RIMS Kôkyûroku Bessatsu
B44 (2013), 213–222

Explicit formulas for Hasse-Witt invariants of cyclotomic function fields with conductor of degree two

By

Daisuke Shiomi *

§ 1. Introduction

Let p be a prime. Let $\mathbb{F}_q$ be the field with $q = p^r$ elements. Let $k = \mathbb{F}_q(T)$ be the rational function field over $\mathbb{F}_q$, and let $A = \mathbb{F}_q[T]$ be the polynomial subring of k. For a monic polynomial $m \in A$, we denote the m-th cyclotomic function field by K_m. For definitions and basic properties of cyclotomic function field, see [Go], [Ha], and [Ro].

Let us denote by J_m the Jacobian of $K_m\bar{\mathbb{F}}_q$, where $\bar{\mathbb{F}}_q$ is an algebraic closure of $\mathbb{F}_q$. For a prime l, it is well-known that the l-primary subgroup $J_m(l)$ of J_m satisfies

$$J_m(l) \simeq \begin{cases} \bigoplus_{i=1}^{2g_m} \mathbb{Q}_l/\mathbb{Z}_l & \text{if } l \neq p, \\ \bigoplus_{i=1}^{\lambda_m} \mathbb{Q}_p/\mathbb{Z}_p & \text{if } l = p, \end{cases}$$

where g_m is the genus of K_m, and λ_m is an integer where $0 \leq \lambda_m \leq g_m$. The integer λ_m is called the Hasse-Witt invariant of K_m.

Kida and Murabayashi gave an explicit formula for g_m for all monic polynomial $m \in A$ as Corollary 1 in the section 2 of [K-M]. Applying their genus formula to the cases of $\deg m = 1$ and $\deg m = 2$, we have gotten

Theorem 1.1. *Let $m \in A$ be a monic polynomial.*

(1) If $\deg m = 1$, then we have $g_m = 0$.

Received March 12, 2012. Revised October 5, 2012 and December 15, 2012.
2000 Mathematics Subject Classification(s): 11M38, 11R60.
Key Words: cyclotomic function field, Jacobian variety.
*Department of Mathematical Sciences, Faculty of Science, Yamagata University, 1-4-12 Kojirakawa-machi, Yamagata, 990-8560, Japan. e-mail: shiomi@sci.kj.yamagata-u.ac.jp

(2) If $\deg m = 2$*, then we have*

$$g_m = \begin{cases} \frac{(q-2)(q+1)}{2} & \text{if } m \text{ is irreducible,} \\ \frac{(q-2)(q-1)}{2} & \text{if } m = P^2 \text{ where } P \text{ is a monic polynomial of degree one,} \\ \frac{(q-2)(q-3)}{2} & \text{if } m = PQ \text{ where } P, Q \text{ are distinct monic polynomials of degree one.} \end{cases}$$

Next we consider the Hasse-Witt invariant case. The main theorem of this paper is the following results.

Theorem 1.2. *Let* $m \in A$ *be a monic polynomial, and* $q = p^r$*.*

(1) If $\deg m = 1$*, then we have* $\lambda_m = 0$*.*

(2) If $\deg m = 2$*, then we have*

$$\lambda_m = \begin{cases} \left(\frac{p(p+1)}{2}\right)^r - q - 1 & \text{if } m \text{ is irreducible,} & \text{(I)} \\ 0 & \text{if } m = P^2 \text{ where } P \text{ is a monic polynomial of degree one,} & \text{(II)} \\ \left(\frac{p(p+1)}{2}\right)^r - 3q + 3 & \text{if } m = PQ \text{ where } P, Q \text{ are distinct monic polynomials of degree one.} & \text{(III)} \end{cases}$$

Remark. Noting that $\lambda_m \leq g_m$, we have $\lambda_m = 0$ if $\deg m = 1$. Hence we obtain the first assertion of Theorem 1.2.

We call K_m ordinary if $\lambda_m = g_m$. By comparing Theorem 1.1 and 1.2, we obtain the following results.

