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Periodicity on poly-Euler numbers and Vandiver type congruence for Euler numbers

By

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Abstract

Poly-Euler numbers are introduced as a generalization of classical Euler numbers. In this article, a periodic property for poly-Euler numbers and Vandiver type congruence for Euler numbers are discussed.

§ 1. Introduction

For every integer k, we define *poly-Euler numbers* $E_n^{(k)}$ $(n = 0, 1, 2, \dots)$ by

$$\frac{\mathrm{Li}_k(1 - e^{-4t})}{4t(\cosh t)} = \sum_{n=0}^{\infty} \frac{E_n^{(k)}}{n!} t^n, \tag{1.1}$$

where

$$\mathrm{Li}_k(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^k} \qquad (|x| < 1,\ k \in \mathbb{Z})$$

is the k-th polylogarithm. Poly-Euler numbers are a generalization of classical Euler numbers E_n defined by

$$\frac{1}{\cosh t} = \sum_{n=0}^{\infty} \frac{E_n}{n!} t^n. \tag{1.2}$$

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Indeed, we easily see that $E_n^{(1)} = E_n$. The manner of generalization using the polylogarithm is due to Kaneko [3]. He introduced the poly-Bernoulli numbers $\mathbb{B}_n^{(k)}$ by

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} = \sum_{n=0}^{\infty} \frac{\mathbb{B}_n^{(k)}}{n!} t^n \qquad (k \in \mathbb{Z})$$

and discovered many meaningful properties of that. Furthermore, combinatorial interpretations for poly-Bernoulli numbers were discovered by Brewbaker [2] and Launois [6] up to the present (see also [10]). In the previous research, the authors presented many properties of poly-Euler numbers ([7] and [8]), for example, explicit formulae, Clausen-von Staudt type formula, and a parity formula. Further we also found certain combinatorial interpretations for poly-Euler numbers.

We should mention that the research on poly-Euler numbers relates to that of multiple L values. In fact, poly-Euler numbers are introduced as special values of a generalized Dirichlet L-function which relates to multiple L-functions (see [9] and [7, 8]). We should also mention Arakawa-Kaneko's zeta-function. Arakawa and Kaneko [1] introduced a zeta-function which relates to the poly-Bernoulli numbers and the multiple zeta-functions. This property has been applied to the research on the duality for multiple zeta-star values (see [4] and [5]). Our L-function would also play a key role in the research on multiple L values.

In this article, we treat a periodic property for poly-Euler numbers with negative index and the Vandiver type congruence for Euler numbers. From the numerical data (see Tables 1 and 2 below), we can find that the one's digits of poly-Euler numbers $(n+1)E_n^{(-k)}$ $(k, n \geq 0)$ change periodically with respect to k and n. We prove this periodicity in the next section. In Section 3, we give the Vandiver type congruence for Euler numbers. Tables 1 and 2 below are the lists of numerical values of poly-Euler numbers $(n+1)E_n^{(-k)}$.

§ 2. Periodicity for the one's digits of poly-Euler numbers

From the numerical data, we can observe many interesting information of poly-Euler numbers. Here, we focus on the one's digits of poly-Euler numbers and prove the periodicity. We start with reviewing an explicit formula for poly-Euler numbers which will be used in the following sections. Hereafter, we put $\widetilde{E}_n^{(k)} := (n+1)E_n^{(k)}$.

Theorem 2.1 (Theorems 3.1 and 6.1 in [7]). *For any non-negative integer n and any integer k, we have*

$$\widetilde{E}_n^{(k)} = \frac{1}{2} \sum_{m=0}^{n+1} \binom{n+1}{m} \mathbb{B}_{n-m+1}^{(k)} 4^{n-m+1}((-1)^m - (-3)^m). \tag{2.1}$$

Table 1. Poly-Euler numbers $\widetilde{E}_n^{(-k)}(=(n+1)E_n^{(-k)})$ $(0 \le k \le 4)$

$n\backslash k$	0	1	2	3	4
0	1	1	1	1	1
1	4	12	28	60	124
2	13	109	493	1837	6253
3	40	888	7192	42840	220120
4	121	6841	95161	865081	6396601
5	364	51012	1189108	16022100	165380884
6	1093	372709	14331493	280592677	3958958053
7	3280	2687088	168625072	4730230320	89841286960
8	9841	19200241	1951326961	77624198641	1961872865521
9	29524	136354812	22314285388	1249160130540	41639632236844
10	88573	964249309	252966361693	19811958812317	864960182738653

