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# An isoperimetric inequality for extremal Sobolev functions

By

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**Abstract**

Let $D \subset \mathbb{R}^n$ be a bounded domain with a Lipschitz boundary, let $1 < p < \frac{2n}{n-2}$, and let $\phi$ minimize the ratio $\|\nabla u\|_{L^2(D)}/\|u\|_{L^p(D)}$ over all functions $u$ vanishing on the boundary of $D$. After presenting a short survey of some results on the value of this minimum, the Sobolev constant, we present a proof of a reverse Hölder inequality for the eigenfunction $\phi$, finding a lower bound for $\|\phi\|_{L^{p-1}}$ in terms of $\|\phi\|_{L^p}$. This result generalizes an inequality due to Payne and Rayner [16, 17] regarding eigenfunctions of the Laplacian.

## § 1. Introduction and statement of results

The eigenvalue problem for the Dirichlet Laplacian $-\Delta$ plays a fundamental role in the study of solutions of both the wave equation and the heat equation in a bounded region $D$ in $\mathbb{R}^n$. In the case of the wave equation, the smallest eigenvalue $\lambda(D)$ determines the fundamental frequency of the domain when the boundary is clamped. In the case of the heat equation, it determines the slowest heat dissipation rate for an initial temperature distribution in a body of shape $D$ when the boundary is held at temperature zero. The principal eigenvalue $\lambda(D)$ is positive and the corresponding eigenfunctions $\phi$ have constant sign. It is the smallest $\lambda$ for which there is a non-trivial solution of the eigenvalue problem $\Delta u + \lambda u = 0$ in $D$ with zero Dirichlet boundary conditions, and can also be viewed in terms of the energy minimization problem

$$\lambda(D) = \inf\left\{ \frac{\int_D |\nabla u|^2\, d\mu}{\int_D u^2\, d\mu} : u \in C_0^\infty(D),\ u \not\equiv 0 \right\}.$$

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Here $d\mu$ refers to Lebesgue measure on $\mathbb{R}^n$.

Minimization of the energy integral $\int_D |\nabla u|^2\, d\mu$ normalized by the $L^1(D)$ norm of the test function $u$ leads to another important quantity in mathematical physics, namely the torsional rigidity. It may be defined by

$$P(D) = 4 \sup \left\{ \frac{\left(\int_D u\, d\mu\right)^2}{\int_D |\nabla u|^2\, d\mu} : u \in C_0^\infty(D),\ u \not\equiv 0 \right\}.$$

The torsional rigidity $P(D)$ of a bounded, simply connected region $D$ in the plane is a measure of the strength under torsion of a beam which has $D$ as its cross section. The corresponding extremal function $\phi$ is known as the torsion function and satisfies $\Delta\phi + 2 = 0$ with zero Dirichlet data. The partial derivatives of $\phi$ give the stresses in the beam under torsion. The solution of this boundary value problem also returns the expected exit time of standard Brownian motion from $D$.

One may work with a more general Rayleigh quotient. Let $1 \leq p < \frac{2n}{n-2}$ (or any $p \geq 1$ if $n = 2$). For this range of exponents, the Sobolev embedding $W_0^{1,2}(D) \hookrightarrow L^p(D)$ is compact, and so the infimum

$$\mathcal{C}_p(D) = \inf \left\{ \frac{\int_D |\nabla u|^2 d\mu}{\left(\int_D |u|^p d\mu\right)^{2/p}} : u \in W_0^{1,2}(D), u \not\equiv 0 \right\} \tag{1.1}$$

is finite and achieved by a nontrivial function $\phi = \phi_p$. Here the Sobolev space $W_0^{1,2}(D)$ is the closure of $\mathcal{C}_0^\infty(D)$ under the norm $\|u\|^2_{W^{1,2}} = \|u\|^2_{L^2} + \|\nabla u\|^2_{L^2}$. From this perspective, $\mathcal{C}_p(D)$ gives the sharp constant $\mathcal{S}_p(D)$ in the Sobolev embedding. In fact, for the above mentioned range of $p$,

$$W_0^{1,2}(D) \subset L^p(D) \ \text{ with } \ \|u\|_{L^p(D)} \leq \mathcal{S}_p \|\nabla u\|_{L^2(D)}, \quad \forall\, u \in W_0^{1,2}(D),$$

so that

$$\mathcal{S}_p(D) = \frac{1}{\sqrt{\mathcal{C}_p(D)}}.$$

This sharp Sobolev constant $\mathcal{S}_p(D)$ and its associated extremal function $\phi_p$ are both the subject of a vast literature, and incorporate much information relating the function theory and the geometry of $D$ (see, for example, [3] or a recent paper by Franzina and Lamberti [9] and the references therein). In particular, a long list of results encompass isoperimetric-type inequalities of various sorts, (see, for example, [16, 6, 1, 5]). Note that $\mathcal{C}_2(D)$ coincides with the principal frequency $\lambda(D)$, while the torsional rigidity is $P(D) = 4/\mathcal{C}_1(D)$.

In general, an extremal function $\phi$ for (1.1) is a solution of the boundary value problem

$$\Delta\phi + \lambda\, \phi^{p-1} = 0, \quad \phi|_{\partial D} = 0. \tag{1.2}$$

Without loss of generality we can take $\phi > 0$ inside $D$. General regularity results imply that $\phi \in C_0^\infty(D)$, and a short integration by parts argument [3, Lemma 2] shows that

$$\lambda = \mathcal{C}_p(D) \left( \int_D \phi^p \, d\mu \right)^{\frac{2-p}{p}} . \tag{1.3}$$

For $1 \leq p \leq 2$, there is a unique positive solution (see, for example, Dai, He, and Hu [8], Colesanti [7], or Pohožaev [18]).