Corollary 1.3. *Let* $m \in A$ *be a monic polynomial of degree two.*

(1) Assume that $q = p$*. Then* K_m *is ordinary if and only if one of the following conditions holds: (a)* $q = 2$*, (b)* m *is irreducible, (c)* $m = PQ$ *where* P, Q *are distinct polynomials of degree one.*

(2) Assume that $q = p^r (r \geq 2)$*. Then* K_m *is not ordinary.*

The second assertion of Corollary 1.3 is generalized as follows:

Theorem 1.4. *Assume that $q = p^r (r \geq 2)$, and $m \in A$ is a monic polynomial. Then K_m is ordinary if and only if* $\deg m = 1$*. In this case, K_m is rational.*

§ 2. Preparations

In this section, we review some basic facts for zeta functions, L-functions, and power residue symbols.

Let us define the zeta function of K_m as follows

$$\zeta(s, K_m) = \prod_{\mathfrak{p}:\text{prime}} \left(1 - \frac{1}{N\mathfrak{p}^s}\right)^{-1},$$

where $\mathfrak{p}$ runs through all primes of K_m, and $N\mathfrak{p}$ is the number of elements of the reduce class field of $\mathfrak{p}$. By the standard fact about the zeta function, there is a polynomial $Z_m(u) \in \mathbb{Z}[u]$ such that

$$\zeta(s, K_m) = \frac{Z_m(q^{-s})}{(1 - q^{-s})(1 - q^{1-s})}.$$

Then we have the following relation between λ_m and $Z_m(u)$.

Theorem 2.1. *(cf. Proposition 11.20 in [Ro]).*

$$\lambda_m = \deg \bar{Z}_m(u),$$

where $\bar{Z}_m(u) \in \mathbb{F}_p[u]$ is the reduction of $Z_m(u)$ modulo p.

Let X_m be the group of all primitive Dirichlet characters modulo m. Then $\zeta(s, K_m)$ can be written as follows

$$\zeta(s, K_m) = \left\{ \prod_{\chi \in X_m} L(s, \chi) \right\} (1 - q^{-s})^{-\frac{[K_m:k]}{q-1}},$$

where $L(s, \chi) = \sum_{\substack{a \in A \\ a:\text{monic}}} \chi(a) q^{-s \deg a}$. (cf. Lemma 2.1 in [Sh1]).

For a character $\chi \in X_m$, we call χ real if $\chi(a) = 1$ for all $a \in \mathbb{F}_q^\times$. Otherwise, we call χ imaginary.

Let $m \in A$ be a monic polynomial of degree d. For $\chi \in X_m$, we put

$$s_i(\chi) = \sum_{\substack{a:\text{monic} \\ \deg a = i}} \chi(a).$$

Then it is known that

- $s_i(\chi) = 0$ if χ is non-trivial and $i \geq \deg f_\chi$,
- $\sum_{i=0}^{d-1} s_i(\chi) = 0$ if χ is non-trivial and real,

where f_χ is the conductor of χ (cf. section 3 in [G-R]). Assume that $d = 2$. Then $L(s, \chi)$ can be calculated as follows

$$L(s,\chi) = \begin{cases} (1-q^{1-s})^{-1} & \text{if } f_\chi = 1, \\ 1 & \text{if } \deg f_\chi = 1, \\ 1-q^{-s} & \text{if } f_\chi = m, \text{ and } \chi \text{ is real}, \\ 1+s_1(\chi)q^{-s} & \text{if } f_\chi = m, \text{ and } \chi \text{ is imaginary}. \end{cases}$$

Hence we obtain

$$Z_m(u) = \prod_{\substack{f_\chi = m \\ \chi:\text{imaginary}}} (1 + s_1(\chi)u). \tag{2.1}$$

In the end of this section, we review a power residue symbol. For an integer $n \geq 2$, let W_n be the set of all n-th roots of unity. Let K be a number field containing W_n, and let $\mathcal{O}_K$ be the ring of integers of K. Let $\mathfrak{p}$ be a prime ideal of K not dividing n. For $\alpha \in \mathcal{O}_K$ which is prime to $\mathfrak{p}$, there exists uniquely $\left(\frac{\alpha}{\mathfrak{p}}\right)_n \in W_n$ satisfying

$$\left(\frac{\alpha}{\mathfrak{p}}\right)_n \equiv \alpha^{(N\mathfrak{p}-1)/n} \mod \mathfrak{p}.$$

We call $\left(\frac{}{\mathfrak{p}}\right)_n$ the power residue symbol mod $\mathfrak{p}$ of order n.

§ 3. A proof of Theorem 1.2

The purpose of this section is to prove the second assertion of Theorem 1.2.

