<table>
<thead>
<tr>
<th>Title</th>
<th>On $v$-adic periods of $t$-motives: a resume (Algebraic Number Theory and Related Topics 2011)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>MISHIBA, Yoshinori</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録別冊 = RIMS Kokyuroku Bessatsu (2013), B44: 59-65</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2013-12</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/209085">http://hdl.handle.net/2433/209085</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>
On $v$-adic periods of $t$-motives: a resume

By

Yoshinori Mishiba*

Abstract

This is a resume of our results ([7]) on $v$-adic periods of $t$-motives, where $v$ is a “finite” place of the rational function field over a finite field. For a $t$-motive $M$, we define $v$-adic periods of $M$ and the fundamental group of the Tannakian category generated by $M$. Our main result is the transcendental degree of the extension generated by the $v$-adic periods is equal to the dimension of the fundamental group.

§1. Introduction

The special values $\zeta(n)$ of the Riemann zeta function are important objects in number theory. However we do not know how many algebraic relations are there among them over $\mathbb{Q}$. Euler proved that if $n \geq 2$ is an even integer, then we have $\zeta(n)/\pi^n \in \mathbb{Q}$. The odd integer points are more mysterious and we have the following conjecture:

Conjecture 1.1. For each integer $n \geq 2$, we have the equality

$$\text{tr.deg}_Q \mathbb{Q}(\pi, \zeta(2), \ldots, \zeta(n)) = n - [n/2].$$

Here, for a real number $x$, we denote by $[x]$ the largest integer not greater than $x$.

To prove Conjecture 1.1 seems to be very difficult, but the function field analogue of this conjecture was proved by Chang and Yu ([4]). We explain this briefly.

Let $\mathbb{F}_q$ be the finite field with $q$ elements, $p$ the characteristic of $\mathbb{F}_q$, $K := \mathbb{F}_q(\theta)$ the rational function field over $\mathbb{F}_q$, and $K_* := \mathbb{F}_q((\theta^{-1}))$ the $\infty$-adic completion of $K$. For each integer $n \geq 1$, the Carlitz zeta value is defined by

$$\zeta_C(n) := \sum_{a \in \mathbb{F}_q[\theta], \text{monic}} a^{-n} \in K_*.$$
This is the function field analogue of the Riemann zeta values. Fix a \((q - 1)\)st root \((-\theta)^{\frac{1}{q-1}}\) of \(-\theta\) and set

\[ \tilde{\pi} := \theta(-\theta)^{\frac{1}{q-1}} \prod_{i=1}^{\infty} (1 - \theta^{1-\theta^i}(-1) \in K_{\infty}|((-\theta)^{\frac{1}{q-1}}). \]

This is an analogue of \(2\pi\sqrt{-1}\). By the definition of \(\zeta_C\), we have the equalities \(\zeta_C(np^n) = \zeta_C(n)p^n\) for all integers \(m, n \geq 1\). Carllitz proved that if \(n\) is divisible by \(q - 1\), then we have \(\zeta_C(n)/\tilde{\pi}^n \in K\). This is an analogue of the corresponding fact about the Riemann zeta values at positive even integers (note that \(q - 1\) and 2 are respectively the cardinalities of \(F_q[\theta]^\times\) and \(\mathbb{Z}^\times\)). Chang and Yu proved that these are essentially the only relations among the special values.

**Theorem 1.2** ([4, Corollary 4.6]). For each integer \(n \geq 1\), we have the equality

\[ \text{tr.deg}_K K(\tilde{\pi}, \zeta_C(1), \ldots, \zeta_C(n)) = n + 1 - [n/p] - [n/(q-1)] + [n/p(q-1)]. \]

The proof of Theorem 1.2 uses Papanikolas’ result on periods of \(t\)-motives. We explain this result. Let \(t\) be a variable independent of \(\theta\). A \(t\)-motive over \(K\) is a free \(K[t]\)-module \(M\) of finite rank equipped with a “Frobenius action” satisfying certain conditions. Let \(C_{\infty}\) be the \(\infty\)-adic completion of an algebraic closure of \(K_{\infty}\) and \(|\cdot|_{\infty}\) its valuation. Set \(T := \{f \in C_{\infty}[t]|f \text{ converges on } |t|_{\infty} \leq 1 \} \) and \(L := \text{Frac}T\) the fraction field of \(T\). For a \(t\)-motive \(M\), a Betti realization \(H_B(M) \subset L \otimes_{K[t]} M\) is defined. This is an \(F_q(t)\)-vector space and we have \(\text{dim}_{F_q(t)} H_B(M) \leq \text{rank}_{K[t]} M\). Assume that the equality holds. Such \(t\)-motives are called \emph{rigid analytically trivial}. Fix bases \(x\) of \(H_B(M)\) and \(m\) of \(M\). We obtain the matrix \(\Psi = (\Psi_{ij})_{i,j} \in \mathrm{G}L_r(L)\) such that \(m = \Psi x\) in \(L \otimes M\), where \(r\) is the rank of \(M\). We can construct a “good” category of rigid analytically trivial \(t\)-motives and this category forms a neutral Tannakian category with fiber functor \(H_B\). Thus we obtain an algebraic group \(\Gamma \subset \mathrm{G}L_r\) over \(F_q(t)\) which corresponds to the Tannakian subcategory generated by \(M\) via the Tannakian duality.

