log |x - y_0|^{-1}$$

is the unique minimal element of $HP_+^0(D_q)$ up to proportionality. Set

$$D_{0,n} = A(n + 3q + 2, 3q + 2),$$
$$D_{j,n} = B_{1/n}((3j, 0)) \setminus \{(3j, 0)\}$$

and

$$F_n = B_{n+3q+2}(0) \setminus (D_q \cup (\bigcup_{j=1}^{q-1} \overline{D_{j,n}})).$$

Set

$$k_n(x) = \begin{cases} 1 & \text{if } x \in \partial D_q \cap \partial F_n, \\ 0 & \text{if } x \in \partial D_{j,n} \cap \partial F_n, \end{cases}$$

and let χ_n be the Dirichlet solution for k_n on F_n. For any integer l, a sequence $\{\chi_n\}_{n=l}^{\infty}$ is increasing and uniformly bounded on F_l, and hence there exists a harmonic function χ on $\Omega \backslash \overline{D_q}$ such that $\{\chi_n\}_{n=1}^{\infty}$ converges to χ locally uniformly on $\Omega \backslash \overline{D_q}$. For a positive number β, set

$$v_\beta(x) = \begin{cases} \chi(x) & \text{if } x \in \Omega \setminus \overline{D_q}, \\ \beta H_q + 1 & \text{if } x \in \overline{D_q}. \end{cases}$$

Then, by the symmetry principle and the minimum principle we can find an appropriate constant β_0 such that v_{β_0} is superharmonic on Ω. Hence $HP_+^0(D_q, \Omega) = HP_+^0(D_q)$. By Proposition 2.4 we get a minimal harmonic function $R(H_q)$ on Ω. Thus there exists positive constant c_1 with $c_1 R(H_q) \in \Delta_{1,\mu}^{\Omega,M}$. Set $h_q = c_1 R_{H_q}$.

By the same argument as that in Example 3.1, we find that $\sharp \Delta_{1,\mu}^{\Omega,M} = q+1$. Let h_j, $0 \le j \le q-1$ be as in Example 3.1. Then, each h_j is identified with a minimal element of $HP_+(\Omega_q)$. For this, suppose that u is a non-negative minimal harmonic function on Ω_q. Then u is a positive harmonic function on Ω. Suppose that v is a positive harmonic function on Ω and $v \le u$ on Ω. Since $\{y_0\}$ is a polar set of Ω_q and v is bounded on a punctured neighborhood of y_0, v has an unique extension v' on y_0 such that $v' \in H(\Omega_q)$ by the removability theorem. Thus $v' \le u$ on Ω_q. By minimality of u we can find a positive constant α with $v' = \alpha u$ on Ω_q, that is, $v = \alpha u$ on Ω. This means that u is a minimal element of $HP_+(\Omega)$, whence each h_j is identified with a minimal element of $HP_+(\Omega_q)$. Consequently,

$$\Delta_{1,\mu}^M(\Omega) = \Delta_{1,\mu}^M(\Omega_q) \cup \{h_q\} = \{h_0, h_1, \dots, h_q\}.$$

By a similar argument as in the preceding paragraph we find that each $h \in HB(\Omega)$ can be identified with an element from $HB(\Omega_q)$. Therefore $\dim HB(\Omega) = q < \infty$ by Example 3.1; and consequently for every $\Phi \in \mathcal{N}$,

$$HB(\Omega) = H_\Phi(\Omega)$$

by Theorem 1.1. However the conditions of Theorem 1.2 do not hold, since

$$h_q \in HP(\Omega) \setminus HB(\Omega).$$

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