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Kyoto University
Periodicity on poly-Euler numbers and Vandiver type congruence for Euler numbers

By

Yasuo Ohno* and Yoshitaka Sasaki**

Abstract

Poly-Euler numbers are introduced as a generalization of classical Euler numbers. In this article, a periodic property for poly-Euler numbers and Vandiver type congruence for Euler numbers are discussed.

§1. Introduction

For every integer $k$, we define poly-Euler numbers $E^{(k)}_n$ ($n = 0, 1, 2, \ldots$) by

\begin{equation}
\frac{\text{Li}_k(1 - e^{-4t})}{4t(\cosh t)} = \sum_{n=0}^{\infty} \frac{E^{(k)}_n}{n!} t^n,
\end{equation}

where

\[ \text{Li}_k(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^k} \quad (|x| < 1, \ k \in \mathbb{Z}) \]

is the $k$-th polylogarithm. Poly-Euler numbers are a generalization of classical Euler numbers $E_n$ defined by

\begin{equation}
\frac{1}{\cosh t} = \sum_{n=0}^{\infty} \frac{E_n}{n!} t^n.
\end{equation}
Indeed, we easily see that $E_n^{(1)} = E_n$. The manner of generalization using the polylogarithm is due to Kaneko [3]. He introduced the poly-Bernoulli numbers $\mathbb{B}_n^{(k)}$ by
\[
\frac{\mathrm{Li}_k(1 - e^{-t})}{1 - e^{-t}} = \sum_{n=0}^{\infty} \frac{\mathbb{B}_n^{(k)}}{n!} t^n \quad (k \in \mathbb{Z})
\]
and discovered many meaningful properties of that. Furthermore, combinatory interpretations for poly-Bernoulli numbers were discovered by Brezbaker [2] and Launois [6] up to the present (see also [10]). In the previous research, the authors presented many properties of poly-Euler numbers ([7] and [8]), for example, explicit formulae, Clausen-von Staudt type formula, and a parity formula. Further we also found certain combinatory interpretations for poly-Euler numbers.

We should mention that the research on poly-Euler numbers relates to that of multiple $L$ values. In fact, poly-Euler numbers are introduced as special values of a generalized Dirichlet $L$-function which relates to multiple $L$-functions (see [9] and [7, 8]). We should also mention Arakawa-Kaneko’s zeta-function. Arakawa and Kaneko [1] introduced a zeta-function which relates to the poly-Bernoulli numbers and the multiple zeta-functions. This property has been applied to the research on the duality for multiple zeta-star values (see [4] and [5]). Our $L$-function would also play a key role in the research on multiple $L$ values.

In this article, we treat a periodic property for poly-Euler numbers with negative index and the Vandiver type congruence for Euler numbers. From the numerical data (see Tables 1 and 2 below), we can find that the one’s digits of poly-Euler numbers $(n + 1)E_{n}^{(-k)}$ $(k, n \geq 0)$ change periodically with respect to $k$ and $n$. We prove this periodicity in the next section. In Section 3, we give the Vandiver type congruence for Euler numbers. Tables 1 and 2 below are the lists of numerical values of poly-Euler numbers $(n + 1)E_{n}^{(-k)}$.

§ 2. Periodicity for the one’s digits of poly-Euler numbers

From the numerical data, we can observe many interesting information of poly-Euler numbers. Here, we focus on the one’s digits of poly-Euler numbers and prove the periodicity. We start with reviewing an explicit formula for poly-Euler numbers which will be used in the following sections. Hereafter, we put $\tilde{E}_n^{(k)} := (n + 1)E_{n}^{(k)}$.