While principal frequency is certainly a special case of the sharp Sobolev constant $\mathcal{C}_p(D)$, it is also the special case $p = 2$ of the eigenvalue of the $p$-Laplacian. This is the first eigenvalue for

$$\operatorname{div}\big(|\nabla u|^{p-2} \nabla u\big) + \lambda \, |u|^{p-2} \, u = 0,$$

and is also given by

$$\lambda_p(D) = \inf \left\{ \frac{\int_D |\nabla u|^p \, d\mu}{\int_D |u|^p \, d\mu} : \, u \in W_0^{1,p}(D), u \not\equiv 0 \right\},$$

see, for example, Lindqvist [12] and Fusco, Maggi, and Pratelli [10]. The most general Dirichlet eigenvalue in this context is probably

$$\lambda_{p,q}(D) = \inf \left\{ \frac{\int_D |\nabla u|^p \, d\mu}{\left( \int_D |u|^q \, d\mu \right)^{p/q}} : \, u \in W_0^{1,p}(D), u \not\equiv 0 \right\},$$

defined for $1 < q < \frac{pn}{n-p}$ if $1 < p < n$, or for any $q > 1$ if $p \geq n$. This eigenvalue appears, for example, in recent work of Brasco [2] in which a rearrangement technique of Kohler-Jobin is used to show that the ball minimizes an appropriate scale invariant quotient of the eigenvalues $\lambda_{p,q}$ and $\lambda_{p,1}$. If one thinks of $1/\lambda_{p,1}$ as a '$p$-torsional rigidity' then this is a generalisation of Kohler-Jobin's famous result [14, 15], in answer to a question of Pólga and Szegő, that balls mimimize principal frequency among all sets of given torsional rigidity. From this general point of view, the eigenvalue for the $p$-Laplacian is $\lambda_p(D) = \lambda_{p,p}(D)$ while the Sobolev constant is $\mathcal{C}_p(D) = \lambda_{2,p}(D)$.

A guiding motivation of the work in this area is to discover what results for principal frequency or for torsional rigidity hold for these more general eigenvalues. Note that $\mathcal{C}_p(D)$ depends monotonically on the domain $D$, and obeys the scaling law

$$\operatorname{Vol}(rD)^{2/p} \mathcal{C}_p(rD) = r^{n-2} \operatorname{Vol}(D)^{2/p} \mathcal{C}_p(D).$$

It can be shown (see [3]) that if $1 \leq p < q$ then

$$\operatorname{Vol}(D)^{2/p} \mathcal{C}_p(D) > \operatorname{Vol}(D)^{2/q} \mathcal{C}_q(D).$$

This extends the inequality $\lambda(D) \, P(D) < 4 \, \operatorname{Area}(D)$, which relates fundamental frequency and torsional rigidity and appears in Pólya and Szegő's book [19].

Various counterparts of the classical isoperimetric inequality play a prominent role. St. Venant's Principle, proved by Pólya in the 1950's, states that among all regions of given volume a ball has the largest torsional rigidity. The ball also has the smallest fundamental frequency among all regions of given volume, which is the famous Faber-Krahn Theorem from the 1920's. These results are subsumed by the following more general isoperimetric inequality: let $D^*$ be a ball with the same volume as $D$ and $p \geq 1$, then

$$\mathcal{C}_p(D) \geq \mathcal{C}_p(D^*)$$

with equality if and only if $D$ itself is a ball (see [3], for example, for a proof).

As is often the case, upper bounds are possible in the case of convex regions. Rather than specifying the volume of the region, however, it is more natural to fix its inradius $R(D)$, this being the supremum radius of all balls contained in $D$. Let $\phi$ be a positive solution of $\Delta\phi + \lambda\,\phi^{p-1} = 0$ on a convex region $D$, with $\phi = 0$ on $\partial D$. Let $\phi_M = \max\{\phi(x) : x \in D\}$. Then

$$\phi_M^{2-p} \leq \frac{2\lambda}{pA_p^2}\, R(D)^2 \quad \text{where} \quad A_p = \int_0^1 \frac{dt}{\sqrt{1-t^p}}.$$

Equality holds in the case of a strip / slab. The proof of this inequality in [3] makes use of Payne's $P$-function as described in Section 6.2.2 of the book [20] by Sperb. This inequality is a generalisation of both Hersch's result [11] for fundamental frequency to the effect that

$$\lambda(D) \geq \frac{\pi^2}{4\,R(D)^2}$$

and of Sperb's result [20] that the maximum value of the torsion function $\phi$ obeys

$$\phi_M \leq R(D)^2.$$

The main focus of the present work begins with a reverse Hölder inequality due to Payne and Rayner [16] for the first Dirichlet eigenfuntion $\phi$ of the Laplacian in a region $D$ in two dimensions. Payne and Rayner proved that

$$\left(\int_D \phi\, d\mu\right)^2 \geq \frac{4\pi}{\lambda(D)} \int_D \phi^2\, d\mu. \tag{1.4}$$

In [4], we extended this inequality to general $p$ recovering, in the case $p = 1$, Saint Venant's Principle that among all planar regions of prescribed area a disk has the largest torsional rigidity. The original inequality (1.4) of Payne and Rayner, and its extension to a range of values of $p$ in [4], are very much two-dimensional results. In [17], Payne and Rayner extended their inequality to higher dimensions, though they describe these extensions as 'not entirely satisfactory'. A more satisfactory extension of (1.4) to higher dimensions was given first by Kohler-Jobin [13] and subsequently strengthened by Chiti [6], using rearrangement techniques of Talenti [21], to more general elliptic operators. Chiti also dealt completely with the case of equality.

In the present work, we return to the original Payne and Rayner inequality (1.4) and prove a version for the Sobolev constant $\mathcal{C}_p(D)$ which is valid in all dimensions and which directly extends the work of Payne and Rayner in [17]. Even if our reverse Hölder inequality for the eigenfunction(s) corresponding to the Sobolev constant $\mathcal{C}_p(D)$ suffers from the same drawbacks as the original higher dimension inequality of Payne and Rayner in [17], it may still be of some interest.

We need to set notation before stating this result. We denote the induced area element on a hypersurface $\Sigma \subset \mathbb{R}^n$ by $d\sigma$. We write the appropriate dimensional volume of a set $\Omega$ as $|\Omega|$, that is if $\Omega \subset \mathbb{R}^n$ is an open set then $|\Omega| = \mu(\Omega)$ and if $\Sigma \subset \mathbb{R}^n$ is a hypersurface then $|\Sigma| = \sigma(\Sigma)$. If $\mathbb{B}_1 \subset \mathbb{R}^n$ is the unit ball, we denote $|\mathbb{B}_1| = \omega_n$, so that $|\mathbb{B}_r| = \omega_n r^n$ and $|\partial\mathbb{B}_r| = n\omega_n r^{n-1}$.