§ 3.1. The case (I)

Assume that m is a monic irreducible polynomial of degree two. Take $\gamma \in \mathbb{F}_{q^2}$ so that $m(\gamma) = 0$. Then $f(T) \mapsto f(\gamma)$ gives rise to an isomorphism $A/mA \xrightarrow{\sim} \mathbb{F}_{q^2}$. Now let $\mathfrak{p}$ be a prime ideal of $K = \mathbb{Q}(e^{2\pi i/(q^2-1)})$ dividing p, and let $\chi_\mathfrak{p} = \left(\frac{}{\mathfrak{p}}\right)_{q^2-1}$ be the power residue symbol mod $\mathfrak{p}$ of order $q^2 - 1$. We see that $\chi_\mathfrak{p}^n$ is real if and only if n is divisible by $q - 1$. Therefore, by the equality (2.1), we have

$$Z_m(u) = \prod_{\substack{0 \leq n \leq q^2-2 \\ n \not\equiv 0 \mod q-1}} (1 + s_1(\chi_\mathfrak{p}^n)u). \tag{3.1}$$

Under the identification $A/mA = \mathbb{F}_{q^2}$, we have an equality

$$s_1(\chi_{\mathfrak{p}}^n) = \sum_{\alpha \in \mathbb{F}_q} (\gamma + \alpha)^n$$

in $\mathbb{F}_{q^2} = \mathcal{O}_K/\mathfrak{p}$.

For $1 \leq n \leq q^2 - 2$ ($n \not\equiv 0 \mod q-1$), we consider the q-adic expansion $n = a(n) + b(n)q$. By the Newton formula, we have gotten

$$\sum_{\alpha \in \mathbb{F}_q} (T+\alpha)^n = -\begin{pmatrix} b(n) \\ q-1-a(n) \end{pmatrix} (T^q - T)^{a(n)+b(n)-(q-1)},$$

as was verified by Gekeler (cf. Corollary 3.14 in [Ge]). Here $\begin{pmatrix} * \\ * \end{pmatrix}$ is a binomial coefficient. This implies an equality

$$s_1(\chi_{\mathfrak{p}}^n) = -\begin{pmatrix} b(n) \\ q-1-a(n) \end{pmatrix} (\gamma^q - \gamma)^{a(n)+b(n)-(q-1)}$$

in $\mathbb{F}_{q^2}$. Notice that $\begin{pmatrix} b(n) \\ q-1-a(n) \end{pmatrix} \equiv 0 \mod p$ for $a(n) + b(n) < q-1$. Therefore, by Theorem 2.1 and the equality (3.1), we obtain

$$\text{(3.2)} \quad \lambda_m = {}^{\#}\left\{ 1 \leq n \leq q^2-2 : a(n)+b(n) > q-1, \ \begin{pmatrix} b(n) \\ q-1-a(n) \end{pmatrix} \not\equiv 0 \mod p \right\},$$

where ${}^{\#}S$ is the number of elements of a set S. Next we will calculate the right side of the equality (3.2). For $1 \leq n \leq q^2 - 2$, we put

$$a(n) = a_0(n) + a_1(n)p + \cdots + a_{r-1}(n)p^{r-1},$$
$$b(n) = b_0(n) + b_1(n)p + \cdots + b_{r-1}(n)p^{r-1},$$

where $0 \leq a_i(n),\ b_i(n) \leq p-1$ ($i = 0, 1, ..., r-1$). Since

$$\begin{aligned} &q - 1 - a(n) \\ &= (p-1-a_0(n)) + (p-1-a_1(n))p + \cdots + (p-1-a_{r-1}(n))p^{r-1}, \end{aligned}$$

we have

$$\begin{pmatrix} b(n) \\ q-1-a(n) \end{pmatrix} \equiv \prod_{i=0}^{r-1} \begin{pmatrix} b_i(n) \\ p-1-a_i(n) \end{pmatrix} \mod p.$$

Hence we obtain the following equivalence

$$\begin{pmatrix} b(n) \\ q-1-a(n) \end{pmatrix} \not\equiv 0 \mod p \Leftrightarrow a_i(n) + b_i(n) \geq p-1 \ \ (0 \leq i \leq r-1).$$