In this situation, Papanikolas proved the following theorem:

**Theorem 1.3** ([8, Theorem 4.3.1, 4.5.10]). Let \(M\), \(\Psi\) and \(\Gamma\) be as above. Then we have

\[ \text{tr.deg}_{K(t)} K(t)(\Psi_{11}, \Psi_{12}, \ldots, \Psi_{rr}) = \dim \Gamma. \]

Note that, each component of \(\Psi\) converges at \(t = \theta\), and moreover Papanikolas proved the equality

\[ \text{tr.deg}_K K(\Psi_{11}|_{t=\theta}, \Psi_{12}|_{t=\theta}, \ldots, \Psi_{rr}|_{t=\theta}) = \dim \Gamma \]

by using the “\(\text{ABP-criterion}\)” ([2]). Anderson and Thakur showed ([3]) that the Carlitz zeta values are described by linear combinations of entries of \(\Psi|_{t=\theta}\) over \(K\) for certain \(t\)-motives. Hence Theorem 1.2 is proved by calculations of algebraic groups.
For a finite place \( v \), there exist \( v \)-adic zeta values and \( v \)-adic realizations of \( t \)-motives. In \([7]\) we proved a \( v \)-adic analogue of Theorem 1.3. However we do not know whether we can apply this result to the \( v \)-adic zeta values.

§2. \( t \)-motives

Before we state our results, we review \( t \)-motives. The notion of \( t \)-motive was introduced by Anderson in \([1]\). Let \( K \) and \( t \) be as in Section 1. We define an endomorphism \( \sigma \) on \( K[t] \) by

\[
\sigma: K[t] \rightarrow K[t]; \quad \sum_i a_i t^i \mapsto \sum_i a_i^q t^i.
\]

**Definition 2.1.** A \( t \)-motive over \( K \) is a free \( K[t] \)-module \( M \) of finite rank equipped with a \( \sigma \)-semilinear map \( \varphi: M \rightarrow M \) such that

- \( \det \varphi = c(t - \theta)^n \) (\( c \in K^\times \), \( n \geq 0 \)),
- \( M \) is finitely generated over \( K[\varphi] \).

Note that \( \det \varphi \) in the first condition is the determinant of the matrix \( A \in \text{GL}_r(K[t]) \) which satisfies \( \varphi \mathfrak{m} = A \mathfrak{m} \) for a fixed basis \( \mathfrak{m} \) of \( M \), where \( r \) is the \( K[t] \)-rank of \( M \). Since \( K[t]^\times = K^\times \), the validity of the first condition is independent of the choice of \( \mathfrak{m} \).

**Remark.** There exists an anti-equivalence of categories between the category of \( t \)-motives over \( K \) and the category of “abelian \( t \)-modules” over \( K \). Roughly speaking, an abelian \( t \)-module is an algebraic group \( \mathbb{G}_a^d \) over \( K \) for some \( d \geq 0 \) equipped with an \( \mathbb{F}_q[t] \)-action which satisfies certain conditions.

**Example 2.2 (Carlitz \( t \)-motive).** As a \( K[t] \)-module, set \( M := K[t] \). We define a \( \varphi \)-action on \( M \) by

\[
\varphi(a) := (t - \theta) \sigma(a)
\]

for each \( a \in M \). This forms a \( t \)-motive. We call this \( t \)-motive the Carlitz \( t \)-motive. The Carlitz \( t \)-motive corresponds to an abelian \( t \)-module \( \mathbb{G}_a \) equipped with an \( \mathbb{F}_q[t] \)-action defined by \( \mathbb{F}_q[t] \rightarrow \text{End}(\mathbb{G}_a); t \mapsto (x \mapsto \theta x + x^q) \).

§3. \( v \)-adic case

Let \( v \in \mathbb{F}_q[t] \) be an irreducible monic polynomial of degree \( d \). Set \( K^{\text{sep}}(t)_v := K^{\text{sep}}(t) \otimes_{K^{\text{sep}}[t]} \varprojlim_n K^{\text{sep}}[t]/v^n \), the \( v \)-adic completion of \( K^{\text{sep}}(t) \), where \( K^{\text{sep}} \) is a separable closure of \( K \). We define \( K(t)_v \) and \( \mathbb{F}_q(t)_v \) similarly. The endomorphism \( \sigma \) on \( K[t] \)
naturally extends to an endomorphism on $K^{\text{sep}}(t)_{v}$, and we have $(K^{\text{sep}}(t)_{v})^\sigma = \mathbb{F}_q(t)_{v}$, where $(-)^\sigma$ is the $\sigma$-fixed part. For a $t$-motive $M$ over $K$, we set
\[ V(M) := (K^{\text{sep}}(t)_{v} \otimes K[t] M)^\sigma \otimes \varphi. \]