**Theorem 1.1.** *Let $p > 1$ and let $D \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Let $\mathcal{C}_p(D)$ be the sharp Sobolev constant defined by* (1.1)*, and let $\phi$ be its associated extremal function. Let $D^*$ be a ball with the same volume as $D$. Then*

$$(1.5)\quad \left(\int_D \phi^{p-1}\, d\mu\right)^2 \geq n^2\omega_n^{2/n}|D|^{\frac{n-2}{n}}\left[\frac{2}{p}\,\frac{1}{\mathcal{C}_p(D)} - \frac{n-2}{n}\,\frac{1}{\mathcal{C}_p(D^*)}\right]\left(\int_D \phi^p\, d\mu\right)^{\frac{2(p-1)}{p}}.$$

*Equality holds if and only if $D$ is a ball.*

The Hölder inequality implies that for any $u \in W_0^{1,2}(D)$ we have

$$\int_D |u|^{p-1}\, d\mu \leq |D|^{1/p}\left(\int_D |u|^p\, d\mu\right)^{\frac{p-1}{p}}.$$

For this reason, upper bounds of the form (1.5) are called reverse-Hölder inequalities.

The drawback with (1.5) is that the inequality becomes trivial if the right hand side is negative, which will be the case if $\mathcal{C}_p(D)$ is large compared with $\mathcal{C}_p(D^*)$. In general, it is not possible to bound $\mathcal{C}_p(D)$ from above in terms of $\mathcal{C}_p(D^*)$ or, in other words, it is not possible to bound $\mathcal{C}_p(D)$ from above in terms of the volume of $D$. Nevertheless, observe that the inequality (1.5) is isoperimetric and that we recover the main inequality of [17] in the case $p = 2$, and that we recover the reverse-Hölder inequality of [4] in the case $n = 2$.

## § 2. Proof of the main theorem

We begin by briefly outlining our strategy for proving (1.5), which we adapted from Payne and Rayner's proof in [17]. Let $M = \sup_{x\in D}\phi(x)$ and, for $0 \leq t \leq M$, we define

$$D_t = \{x \in D : \phi(x) > t\}, \qquad \Sigma_t = \{x \in D : \phi(x) = t\}.$$

By Sard's theorem, we have $\Sigma_t = \partial D_t$ for almost every value of $t$. To prove (1.5) we define the auxilliary function

$$H(t) = \int_{D_t} \phi^{p-1}\, d\mu = \int_t^M \tau^{p-1} \int_{\Sigma_\tau} \frac{d\sigma}{|\nabla \phi|}\, d\tau, \quad t \in [0, M].$$

In Section 2.1 we derive lower bounds for the second derivative of $H$, and in Section 2.2 we integrate these to obtain several integral inequalites for $H$ and for powers of $\phi$. In Section 2.3 we examine a one-dimensional eigenvalue problem which arises in the course of the proof and identify its solution in terms of $\mathcal{C}_p(D^*)$. The proof of (1.5) is completed in Section 2.4.

### § 2.1. Differential inequalities

We let $V(t) = |D_t|$. Then, by the co-area formula,

$$V'(t) = -\int_{\Sigma_t} \frac{d\sigma}{|\nabla \phi|} < 0.$$

Thus $V$ is a monotone function of $t$, and we can invert it to obtain $t = t(V)$, with

$$\frac{dt}{dV} = \frac{1}{V'(t)} = -\frac{1}{\int_{\Sigma_t} \frac{d\sigma}{|\nabla \phi|}}.$$

This, in turn, implies that

$$\frac{dH}{dV} = \frac{dH}{dt}\frac{dt}{dV} = \left(-t^{p-1}\int_{\Sigma_t} \frac{d\sigma}{|\nabla \phi|}\right) \cdot \left(-\frac{1}{\int_{\Sigma_t} \frac{d\sigma}{|\nabla \phi|}}\right) = t^{p-1}, \tag{2.1}$$

an identity which will prove useful at several points in our computations. Taking one more derivative shows that

$$\frac{d^2 H}{dV^2} = \frac{d}{dV}\left(t^{p-1}\right) = -\frac{(p-1)\, t^{p-2}}{\int_{\Sigma_t} \frac{d\sigma}{|\nabla \phi|}}.$$

**Lemma 2.1.** *The function $H$ satisfies*

$$\frac{d^2 H}{dV^2} \geq -(p-1)\left(t(V)\right)^{p-2} \frac{\lambda\, H(V)}{n^2 \omega_n^{2/n} V^{\frac{2(n-1)}{n}}}, \quad V \in \left[0, |D|\right]. \tag{2.2}$$

*with the boundary conditions $H(0) = 0$ and $H'(|D|) = 0$. Moreover, equality in* (2.2) *forces $D$ to be a ball, and forces the function $\phi$ to be radially symmetric.*

*Proof.* By the Cauchy-Schwarz inequality,

$$|\Sigma_t|^2 \leq \left(\int_{\Sigma_t} |\nabla \phi|\, d\sigma\right)\left(\int_{\Sigma_t} \frac{d\sigma}{|\nabla \phi|}\right),$$

which we can rearrange to read

$$\int_{\Sigma_t} \frac{d\sigma}{|\nabla\phi|} \;\geq\; \frac{|\Sigma_t|^2}{\int_{\Sigma_t} |\nabla\phi|\, d\sigma}. \tag{2.3}$$

Since $\Sigma_t$ is a level-set of $\phi$, we may use the divergence theorem and (1.2) to obtain

$$\begin{aligned}\int_{\Sigma_t} |\nabla\phi|\, d\sigma &= -\int_{\Sigma_t} \frac{\partial\phi}{\partial\eta}\, d\sigma = -\int_{D_t} \Delta\phi\, d\mu \\ &= \lambda \int_{D_t} \phi^{p-1}\, d\mu = \lambda\, H(t).\end{aligned}$$

Combining this with (2.3) we obtain

$$\int_{\Sigma_t} \frac{d\sigma}{|\nabla\phi|} \;\geq\; \frac{|\Sigma_t|^2}{\lambda\, H(t)}. \tag{2.4}$$

By the classical isoperimetric inequality,

$$|\Sigma_t|^2 \geq n^2\omega_n^{2/n}|D_t|^{\frac{2(n-1)}{n}} = n^2\omega_n^{2/n}V^{\frac{2(n-1)}{n}}. \tag{2.5}$$

Together with (2.4), this shows that

$$\begin{aligned}\frac{d^2H}{dV^2} &= -(p-1)\frac{t^{p-2}}{\int_{\Sigma_t}\frac{d\sigma}{|\nabla\phi|}} \geq -(p-1)\, t^{p-2}\,\frac{\lambda H(V)}{|\Sigma_t|^2} \\ &\geq -(p-1)\, t^{p-2}\,\frac{\lambda\, H(V)}{n^2\omega_n^{2/n}V^{\frac{2(n-1)}{n}}}.\end{aligned}$$