We see that

$$\left(\frac{p(p+1)}{2}\right)^r = \# \left\{n \in [0, q^2-1] \ : \ a_i(n) + b_i(n) \geq p-1 \ \ (0 \leq i \leq r-1)\right\},$$
$$q = \# \left\{n \in [0, q^2-1] \ : \ a(n) + b(n) = q-1\right\},$$
$$1 = \# \left\{n \in [0, q^2-1] \ : \ a(n) + b(n) = 2(q-1)\right\},$$

where $[0, q^2-1] = \{0, 1, 2, ..., q^2-1\}$. Therefore we have

$$\lambda_m = \left(\frac{p(p+1)}{2}\right)^r - q - 1.$$

§ 3.2. The case (II)

Let $\alpha \in \mathbb{F}_q$ and $m(T) = (T-\alpha)^2$. Let ε denote the image of $T-\alpha$ in A/mA. Then $f(T) \mapsto f(\alpha) + f'(\alpha)\varepsilon$ gives rise to an isomorphism $A/mA \xrightarrow{\sim} \mathbb{F}_q[\varepsilon]$. It follows that any character $\chi : (A/mA)^\times \to \mathbb{C}^\times$ is given by $f(T) \mapsto \eta(f(\alpha))\psi(f'(\alpha)/f(\alpha))$, where η is a multiplicative character of $\mathbb{F}_q$, and ψ is an additive character of $\mathbb{F}_q$. Furthermore $s_1(\chi)$ is nothing but the Gauss sum $G(\eta^{-1}, \psi)$. It is readily seen that

- χ is trivial $\Leftrightarrow$ η is trivial and ψ is trivial,
- $\deg f_\chi = 1 \Leftrightarrow \eta$ is non-trivial and ψ is trivial,
- $f_\chi = m$ and χ is real $\Leftrightarrow$ η is trivial and ψ is non-trivial,
- $f_\chi = m$ and χ is imaginary $\Leftrightarrow$ η is non-trivial and ψ is non-trivial.

By the equality (2.1), we have

$$Z_m(u) = \prod (1 + G(\eta^{-1}, \psi)u),$$

where η runs through all non-trivial multiplicative characters of $\mathbb{F}_q$, and ψ runs through all non-trivial additive characters of $\mathbb{F}_q$.

Let $\mathfrak{p}$ be a prime ideal of $\mathbb{Q}(e^{2\pi i/p}, e^{2\pi i/(q-1)})$ dividing p. If η is non-trivial and ψ is non-trivial, then we have $G(\eta^{-1}, \psi) \in \mathfrak{p}$ by the Stickelberger theorem for Gauss sums (cf. Theorem 11.2.1 in [B-E-W]). Hence we obtain $\lambda_m = 0$ by Theorem 2.1. This completes the proof of the case (II).

Remark. I appreciate that the referee taught me the above proof. We can generalize the case (II) as follows: $\lambda_{P^n} = 0$ $(n \geq 0)$ if P is a monic polynomial of degree one (cf. Proposition 3.2 in [Sh1]).

§ 3.3. The case (III)

Let $\alpha, \beta \in \mathbb{F}_q$ ($\alpha \neq \beta$) and $m(T) = (T-\alpha)(T-\beta)$. Then $f(T) \mapsto (f(\alpha), f(\beta))$ gives rise to an isomorphism $(A/mA)^\times \xrightarrow{\sim} \mathbb{F}_q^\times \times \mathbb{F}_q^\times$. It follows that any character $\chi : (A/mA)^\times \to \mathbb{C}^\times$ is given by $f(T) \mapsto \chi_1(f(\alpha))\chi_2(f(\beta))$, where χ_1 and χ_2 are the multiplicative characters of $\mathbb{F}_q$. Furthermore we have an equality

$$s_1(\chi) = \chi_2(-1)(\chi_1\chi_2)(\alpha-\beta)J(\chi_1,\chi_2), \tag{3.3}$$

where $J(\chi_1,\chi_2)$ denotes the Jacobi sum associated to χ_1 and χ_2. It is readily seen that

- χ is trivial $\Leftrightarrow$ χ_1 and χ_2 are trivial,
- $\deg f_\chi = 1 \Leftrightarrow$ one of χ_1 and χ_2 is non-trivial and the other is trivial,
- $f_\chi = m$ and χ is real $\Leftrightarrow$ χ_1 and χ_2 are non-trivial and $\chi_1\chi_2$ is trivial ,
- $f_\chi = m$ and χ is imaginary $\Leftrightarrow$ χ_1, χ_2, and $\chi_1\chi_2$ are non-trivial.