Table 2. Poly-Euler numbers $\widetilde{E}_n^{(-k)}(=(n+1)E_n^{(-k)})$ $(5 \le k \le 7)$

$n\backslash k$	5	6	7
0	1	1	1
1	252	508	1020
2	20269	63853	197677
3	1040088	4666072	20235480
4	41968441	255205561	1474388281
5	1460924052	11672605588	87058925460
6	46088370469	473519630053	4461656417317
7	1356820880688	17643342363952	206695980640560
8	37978754131441	617481161118961	8884959409517041
9	1023561900404652	20608543411101868	360705084750192300
10	26796243596416669	662962530489535453	14004154117362681757

In particular, for poly-Euler numbers with non-positive index, the above formula can be rewritten as

$$\widetilde{E}_n^{(-k)} = \frac{(-1)^k}{2}\sum_{l=0}^{k}(-1)^l l!\left\{ {k \atop l} \right\}\{(4l+3)^{n+1}-(4l+1)^{n+1}\} \quad (k \geq 0). \tag{2.2}$$

The following theorem gives the periodicity for the one's digits of poly-Euler numbers:

Theorem 2.2. *For any non-negative integers n, n', k and k' with $n \equiv n', k \equiv k'$* $\pmod 4$, *we have*

$$\widetilde{E}_n^{(-k)} \equiv \widetilde{E}_{n'}^{(-k')} \pmod{10}.$$

In particular, we have $\widetilde{E}_n^{(-k)} \equiv 6\mathcal{E}_n^{(-k)} + 5\delta_n \pmod{10}$, where δ_n takes 1 if n is even and 0 otherwise, and

$$\mathcal{E}_n^{(-k)} = \begin{cases} -(3^n+3) & \text{for } k \equiv 0 \pmod 4, \\ 2^n & \text{for } k \equiv 1 \pmod 4, \\ -2^n + (1+(-1)^n) & \text{for } k \equiv 2 \pmod 4, \\ (2^n+2)(1+(-1)^n) & \text{for } k \equiv 3 \pmod 4. \end{cases}$$

Proof. The authors had shown the parity of poly-Euler numbers in [8]. Namely, we have $\widetilde{E}_n^{(-k)} \equiv \delta_n \pmod 2$. Therefore we need to show that $\widetilde{E}_n^{(-k)} \equiv \mathcal{E}_n^{(-k)} \pmod 5$. From (2.2), we have

$$2\widetilde{E}_n^{(-k)} \equiv (-1)^k \sum_{l=0}^{k}(-1)^l l!\left\{ {k \atop l} \right\}\alpha(n,l) \pmod 5, \tag{2.3}$$

where $\alpha(n,l) := (3-l)^{n+1} - (1-l)^{n+1}$. Note that $\left\{ {k \atop l} \right\} = 0$ for $0 \leq k < l$ and

$$\left\{ {k \atop l} \right\} \equiv \left\{ {k' \atop l} \right\} \pmod p$$

holds for any odd prime p and any non-negative integers l, k and k' with $k \equiv k' \pmod{p-1}$. Therefore, when we put $k \equiv a$, $n \equiv b \pmod 4$ $(a, b \in \{0,1,2,3\})$, the above formula becomes

$$2\widetilde{E}_n^{(-k)} \equiv (-1)^a \sum_{l=0}^{a}(-1)^l l!\left\{ {a \atop l} \right\}\alpha(b,l) \pmod 5,$$

which gives $\widetilde{E}_n^{(-k)} \equiv \mathcal{E}_n^{(-k)} \pmod 5$. It follows that $\widetilde{E}_n^{(-k)} \equiv 6\mathcal{E}_n^{(-k)} + 5\delta_n \pmod{10}$ from the Chinese remainder theorem. Furthermore, we have $\widetilde{E}_n^{(-k)} \equiv \widetilde{E}_{n'}^{(-k')} \pmod{10}$

for any non-negative integers n, n', k and k' with $n \equiv n', k \equiv k' \pmod 4$, since $\widetilde{E}_n^{(-k)} \equiv \widetilde{E}_{n'}^{(-k')} \pmod 2$ and $\widetilde{E}_n^{(-k)} \equiv \widetilde{E}_{n'}^{(-k')} \pmod 5$. Thus we obtain Theorem 2.2. □