Notice that the boundary conditions for this differential inequality are

$$H(0) = 0, \qquad H'(|D|) = t^{p-1}\big|_{t=0} = 0. \tag{2.6}$$

Moreover, we only have equality in (2.2) for each $V$ in $\big[0, |D|\big]$ if we have equality in (2.5) for almost every $t$, which in turn implies that $\Sigma_t$ is a round sphere for almost every $t \in [0, M]$. This is possible only if $D$ is itself a ball. Also, equality in (2.2) forces equality in (2.3), which implies $|\nabla\phi|$ must be constant on each sphere $\Sigma_t$, and so $\phi$ must be radial. □

We change variables by letting $\rho = (V/\omega_n)^{1/n}$ be the volume radius of $D_t$, so that $V = |D_t| = \omega_n\rho^n$. We also define $\rho_M = (|D|/\omega_n)^{1/n}$. As a function of $\rho$, the function $H$ satisfies the boundary conditions

$$H(0) = H'(0) = \cdots = H^{(n-1)}(0) = 0, \qquad H'(\rho_M) = 0. \tag{2.7}$$

**Lemma 2.2.**

$$\frac{d}{d\rho}\left[\left(\rho^{1-n}\frac{dH}{d\rho}\right)^{\frac{1}{p-1}}\right] \geq -\lambda\,(n\omega_n)^{\frac{2-p}{p-1}}\,\rho^{1-n}\,H(\rho), \quad 0<\rho<\rho_M. \tag{2.8}$$

*Proof.* Taking derivatives, we see that

$$\frac{dH}{dV} = \frac{\rho^{1-n}}{n\omega_n}\frac{dH}{d\rho}, \quad \frac{d^2H}{dV^2} = \frac{\rho^{1-n}}{n\omega_n}\frac{d}{d\rho}\left(\frac{\rho^{1-n}}{n\omega_n}\frac{dH}{d\rho}\right). \tag{2.9}$$

Substituting these expressions in (2.2) gives

$$\frac{d}{d\rho}\left(\rho^{1-n}\frac{dH}{d\rho}\right) \geq -(p-1)\,\lambda\, t^{p-2}\,\rho^{1-n}\,H(\rho). \tag{2.10}$$

However,

$$t^{p-1} = \frac{dH}{dV} = \frac{\rho^{1-n}}{n\omega_n}\,\frac{dH}{d\rho},$$

so that (2.10) becomes

$$\frac{d}{d\rho}\left(\rho^{1-n}\,\frac{dH}{d\rho}\right) \geq -(p-1)\,\lambda\,(n\omega_n)^{\frac{2-p}{p-1}}\left(\rho^{1-n}\,\frac{dH}{d\rho}\right)^{\frac{p-2}{p-1}}\rho^{1-n}\,H(\rho).$$

This we can rewrite as

$$\frac{1}{p-1}\,\frac{\frac{d}{d\rho}\left(\rho^{1-n}\,\frac{dH}{d\rho}\right)}{\left(\rho^{1-n}\,\frac{dH}{d\rho}\right)^{\frac{p-2}{p-1}}} = \frac{d}{d\rho}\left[\left(\rho^{1-n}\frac{dH}{d\rho}\right)^{\frac{1}{p-1}}\right] \geq -\lambda\,(n\omega_n)^{\frac{2-p}{p-1}}\,\rho^{1-n}\,H(\rho).$$

□

**Remark 1.** Since (2.8) is really the same as (2.2) rewritten in different variables, equality holds in (2.8) for $0<\rho<\rho_M$ if and only if $D$ is a ball and $\phi$ is radial.

## § 2.2. Integral inequalities

In this section we integrate (2.2) and (2.8) to obtain inequalities for the integral of $H$ and the integral of powers of $\phi$. As each of these inequalities is an integrated form of (2.2) and (2.8), equality holds if and only if $D$ is a ball and $\phi$ is radial.

**Lemma 2.3.**

$$\begin{aligned}\left(\int_D \phi^{p-1}\,d\mu\right)^2 \geq{}& \frac{2n^2\omega_n^{2/n}|D|^{\frac{n-2}{n}}}{p\,\mathcal{C}_p(D)}\left(\int_D \phi^p\,d\mu\right)^{\frac{2(p-1)}{p}} \\ &-\frac{n-2}{n}|D|^{\frac{n-2}{n}}\int_0^{|D|} V^{\frac{2(1-n)}{n}}H^2(V)\,dV.\end{aligned} \tag{2.11}$$

*Proof.* We multiply the inequality (2.2) by $\frac{p}{p-1}V\left(\frac{dH}{dV}\right)^{1/(p-1)}$ and integrate from 0 to $|D|$. Upon integration, the left hand side becomes

$$\begin{aligned}\int_0^{|D|}\frac{p}{p-1}V\left(\frac{dH}{dV}\right)^{1/(p-1)}\frac{d^2H}{dV^2}\,dV &= \int_0^{|D|}V\frac{d}{dV}\left[\left(\frac{dH}{dV}\right)^{p/(p-1)}\right]dV\\ &= V\left(\frac{dH}{dV}\right)^{p/(p-1)}\Bigg|_0^{|D|} - \int_0^{|D|}\left(\frac{dH}{dV}\right)^{p/(p-1)}dV\\ &= -\int_0^{|D|}\left(t^{p-1}(V)\right)^{p/(p-1)}dV\\ &= -\int_0^{|D|}t^p(V)dV = -\int_D\phi^p\,d\mu.\end{aligned}$$

The boundary terms in the integration by parts vanished since $H'(|D|)=0$, while (2.1) was used at the third step. On the other hand, using (2.1) again, the right hand side becomes

$$\begin{aligned}&-\frac{p\,\lambda}{n^2\omega_n^{2/n}}\int_0^{|D|}V\left(\frac{dH}{dV}\right)^{1/(p-1)}t^{p-2}\,H(t)\,V^{\frac{2(1-n)}{n}}\,dV\\ &\qquad= -\frac{p\,\lambda}{n^2\omega_n^{2/n}}\int_0^{|D|}t^{p-2}\left(\frac{dH}{dV}\right)^{\frac{2-p}{p-1}}V^{\frac{2-n}{n}}H(V)\frac{dH}{dV}\,dV\\ &\qquad= -\frac{p\,\lambda}{n^2\omega_n^{2/n}}\int_0^{|D|}t^{p-2}(t^{p-1})^{\frac{2-p}{p-1}}V^{\frac{2-n}{n}}H(V)\frac{dH}{dV}\,dV\\ &\qquad= -\frac{p\,\lambda}{n^2\omega_n^{2/n}}\int_0^{|D|}V^{\frac{2-n}{n}}H(V)\frac{dH}{dV}\,dV\end{aligned}$$