Let $\mathfrak{p}$ be a prime ideal of $\mathbb{Q}(e^{2\pi i/(q-1)})$ above p. Let $\chi_\mathfrak{p} = \left(\frac{\ }{\mathfrak{p}}\right)_{q-1}$ be the power residue symbol mod $\mathfrak{p}$ of order $q-1$. Then we have the following one to one corresponding

$$\left\{\chi \in X_m : \begin{array}{l}\chi \text{ is imaginary of} \\ \text{conductor } m\end{array}\right\} \overset{1:1}{\longleftrightarrow} \left\{(\chi_\mathfrak{p}^{n_1}, \chi_\mathfrak{p}^{n_2}) : \begin{array}{l} 1 \le n_1, n_2 \le q-2, \\ n_1+n_2 \not\equiv 0 \mod q-1\end{array}\right\}.$$

By the equalities (2.1) and (3.3), we have

$$\begin{aligned}\lambda_m &= \deg(Z_m(u) \mod \mathfrak{p}) \\ &= \sum_{\substack{1\le n_1\le q-2 \\ 1\le n_2 \le q-2 \\ n_1+n_2\not\equiv 0 \mod q-1}} \deg(1+\chi_\mathfrak{p}^{n_2}(-1)\chi_\mathfrak{p}^{n_1+n_2}(\alpha-\beta)J(\chi_\mathfrak{p}^{n_1},\chi_\mathfrak{p}^{n_2})u \mod \mathfrak{p}).\end{aligned}$$

Next we will calculate $\mathrm{ord}_\mathfrak{p} J(\chi_\mathfrak{p}^{n_1},\chi_\mathfrak{p}^{n_2})$, where $\mathrm{ord}_\mathfrak{p}$ is the valuation of $\mathfrak{p}$. For an integer $n \in \mathbb{Z}$, we define $L(n) \in \mathbb{Z}$ as follows

$$0 \le L(n) < q-1, \quad L(n) \equiv n \mod q-1.$$

We consider the p-adic expansion

$$L(n) = a_0(n) + a_1(n)p + \cdots + a_{r-1}(n)p^{r-1} \quad (0 \le a_i(n) < p).$$

Define $l(n)$ as follows

$$l(n) = a_0(n) + a_1(n) + \cdots + a_{r-1}(n).$$

For $1 \leq n_1, n_2 \leq q-2$ $(n_1+n_2 \neq q-1)$, it is known as the Stickelberger theorem that

$$\begin{aligned}\mathrm{ord}_{\mathfrak{p}} J(\chi_{\mathfrak{p}}^{n_1}, \chi_{\mathfrak{p}}^{n_2}) &= r - \frac{l(n_1)+l(n_2)-l(n_1+n_2)}{p-1} \\ &= r - {}^{\#}\left\{0 \leq i \leq r-1 : L(n_1 p^i) + L(n_2 p^i) > q-1\right\}\end{aligned}$$

(cf. Corollary 11.2.4 and Theorem 11.2.9 in [B-E-W]). Noting that

$$J(\chi_{\mathfrak{p}}^{n_1}, \chi_{\mathfrak{p}}^{n_2}) J(\chi_{\mathfrak{p}}^{q-1-n_1}, \chi_{\mathfrak{p}}^{q-1-n_2}) = q,$$

we obtain

$$\begin{aligned}\lambda_m &= {}^{\#}\left\{(n_1,n_2) \in [1,q-2]^2 : \begin{array}{l} n_1+n_2 \not\equiv 0 \mod q-1, \\ \mathrm{ord}_{\mathfrak{p}} J(\chi_{\mathfrak{p}}^{n_1}, \chi_{\mathfrak{p}}^{n_2}) = 0 \end{array}\right\} \\ &= {}^{\#}\left\{(n_1,n_2) \in [1,q-2]^2 : \begin{array}{l} n_1+n_2 \not\equiv 0 \mod q-1, \\ \mathrm{ord}_{\mathfrak{p}} J(\chi_{\mathfrak{p}}^{n_1}, \chi_{\mathfrak{p}}^{n_2}) = r \end{array}\right\} \\ &= {}^{\#}\left\{(n_1,n_2) \in [1,q-2]^2 : \begin{array}{l} n_1+n_2 \not\equiv 0 \mod q-1, \\ l(n_1)+l(n_2) = l(n_1+n_2) \end{array}\right\}\end{aligned}$$

by the Stickelberger theorem. We see that

$$\begin{aligned}& l(n_1)+l(n_2) = l(n_1+n_2) \\ \iff & L(n_1 p^{r-1-i}) + L(n_2 p^{r-1-i}) \leq q-1 \ (i = 0,1,2,...,r-1) \\ \iff & a_i(n_1) + a_i(n_2) \leq p-1 \ (i = 0,1,2,...,r-1).\end{aligned}$$