§ 3. Vandiver type congruence for the Euler numbers

Kaneko [3] showed the following congruence for the Bernoulli numbers which is originally due to Vandiver from the viewpoint of the poly-Bernoulli numbers: For any odd prime p and positive integer $n(\leq p-2)$,

$$B_n \equiv \sum_{l=1}^{p-2} H_l (l+1)^n \pmod p,$$

where $H_n := \sum_{i=1}^{n} i^{-1}$ is the n-th harmonic number and B_n is the n-th Bernoulli number defined by

$$\frac{te^t}{e^t-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n.$$

Similarly, we obtain the following Vandiver type congruence for the Euler numbers from the viewpoint of poly-Euler numbers.

Theorem 3.1. *For any odd prime p and non-negative integer n not exceeding $p-3$, we have*

$$(n+1)E_n \equiv \sum_{l=1}^{(p-1)/2} H_l \widetilde{A}(n,l) \pmod p,$$

where

$$\widetilde{A}(n,l) = \begin{cases} 0 & \text{when } n \text{ is odd}, \\ 1 & \text{when } l = (p-1)/2 \text{ and } n \text{ is even}, \\ 2\displaystyle\sum_{j=0}^{n/2} \binom{n+1}{2j+1} (4l+2)^{n-2j} & \text{otherwise}. \end{cases}$$

Proof. In [3], Kaneko showed an explicit formula for the poly-Bernoulli numbers: For any integers k and $n \geq 0$, we have

$$\mathbb{B}_n^{(k)} = (-1)^n \sum_{m=0}^{n} \frac{(-1)^m m! \left\{ {n \atop m} \right\}}{(m+1)^k}.$$

From this formula, we see that $\mathbb{B}_n^{(1)} \equiv \mathbb{B}_n^{(2-p)} \pmod p$ for $n = 0, 1, \ldots, p-2$ and any odd prime p. Consequently, $\widetilde{E}_n^{(1)} \equiv \widetilde{E}_n^{(2-p)} \pmod p$ for $n = 0, 1, \ldots, p-3$ from Theorem 2.1.

Hereafter we use another explicit formula for poly-Euler numbers which is a modified version of (2.2) (see Corollary 6.6 in [7]):

$$\widetilde{E}_n^{(-k)} = (-1)^k \sum_{l=0}^{k} (-1)^l l! \left\{ {k \atop l} \right\} \sum_{\substack{m=1 \\ m:odd}}^{n+1} \binom{n+1}{m} (4l+2)^{n+1-m}.$$

Since $\widetilde{E}_n^{(1)} \equiv \widetilde{E}_n^{(2-p)} \pmod p$, we have

$$\widetilde{E}_n^{(1)} \equiv -\sum_{l=0}^{p-2} (-1)^l l! \left\{ {p-2 \atop l} \right\} A(n,l) \pmod p. \tag{3.1}$$

Here, we have put

$$A(n,l) := \sum_{j=0}^{[n/2]} \binom{n+1}{2j+1} (4l+2)^{n-2j}.$$

Since $4(p-l-1)+2 \equiv -(4l+2) \pmod p$, we have

$$A(n,l) \equiv \begin{cases} 0 \pmod p & \text{for } l=(p-1)/2 \text{ and } n \text{ is odd,} \\ 1 \pmod p & \text{for } l=(p-1)/2 \text{ and } n \text{ is even,} \\ (-1)^n A(n,p-l-1) \pmod p & \text{otherwise.} \end{cases}$$

Furthermore we remark that

$$(-1)^l l! \left\{ {p-2 \atop l} \right\} \equiv (-1)^{p-l-1} (p-l-1)! \left\{ {p-2 \atop p-l-1} \right\} \pmod p$$

holds for any positive integer l less than $p-1$ (see Lemma 7.6 in [7]). Hence (3.1) is rewritten as

$$\widetilde{E}_n^{(1)} \equiv -\sum_{l=1}^{(p-1)/2} (-1)^l l! \left\{ {p-2 \atop l} \right\} \widetilde{A}(n,l) \pmod p.$$

Thus the theorem is proved by using the following lemma:

Lemma 3.2 (Lemma 2 in [3]). *Suppose p is an odd prime, and $1 \le l \le p-2$. Then,*

$$(-1)^{l-1} l! \left\{ {p-2 \atop l} \right\} \equiv H_l \pmod p.$$

□

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