We combine these last two equations and replace $\lambda$ by $\mathcal{C}_p(D)\left(\int_D\phi^p d\mu\right)^{(2-p)/p}$ to obtain

$$\begin{aligned}\left(\int_D\phi^p\,d\mu\right)^{\frac{2(p-1)}{p}} &\le \frac{p\,\mathcal{C}_p(D)}{n^2\omega_n^{2/n}}\int_0^{|D|}V^{\frac{2-n}{n}}H(V)\,\frac{dH}{dV}\,dV\\ &= \frac{p\,\mathcal{C}_p(D)}{2n^2\omega_n^{2/n}}\int_0^{|D|}V^{\frac{2-n}{n}}\frac{d}{dV}\left(H^2(V)\right)dV\\ &= \frac{p\,\mathcal{C}_p(D)}{2n^2\omega_n^{2/n}}\left[|D|^{\frac{2-n}{n}}\left(\int_D\phi^{p-1}d\mu\right)^2+\frac{n-2}{n}\int_0^{|D|}H^2(V)\,V^{\frac{2(1-n)}{n}}dV\right],\end{aligned}$$

which we can rearrange to give (2.11). □

**Lemma 2.4.**

$$\int_0^{\rho_M}\rho^{\frac{1-n}{p-1}}\left(\frac{dH}{d\rho}\right)^{\frac{p}{p-1}}d\rho \;\le\; \lambda\,(n\omega_n)^{\frac{2-p}{p-1}}\int_0^{\rho_M}\rho^{1-n}\,H^2(\rho)\,d\rho. \tag{2.12}$$

*Equality holds if and only if $D$ is a ball.*

*Proof.* We mutliply (2.8) by $H$ and integrate from 0 to $\rho_M$. The boundary conditions (2.7) imply that $\rho^{1-n}\,\frac{dH}{d\rho}$ is bounded at 0. Hence the boundary terms vanish in the integration parts below, and we obtain that

$$\begin{aligned}\int_0^{\rho_M} \rho^{\frac{1-n}{p-1}}\left(\frac{dH}{d\rho}\right)^{\frac{p}{p-1}} d\rho &= \int_0^{\rho_M}\left[\rho^{1-n}\,\frac{dH}{d\rho}\right]^{\frac{1}{p-1}}\frac{dH}{d\rho}\,d\rho \\ &\leq \lambda\,(n\omega_n)^{\frac{2-p}{p-1}}\int_0^{\rho_M}\rho^{1-n}\,H^2(\rho)\,d\rho.\end{aligned}$$

□

**Lemma 2.5.** *With $\phi$, $H$, and $\rho$ defined as above,*

$$\int_0^{\rho_M} \rho^{\frac{1-n}{p-1}}\left(\frac{dH}{d\rho}\right)^{\frac{p}{p-1}} d\rho = (n\omega_n)^{\frac{1}{p-1}}\int_D \phi^p\,d\mu. \tag{2.13}$$

*Proof.* We use (2.1) and (2.9) to conclude that

$$\begin{aligned}\int_0^{\rho_M} \rho^{\frac{1-n}{p-1}}\left(\frac{dH}{d\rho}\right)^{\frac{p}{p-1}} d\rho &= \int_0^{\rho_M}(\rho^{1-n})^{\frac{p}{p-1}}\left(\frac{dH}{d\rho}\right)^{\frac{p}{p-1}}\rho^{n-1}\,d\rho \\ &= \int_0^{|D|}\left(\rho^{1-n}\,\frac{dH}{d\rho}\right)^{\frac{p}{p-1}}\frac{dV}{n\omega_n} \\ &= \int_0^{|D|}\left(n\omega_n\frac{dH}{dV}\right)^{\frac{p}{p-1}}\frac{dV}{n\omega_n} \\ &= (n\omega_n)^{\frac{1}{p-1}}\int_0^{|D|}\left(\frac{dH}{dV}\right)^{\frac{p}{p-1}} dV \\ &= (n\omega_n)^{\frac{1}{p-1}}\int_0^{|D|}(t^{p-1})^{\frac{p}{p-1}}\,dV \\ &= (n\omega_n)^{\frac{1}{p-1}}\int_0^{|D|} t^p\,dV = (n\omega_n)^{\frac{1}{p-1}}\int_D \phi^p\,d\mu.\end{aligned}$$

□

**Corollary 2.6.**

$$n\omega_n\left(\int_D \phi^p\,d\mu\right)^{\frac{2(p-1)}{p}} \leq\ \mathcal{C}_p(D)\int_0^{\rho_M}\rho^{1-n}H^2(\rho)\,d\rho. \tag{2.14}$$

*Moreover, we have equality if and only if $D$ is a ball and $\phi$ is radial.*

*Proof.* Combine (2.12), (2.13), and (1.3). □

**Lemma 2.7.**

$$\left(\int_D \phi^{p-1}\, d\mu\right)^2 \geq \frac{2n^2\omega_n^{2/n}|D|^{\frac{n-2}{n}}}{p\,\mathcal{C}_p(D)} \left(\int_D \phi^p d\mu\right)^{\frac{2(p-1)}{p}} - (n-2)\,\omega_n^{\frac{2-n}{n}}\,|D|^{\frac{n-2}{n}} \int_0^{\rho_M} \rho^{1-n} H^2(\rho)\, d\rho. \tag{2.15}$$

*Equality holds if and only if $D$ is a ball.*

*Proof.* Since $\rho(V) = \big(V/\omega_n\big)^{1/n}$, we have

$$n\omega_n^{1/n} \frac{d\rho}{dV} V^{\frac{n-1}{n}} = 1,$$

so that

$$\begin{aligned}\int_0^{|D|} V^{\frac{2(1-n)}{n}} H^2(V)\, dV &= \int_0^{|D|} H^2(\rho)\, \omega_n^{\frac{2(1-n)}{n}} \rho^{2(1-n)}\, n\omega_n^{1/n}\, (\omega_n \rho^n)^{\frac{n-1}{n}} \frac{d\rho}{dV}\, dV \\ &= n\omega_n^{\frac{2-n}{n}} \int_0^{\rho_M} \rho^{1-n} H^2(\rho)\, d\rho.\end{aligned}$$

Using this identity in (2.11) gives (2.15). □

### § 2.3. An auxiliary one dimensional eigenvalue problem

Motivated by (2.12) and (2.7), we define $\Lambda_*$ by

$$\Lambda_* = \inf \left\{ \left( \int_0^{\rho_M} \rho^{\frac{1-n}{p-1}}\, f'(\rho)^{\frac{p}{p-1}}\, d\rho \right)^{\frac{2(p-1)}{p}} \Big/ \int_0^{\rho_M} \rho^{1-n}\, f^2(\rho)\, d\rho \right\} \tag{2.16}$$

where the infimum is over all functions on $[0, \rho_M]$ for which

$$f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0 = f'(\rho_M), \quad f \not\equiv 0. \tag{2.17}$$

**Remark 2.** Notice that we have rescaled the numerator to make the quotient scale-invariant. This does not, however, affect the Euler-Lagrange equation involved.