Therefore we have

$$\lambda_m = {}^{\#}\left\{(n_1,n_2) \in [1,q-2]^2 : \begin{array}{l} n_1+n_2 \not\equiv 0 \mod q-1, \\ a_i(n_1)+a_i(n_2) \leq p-1 \ (0 \leq i \leq r-1) \end{array}\right\}.$$

Notice that

$$\begin{aligned}\left(\frac{p(p+1)}{2}\right)^r &= {}^{\#}\left\{(n_1,n_2) \in [0,q-1]^2 \ : \ a_i(n_1)+a_i(n_2) \leq p-1 \ \ (0 \leq i \leq r-1)\right\}, \\ 3q-3 &= {}^{\#}\left\{(n_1,n_2) \in [0,q-1]^2 \ : \ n_1 = 0 \text{ or } n_2 = 0 \text{ or } n_1+n_2 = q-1\right\}.\end{aligned}$$

Hence we have

$$\lambda_m = \left(\frac{p(p+1)}{2}\right)^r - 3q + 3.$$

§ 4. A proof of Theorem 1.4

The purpose of this section is to prove Theorem 1.4.

Lemma 4.1. *Let m_1, m_2 be monic polynomials such that $m_1 | m_2$. If K_{m_2} is ordinary, then K_{m_1} is also ordinary.*

This follows from the following general result.

Lemma 4.2. *Let k be a field of characteristic p, and let $\pi : Y \to X$ be a finite covering of projective non-singular curves over k. If Y is ordinary, then X is also ordinary.*

Proof. We give a proof for the reader's convenience. Let A, B be the Jacobians of X, Y, respectively. Then π induces the homomorphism of abelian varieties $\pi^* : A \to B$, and the embedding of p-divisible groups $\pi^* : T_pA \to T_pB$. Assume that Y is ordinary. Then each slope of T_pB is only 0 or 1. Hence each slope of T_pA is only 0 or 1. Therefore X is also ordinary. $\square$

Now we prove Theorem 1.4.

Proof. Assume that $\deg m = 1$. Then we have $g_m = \lambda_m = 0$. Hence K_m is ordinary. Conversely, we assume that K_m is ordinary. Let $m = Q_1^{n_1} Q_2^{n_2} \cdots Q_t^{n_t}$ be the irreducible factorization, where $Q_1, Q_2, ..., Q_t$ are distinct monic polynomials. By Lemma 4.1, $K_{Q_i^{n_i}}$ is ordinary for each i. It follows from Corollary 1.3 and Corollary 3.1 in [Sh2] that $\deg Q_i = 1$ and $n_i = 1$. Now suppose that $t \geq 2$. Again by Lemma 4.1, $K_{Q_1 Q_2}$ is ordinary, which contradicts to the second assertion of Corollary 1.3. $\square$

Acknowledgements

The author wish to thank the referee for many helpful suggestions.

References

[B-E-W] B. C. Berndt, R. J. Evans and K. S. Williams, Gauss and Jacobi sums, Canadian Mathematical Society Series of Monographs and Advanced Texts, Wiley-Interscience Publication, New York, 1998.

[Ge] E.-U. Gekeler, On power sums of polynomials over finite fields, J. Number Theory, **30** (1988), 11–26.

[G-R] S. Galovich and M. Rosen, The class number of cyclotomic function fields, J. Number Theory, **13** (1981), 363–375.

[Go] D. Goss, Basic Structures of Function Field Arithmetic, Springer, Berlin, 1998.

[Ha] D.R. Hayes, Explicit class field theory for rational function fields, Trans. Amer. Math. Soc., **189** (1974), 77–91.

[K-M] M. Kida and N. Murabayashi, Cyclotomic function fields with divisor class number one, Tokyo J. Math., **14** (1991), 45–56.

[Ro] M. Rosen, Number Theory in Function Fields, Springer, Berlin, 2002.

[Sh1] D. Shiomi, The Hasse-Witt invariant of cyclotomic function fields, Acta Arith., **150** (2011), 227–240.
[Sh2] D. Shiomi, Ordinary cyclotomic function fields, J. Number Theory, **133** (2013), 523–533.