**Theorem 2.1** (Theorems 3.1 and 6.1 in [7]). For any non-negative integer $n$ and any integer $k$, we have
\[
(2.1) \quad \tilde{E}_n^{(k)} = \frac{1}{2} \sum_{m=0}^{n+1} \binom{n+1}{m} \mathbb{B}_{n-m+1}^{(k)} 4^{n-m+1}((-1)^m - (-3)^m).
\]
Table 1. Poly-Euler numbers $\overline{E}_{n}^{(-k)} = (n+1)E_{n}^{(-k)}$ $(0 \leq k \leq 4)$

<table>
<thead>
<tr>
<th>$n \backslash k$</th>
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<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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<td>4</td>
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<td>19811958812317</td>
<td>864960182738653</td>
</tr>
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</table>

Table 2. Poly-Euler numbers $\overline{E}_{n}^{(-k)} = (n+1)E_{n}^{(-k)}$ $(5 \leq k \leq 7)$

<table>
<thead>
<tr>
<th>$n \backslash k$</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
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<td>1</td>
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<td>14004154117362681757</td>
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In particular, for poly-Euler numbers with non-positive index, the above formula can be rewritten as

\[
\tilde{E}_n^{(-k)} = \frac{(-1)^k}{2} \sum_{l=0}^{k} (-1)^l l! \left\{ \begin{array}{l} k \\ l \end{array} \right\} (4l + 3)^{n+1} - (4l + 1)^{n+1} \quad (k \geq 0).
\]

The following theorem gives the periodicity for the one’s digits of poly-Euler numbers:

**Theorem 2.2.** For any non-negative integers \( n, n', k \) and \( k' \) with \( n \equiv n' \pmod{4} \), we have

\[
\tilde{E}_n^{(-k)} \equiv \tilde{E}_{n'}^{(-k')} \pmod{10}.
\]

In particular, we have \( \tilde{E}_n^{(-k)} \equiv 6\mathcal{E}_n^{(-k)} + 5\delta_n \pmod{10} \), where \( \delta_n \) takes 1 if \( n \) is even and 0 otherwise, and

\[
\mathcal{E}_n^{(-k)} = \begin{cases} 
-(3^n + 3) & \text{for } k \equiv 0 \pmod{4}, \\
2^n & \text{for } k \equiv 1 \pmod{4}, \\
-2^n + (1 + (-1)^n) & \text{for } k \equiv 2 \pmod{4}, \\
(2^n + 2)(1 + (-1)^n) & \text{for } k \equiv 3 \pmod{4}.
\end{cases}
\]

**Proof.** The authors had shown the parity of poly-Euler numbers in \([8]\). Namely, we have \( \tilde{E}_n^{(-k)} \equiv \delta_n \pmod{2} \). Therefore we need to show that \( \tilde{E}_n^{(-k)} \equiv \mathcal{E}_n^{(-k)} \pmod{5} \).

From (2.2), we have

\[
2\tilde{E}_n^{(-k)} \equiv (-1)^k \sum_{l=0}^{k} (-1)^l l! \left\{ \begin{array}{l} k \\ l \end{array} \right\} \alpha(n, l) \pmod{5},
\]

where \( \alpha(n, l) := (3 - l)^{n+1} - (1 - l)^{n+1} \). Note that \( \left\{ \begin{array}{l} k \\ l \end{array} \right\} = 0 \) for \( 0 \leq k < l \) and

\[
\left\{ \begin{array}{l} k \\ l \end{array} \right\} \equiv \left\{ \begin{array}{l} k' \\ l \end{array} \right\} \pmod{p}
\]

holds for any odd prime \( p \) and any non-negative integers \( l, k \) and \( k' \) with \( k \equiv k' \pmod{p - 1} \). Therefore, when we put \( k \equiv a, n \equiv b \pmod{4} \) \((a, b \in \{0, 1, 2, 3\})\), the above formula becomes

\[
2\tilde{E}_n^{(-k)} \equiv (-1)^a \sum_{l=0}^{a} (-1)^l l! \left\{ \begin{array}{l} a \\ l \end{array} \right\} \alpha(b, l) \pmod{5},
\]

which gives \( \tilde{E}_n^{(-k)} \equiv \mathcal{E}_n^{(-k)} \pmod{5} \). It follows that \( \tilde{E}_n^{(-k)} \equiv 6\mathcal{E}_n^{(-k)} + 5\delta_n \pmod{10} \) from the Chinese remainder theorem. Furthermore, we have \( \tilde{E}_n^{(-k)} \equiv \tilde{E}_{n'}^{(-k')} \pmod{10} \).
for any non-negative integers \(n, n', k\) and \(k'\) with \(n \equiv n', k \equiv k' \pmod{4}\), since \(E_n^{(-k)} \equiv E_{n'}^{(-k')} \pmod{2}\) and \(E_n^{(-k)} \equiv E_{n'}^{(-k')} \pmod{5}\). Thus we obtain Theorem 2.2.