**Lemma 2.8.** *The Euler-Lagrange equation for the variational problem* (2.16)*, with the boundary conditions* (2.17)*, is*

$$f''(\rho) - \frac{n-1}{\rho}\, f'(\rho) + \Lambda\big[\rho^{1-n} f'(\rho)\big]^{\frac{p-2}{p-1}}\, f(\rho) = 0. \tag{2.18}$$

*Proof.* Since the ratio defining $\Lambda_*$ is scale-invariant, we may restrict our attention to either of the constrained critical point problems:

$$\text{minimize} \int_0^{\rho_M} \rho^{\frac{1-n}{p-1}}\, f'(\rho)^{\frac{p}{p-1}}\, d\rho \ \text{ subject to} \int_0^{\rho_M} \rho^{1-n} f^2(\rho)\, d\rho \ = \ \text{constant}$$

or

$$\text{maximize} \int_0^{\rho_M} \rho^{1-n} f^2(\rho)\, d\rho \;\; \text{subject to} \int_0^{\rho_M} \rho^{\frac{1-n}{p-1}} \, f'(\rho)^{\frac{p}{p-1}} \, d\rho \; = \; \text{constant.}$$

Regardless, the method of Lagrange multipliers implies that a constrained critical point $f$ satisfies

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} \int_0^{\rho_M} \rho^{\frac{1-n}{p-1}} \left( \frac{df}{d\rho} + \epsilon\, \frac{dg}{d\rho} \right)^{\frac{p}{p-1}} d\rho = \Lambda\, \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \int_0^{\rho_M} \rho^{1-n} \left[ f(\rho) + \epsilon\, g(\rho) \right]^2 d\rho,$$

for any admissible $g$. On evaluating these derivatives, using the boundary conditions (2.17) to see that $\rho^{1-n}\, f'(\rho)$ is bounded at 0 and that consequently the boundary terms arising from integration by parts vanish, we obtain

$$\begin{aligned} &2\Lambda \int_0^{\rho_M} \rho^{1-n}\, f(\rho)\, g(\rho)\, d\rho \\ &\quad = \frac{p}{p-1} \int_0^{\rho_M} \rho^{\frac{1-n}{p-1}} \left( \frac{df}{d\rho} \right)^{\frac{1}{p-1}} \frac{dg}{d\rho}\, d\rho \\ &\quad = -\frac{p}{p-1} \int_0^{\rho_M} g(\rho) \frac{d}{d\rho} \left[ \rho^{\frac{1-n}{p-1}} \left( \frac{df}{d\rho} \right)^{\frac{1}{p-1}} \right] d\rho \\ &\quad = -\frac{p}{p-1} \int_0^{\rho_M} g(\rho) \left[ \frac{1}{p-1}\, \rho^{\frac{1-n}{p-1}} \left( \frac{df}{d\rho} \right)^{\frac{2-p}{p-1}} \frac{d^2 f}{d\rho^2} + \frac{1-n}{p-1}\, \rho^{\frac{2-p-n}{p-1}} \left( \frac{df}{d\rho} \right)^{\frac{1}{p-1}} \right] d\rho. \end{aligned}$$

This must hold for all choices of $g$, hence (absorbing a factor of $2(p-1)^2/p$ into the Lagrange multiplier $\Lambda$) we must have

$$\begin{aligned} 0 &= \rho^{\frac{1-n}{p-1}} f'(\rho)^{\frac{2-p}{p-1}} f''(\rho) - (n-1)\, \rho^{\frac{2-p-n}{p-1}} f'(\rho)^{\frac{1}{p-1}} + \Lambda\, \rho^{1-n}\, f(\rho) \\ &= \rho^{\frac{1-n}{p-1}} f'(\rho)^{\frac{2-p}{p-1}} \left[ f''(\rho) - (n-1)\rho^{-1} f'(\rho) + \Lambda \left[ \rho^{1-n}\, f'(\rho) \right]^{\frac{p-2}{p-1}} f(\rho) \right], \end{aligned}$$

as claimed. $\square$

**Lemma 2.9.** *Let $D^*$ be the ball $\mathbb{B}_{\rho_M}$ of radius $\rho_M$. Then,*

$$\Lambda_* \leq (n\omega_n)^{\frac{2-p}{p}} \mathcal{C}_p(D^*). \tag{2.19}$$

*Proof.* We use the function $H(\rho)$ for the ball $\mathbb{B}_{\rho_M}$ as a test function for the quotient defining $\Lambda_*$ and use (2.12), (1.3), and (2.13) in the case of a ball:

$$\begin{aligned}
\Lambda_* &\leq \left(\int_0^{\rho_M} \rho^{\frac{1-n}{p-1}} H'(\rho)^{\frac{p}{p-1}}\, d\rho\right)^{\frac{2(p-1)}{p}} \Big/ \int_0^{\rho_M} \rho^{1-n}\, H^2(\rho)\, d\rho \\
&\leq \lambda\, (n\omega_n)^{\frac{2-p}{p-1}} \left(\int_0^{\rho_M} \rho^{\frac{1-n}{p-1}} H'(\rho)^{\frac{p}{p-1}}\, d\rho\right)^{\frac{p-2}{p}} \\
&= (n\omega_n)^{\frac{2-p}{p-1}}\, \mathcal{C}_p(D^*) \left(\int_{D^*} \phi^p\, d\mu\right)^{\frac{2-p}{p}} \left[(n\omega_n)^{\frac{1}{p-1}} \int_{D^*} \phi^p\, d\mu\right]^{\frac{p-2}{p}} \\
&= (n\omega_n)^{\frac{2-p}{p}}\, \mathcal{C}_p(D^*).
\end{aligned}$$

□

In order to obtain a lower bound for $\Lambda_*$ in terms of $\mathcal{C}_p(D^*)$, we first need to relate the particular $\Lambda$ occurring in the Euler-Lagrage equation (2.18) to the eigenvalue $\Lambda_*$, just as (1.3) relates the number $\lambda$ occurring in the Euler-Lagrange equation (1.2) to the eigenvalue $\mathcal{C}_p(D)$.

**Lemma 2.10.** *Let $f$ be a minimizer for $\Lambda_*$ given by* (2.16) *with the boundary conditions* (2.17) *and satisfy the Euler-Lagrange equation* (2.18), *written as*

$$\frac{d}{d\rho}\left[\rho^{\frac{1-n}{p-1}} f'(\rho)^{\frac{1}{p-1}}\right] + \Lambda\, \rho^{1-n}\, f(\rho) = 0. \tag{2.20}$$

*Then*

$$\Lambda = \Lambda_* \left(\int_0^{\rho_M} \rho^{\frac{1-n}{p-1}}\, f'(\rho)^{\frac{p}{p-1}}\, d\rho\right)^{\frac{2-p}{p}}. \tag{2.21}$$

*Proof.* Multiply the Euler-Lagrange equation (2.20) across by $f(\rho)$ and integrate from 0 to $\rho_M$ to obtain

$$\int_0^{\rho_M} f(\rho)\, \frac{d}{d\rho}\left[\rho^{\frac{1-n}{p-1}} f'(\rho)^{\frac{1}{p-1}}\right] d\rho + \Lambda \int_0^{\rho_M} \rho^{1-n}\, f(\rho)^2\, d\rho = 0.$$

Integrating by parts in the first term and using the boundary conditions (2.17) gives

$$\int_0^{\rho_M} f(\rho)\, \frac{d}{d\rho}\left[\rho^{\frac{1-n}{p-1}} f'(\rho)^{\frac{1}{p-1}}\right] d\rho = -\int_0^{\rho_M} \rho^{\frac{1-n}{p-1}} f'(\rho)^{\frac{p}{p-1}}\, d\rho,$$

from which it follows that

$$\Lambda = \int_0^{\rho_M} \rho^{\frac{1-n}{p-1}} f'(\rho)^{\frac{p}{p-1}}\, d\rho \Big/ \int_0^{\rho_M} \rho^{1-n}\, f(\rho)^2\, d\rho.$$

We can use (2.16) to write $\int_0^{\rho_M} \rho^{1-n} f^2(\rho)\, d\rho$ in terms of $\Lambda_*$ since $f$ is a minimizer for this Rayleigh quotient, leading to

$$\Lambda = \Lambda_* \left( \int_0^{\rho_M} \rho^{\frac{1-n}{p-1}} f'(\rho)^{\frac{p}{p-1}}\, d\rho \right)^{1-\frac{2(p-1)}{p}},$$

which is (2.21). □

**Lemma 2.11.**

$$\mathcal{C}_p(D^*) \le (n\omega_n)^{\frac{p-2}{p}} \Lambda_*. \tag{2.22}$$

*Proof.* Let $f$ be a minimizer for the generalized Rayleigh quotient (2.16) defining $\Lambda_*$. Set

$$\psi(\rho) = \int_\rho^{\rho_M} r^{1-n} f(r)\, dr, \qquad 0 \le \rho \le \rho_M,$$

so that $\psi(\rho_M) = 0$. Then $\psi(\rho)$ (where $\rho = |x|$ for $x \in D^*$) is an admissible test function for the quotient defining $\mathcal{C}_p(D^*)$, from which it follows that

$$\mathcal{C}_p(D^*) \le (n\omega_n)^{\frac{p-2}{p}} \int_0^{\rho_M} \rho^{n-1} \psi'(\rho)^2\, d\rho \Bigg/ \left( \int_0^{\rho_M} \rho^{n-1} \psi(\rho)^p\, d\rho \right)^{2/p}. \tag{2.23}$$

Now

$$\int_0^{\rho_M} \rho^{n-1} \psi'(\rho)^2\, d\rho = \int_0^{\rho_M} \rho^{n-1} \left[\rho^{1-n} f(\rho)\right]^2\, d\rho = \int_0^{\rho_M} \rho^{1-n} f(\rho)^2\, d\rho. \tag{2.24}$$

Next, using the Euler-Lagrange equation (2.20),

$$\begin{aligned} \psi(\rho) &= \int_\rho^{\rho_M} r^{1-n} f(r)\, dr \\ &= -\frac{1}{\Lambda} \int_\rho^{\rho_M} \frac{d}{dr} \left[ r^{\frac{1-n}{p-1}} f'(r)^{\frac{1}{p-1}} \right] dr \\ &= -\frac{1}{\Lambda}\, r^{\frac{1-n}{p-1}} f'(r)^{\frac{1}{p-1}} \Big|_{r=\rho}^{r=\rho_M} \\ &= \frac{1}{\Lambda} \rho^{\frac{1-n}{p-1}} f'(\rho)^{\frac{1}{p-1}}, \end{aligned}$$

where we used $f'(\rho_M) = 0$. From this we obtain that

$$\begin{aligned} \int_0^{\rho_M} \rho^{n-1} \psi(\rho)^p\, d\rho &= \int_0^{\rho_M} \rho^{n-1} \frac{1}{\Lambda^p} \rho^{\frac{p(1-n)}{p-1}} f'(\rho)^{\frac{p}{p-1}}\, d\rho \\ &= \frac{1}{\Lambda^p} \int_0^{\rho_M} \rho^{\frac{1-n}{p-1}} f'(\rho)^{\frac{p}{p-1}}\, d\rho. \end{aligned} \tag{2.25}$$

With the help of the identities (2.24) and (2.25), we can write the numerator and the denominator of the right hand side of (2.23) in terms of the minimizer $f$ for $\Lambda_*$. We find, using that $f$ minimizes the quotient for $\Lambda_*$ at the second step and using (2.21) at the last step, that

$$\mathcal{C}_p(D^*) \le (n\omega_n)^{\frac{p-2}{p}} \int_0^{\rho_M} \rho^{1-n} f(\rho)^2\, d\rho \Big/ \left( \frac{1}{\Lambda^p} \int_0^{\rho_M} \rho^{\frac{1-n}{p-1}} f'(\rho)^{\frac{p}{p-1}}\, d\rho \right)^{\frac{2}{p}} \tag{2.26}$$

$$= (n\omega_n)^{\frac{p-2}{p}} \frac{\Lambda^2}{\Lambda_*} \left( \int_0^{\rho_M} \rho^{\frac{1-n}{p-1}} f'(\rho)^{\frac{p}{p-1}}\, d\rho \right)^{\frac{2(p-1)}{p} - \frac{2}{p}} \tag{2.27}$$

$$= (n\omega_n)^{\frac{p-2}{p}} \frac{1}{\Lambda_*} \left[ \Lambda \left( \int_0^{\rho_M} \rho^{\frac{1-n}{p-1}} f'(\rho)^{\frac{p}{p-1}}\, d\rho \right)^{\frac{p-2}{p}} \right]^2 \tag{2.28}$$

$$= (n\omega_n)^{\frac{p-2}{p}} \frac{\Lambda_*^2}{\Lambda_*} = (n\omega_n)^{\frac{p-2}{p}} \Lambda_*. \tag{2.29}$$