\[\square\]

§ 3. Vandiver type congruence for the Euler numbers

Kaneko [3] showed the following congruence for the Bernoulli numbers which is originally due to Vandiver from the viewpoint of the poly-Bernoulli numbers: For any odd prime \(p\) and positive integer \(n(\leq p - 2)\),

\[B_n \equiv \sum_{l=1}^{p-2} H_l (l+1)^n \pmod{p},\]

where \(H_n := \sum_{i=1}^{n} i^{-1}\) is the \(n\)-th harmonic number and \(B_n\) is the \(n\)-th Bernoulli number defined by

\[\frac{te^t}{e^t-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n.\]

Similarly, we obtain the following Vandiver type congruence for the Euler numbers from the viewpoint of poly-Euler numbers.

**Theorem 3.1.** For any odd prime \(p\) and non-negative integer \(n\) not exceeding \(p - 3\), we have

\[(n+1)E_n \equiv \sum_{l=1}^{(p-1)/2} H_l \tilde{A}(n, l) \pmod{p},\]

where

\[\tilde{A}(n, l) = \begin{cases} 
0 & \text{when } n \text{ is odd,} \\
1 & \text{when } l = (p - 1)/2 \text{ and } n \text{ is even,} \\
2 \sum_{j=0}^{n/2} \binom{n + 1}{2j + 1} (4l + 2)^{n - 2j} & \text{otherwise.} 
\end{cases}\]

**Proof.** In [3], Kaneko showed an explicit formula for the poly-Bernoulli numbers: For any integers \(k\) and \(n \geq 0\), we have

\[B_n^{(k)} = (-1)^n \sum_{m=0}^{n} \frac{(-1)^m m! \{n\}}{(m + 1)^k}.\]

From this formula, we see that \(B_n^{(1)} \equiv B_n^{(2-p)} \pmod{p}\) for \(n = 0, 1, \ldots, p - 2\) and any odd prime \(p\). Consequently, \(\tilde{E}_n^{(1)} \equiv \tilde{E}_n^{(2-p)} \pmod{p}\) for \(n = 0, 1, \ldots, p - 3\) from Theorem 2.1.
Hereafter we use another explicit formula for poly-Euler numbers which is a modified version of (2.2) (see Corollary 6.6 in [7]):

\[
\widetilde{E}_n^{(-k)} = (-1)^k \sum_{l=0}^{k} (-1)^l l! \binom{k}{l} \sum_{m=1}^{n+1} \binom{n+1}{m} (4l + 2)^{n+1-m}.
\]

Since \(\widetilde{E}_n^{(1)} \equiv \widetilde{E}_n^{(2-p)} \pmod{p}\), we have

\[(3.1) \quad \widetilde{E}_n^{(1)} \equiv - \sum_{l=0}^{p-2} (-1)^l l! \binom{p-2}{l} A(n, l) \pmod{p}.
\]

Here, we have put

\[
A(n, l) := \sum_{j=0}^{[n/2]} \binom{n+1}{2j+1} (4l+2)^{n-2j}.
\]

Since \(4(p-l-1)+2 \equiv -(4l+2) \pmod{p}\), we have

\[
(-1)^{l}l! \binom{p-2}{l} \equiv (-1)^{p-l-1} (p-l-1)! \binom{p-2}{p-l-1} \pmod{p}
\]

holds for any positive integer \(l\) less than \(p-1\) (see Lemma 7.6 in [7]). Hence (3.1) is rewritten as

\[
\widetilde{E}_n^{(1)} \equiv - \sum_{l=1}^{(p-1)/2} (-1)^l l! \binom{p-2}{l} A(n, l) \pmod{p}.
\]

Thus the theorem is proved by using the following lemma:

**Lemma 3.2** (Lemma 2 in [3]). Suppose \(p\) is an odd prime, and \(1 \leq l \leq p-2\). Then,

\[
(-1)^{l-1} l! \binom{p-2}{l} \equiv H_l \pmod{p}.
\]

\[\square\]

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