□

## § 2.4. Completion of the proof of Theorem 1.1

We are now finally in a position to complete the proof of Theorem 1.1. Indeed, since $H(\rho)$ is an admissible function for $\Lambda^*$ as defined by (2.16), we have

$$\begin{aligned}
\int_0^{\rho_M} \rho^{1-n} H^2(\rho)\, d\rho &\le \frac{1}{\Lambda_*} \left( \int_0^{\rho_M} \rho^{\frac{1-n}{p-1}} \left( \frac{dH}{d\rho} \right)^{\frac{p}{p-1}} d\rho \right)^{\frac{2(p-1)}{p}} \\
&= \frac{1}{\Lambda_*} \left( (n\omega_n)^{\frac{1}{p-1}} \int_D \phi^p d\mu \right)^{\frac{2(p-1)}{p}} \\
&= \frac{1}{\Lambda_*} (n\omega_n)^{2/p} \left( \int_D \phi^p d\mu \right)^{\frac{2(p-1)}{p}} \\
&= \frac{n\omega_n}{\mathcal{C}_p(D^*)} \left( \int_D \phi^p d\mu \right)^{\frac{2(p-1)}{p}}
\end{aligned}$$

where we used the identity $\Lambda_* = (n\omega_n)^{\frac{2-p}{p}} \mathcal{C}_p(D^*)$ resulting from (2.19) and (2.22). Moreover, equality holds if and only if $D$ is a ball and $\phi$ is radial. The identity (2.13) was used at the second step above. Substituting this last inequality into (2.15), we have

$$\begin{aligned}
\left( \int_D \phi^{p-1} d\mu \right)^2 \ \ge\ & \frac{2n^2\omega_n^{2/n}}{p\,\mathcal{C}_p(D)} |D|^{\frac{n-2}{n}} \left( \int_D \phi^p d\mu \right)^{\frac{2(p-1)}{p}} \\
& - (n-2)\, \omega_n^{\frac{2-n}{n}} |D|^{\frac{n-2}{n}} \frac{n\omega_n}{\mathcal{C}_p(D^*)} \left( \int_D \phi^p d\mu \right)^{\frac{2(p-1)}{p}} .
\end{aligned}$$

The main inequality (1.5) follows with equality if and only if $D$ is a ball. $\square$

## References


[1] A. Alvino, V. Ferone and G. Trombetti, *On the properties of some nonlinear eigenvalues*, SIAM J. Math. Anal. **29** (1998), 437–451.

[2] L. Brasco, *On torsional rigidity and principal frequencies: an invitation to the Kohler-Jobin rearrangement technique*, preprint, January 2013.

[3] T. Carroll and J. Ratzkin, *Interpolating between torsional rigidity and principal frequency*, J. Math. Anal. Appl. **379** (2011), 818–826.

[4] ———, *Two isoperimetric inequalities for the Sobolev constant*, Z. Angew. Math. Phys. **63**, (2012), 855–863.

[5] A. Chianchi, N. Fusco, F. Maggi and A. Pratelli, *The sharp Sobolev inequality in quantitative form*, J. Eur. Math. Soc. **11** (2009), 1105–1139.

[6] G. Chiti, *A reverse Hölder inequality for the eigenfunctions of linear second order elliptic operators*, Z. Angew. Math. Phys. **33** (1982), 143–148.

[7] A. Colesanti, *Brunn-Minkowski inequalities for variational functionals and related problems*, Adv. Math. **194** (2005), 105–140.

[8] Q. Dai, R. He and H. Hu, *Isoperimetric inequalities and sharp estimates for positive solution of sublinear elliptic equations*, preprint, `arXiv:AP/1003.3768`

[9] G. Franzina and P. D. Lamberti, *Existence and uniqueness for a p-Laplacian nonlinear eigenvalue problem*, Electron. J. Differential Equations **26** (2010), 10 pp.

[10] N. Fusco, F. Maggi and A. Pratelli, *Stability estimates for certain Faber-Krahn, isocapacitary and Cheeger inequalities*, Ann. Sc. Norm. Super. Pisa Cl. Sci. **8** (2009), 51–71.

[11] J. Hersch, *Sur la fréquence fondamentale d'une membrane vibrante: évaluations par défault et principe de maximum*, Z. Angew. Math. Mech. **11** (1960), 387–413.

[12] P. Lindqvist, *On non-linear Rayleigh quotients*, Potential Anal. **2** (1993), 199–218.

[13] M. -T. Kohler-Jobin, *Sur la première fonction propre d'une membrane: une extension à N dimensions de l'inégalité isopérimétrique de Payne-Rayner*, Z. Angew. Math. Phys. **28** (1977), 1137–1140.

[14] ———, *Démonstration de l'inégalité isopérimétrique $P\lambda^2 \geq \pi j_0^4/2$, conjecturée par Pólya et Szegő*, C. R. Acad. Sci. Paris Sér. A-B **281** (1975), A119–A121.

[15] ———, *Une méthode de comparaison isopérimétrique de fonctionnelles de domaines de la physique mathématique. I. Une démonstration de la conjecture isopérimétrique $P\lambda^2 \geq \pi j_0^4/2$ de Pólya et Szegő*, Z. Angew. Math. Phys. **29** (1978), 757–766.

[16] L. E. Payne and M. E. Rayner, *An isoperimetric inequality for the first eigenfunction in the fixed membrane problem*, Z. Angew. Math. Phys. **23** (1972), 13–15.

[17] ———, *Some isoperimetric norm bounds for solutions of the Helmholtz equation*, Z. Angew. Math. Phys. **24** (1973), 105–110.

[18] S. Pohožaev, *Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$*, Soviet Math. Doklady **6** (1965), 1408–1411.

[19] G. Pólya and G. Szegő, *Isoperimetric Inequalities in Mathematical Physics*, Princeton University Press, (1951).

[20] R. Sperb, *Maximum principles and their applications.* Academic Press, (1981).

[21] G. Talenti, *Elliptic equations and rearrangements*, Ann. Sc. Norm. Super. Pisa Cl. Sci. **3** (1976), 697–718.