We call $V(M)$ the $v$-adic realization of $M$. This is an $\mathbb{F}_q(t)_{v}$-vector space and the absolute Galois group $G_K := \text{Gal}(K^{\text{sep}}/K)$ of $K$ acts on $V(M)$ naturally. For any $t$-motive $M$, we can prove that $\dim_{\mathbb{F}_q(t)_{v}} V(M) = \text{rank}_{K[t]} M$. Thus we can fix bases $\mathbf{x}$ of $V(M)$ and $\mathbf{m}$ of $M$, we obtain the matrix $\Psi = (\Psi_{ij})_{i,j} \in \text{GL}_r(K^{\text{sep}}(t)_{v})$ such that $\mathbf{m} = \Psi \mathbf{x}$ in $K^{\text{sep}}(t)_{v} \otimes M$. Each component of the matrix $\Psi$ is called a $v$-adic period of $M$. If we factorize $v = \prod_{l \in \mathbb{Z}/d} (t - \lambda_l)$ in $\mathbb{F}_q$, we have the decomposition $K^{\text{sep}}(t)_{v} = \prod_l K^{\text{sep}}((t - \lambda_l))$. Thus we can write $\Psi_{ij} = (\Psi_{ijl})_{l \in \mathbb{Z}/d}$ where $\Psi_{ijl} \in K^{\text{sep}}((t - \lambda_l))$ for each $i$, $j$ and $l$.

To construct the $v$-adic analogue of the algebraic group $\Gamma$, we consider a certain subcategory of the category of $\varphi$-modules over $K(t)_{v}$. A $\varphi$-module over $K(t)_{v}$ is a finite-dimensional $K(t)_{v}$-vector space $N$ equipped with a $\sigma$-semilinear map $\varphi : N \to N$. A morphism between $\varphi$-modules is a $K(t)_{v}$-linear map which commutes with $\varphi$’s. For a $\varphi$-module $N$, we set
\[ V(N) := (K^{\text{sep}}(t)_{v} \otimes K(t)_{v} N)^\sigma \otimes \varphi. \]

We have a natural injection
\[ K^{\text{sep}}(t)_{v} \otimes \mathbb{F}_q(t)_{v} V(N) \to K^{\text{sep}}(t)_{v} \otimes K(t)_{v} N. \]

Let $\mathcal{C}$ be the full subcategory of the category of $\varphi$-modules over $K(t)_{v}$ whose objects are the $\varphi$-modules such that the above map is an isomorphism. We can prove that $K(t)_{v} \otimes K[t] M$ is an object of $\mathcal{C}$ for each $t$-motive $M$. Recall that a neutral Tannakian category over a field $k$ is a rigid abelian $k$-linear tensor category $\mathcal{A}$ for which $k \xrightarrow{\sim} \text{End}(\mathbf{1})$ and there exists an exact faithful $k$-linear tensor functor $\omega : \mathcal{A} \to \text{Vec}(k)$, where $\mathbf{1}$ is the unit object of $\mathcal{A}$ and $\text{Vec}(k)$ is the category of finite-dimensional $k$-vector spaces (cf. [5, Definition 2.19]). Any such functor $\omega$ is said to be a fiber functor for $\mathcal{A}$. We can prove that the category $\mathcal{C}$ forms a neutral Tannakian category over $\mathbb{F}_q(t)_{v}$, and the functor $V$ is a fiber functor for $\mathcal{C}$. For a $t$-motive $M$ over $K$, we set $\mathcal{C}_M$ to be the Tannakian subcategory of $\mathcal{C}$ generated by $K(t)_{v} \otimes M$. Let $\Gamma_v$ be the algebraic group over $\mathbb{F}_q(t)_{v}$ corresponding to $\mathcal{C}_M$ via the Tannakian duality. Thus we obtain the matrix $\Psi$ and the algebraic group $\Gamma_v$ from a $t$-motive $M$. We have the following theorem, which is a $v$-adic analogue of Theorem 1.3:

**Theorem 3.1** ([7, Theorem 4.14, 5.15]). Let $M$, $\Psi$ and $\Gamma_v$ be as above. Then we have
\[ \text{tr.deg}_{K(t)_{v}} K(t)_{v}(\Psi_{11l}, \Psi_{12l}, \ldots, \Psi_{rrl}) = \dim \Gamma_v. \]
for all $l \in \mathbb{Z}/d$.

**Example 3.2.** Let $M$ be the Carlitz $t$-motive defined in Example 2.2. Then we have $\Gamma_v = \mathbb{G}_m$, the multiplicative group over $\mathbb{F}_q(t)_v$, and the transcendental degree is one.

By using Theorem 3.1, we can prove the following proposition:

**Proposition 3.3** ([7, Proposition 6.4, Corollary 7.4]). Fix an integer $n \geq 1$.

1. For each $\alpha \in K$, there exists an element $L_{\alpha,n} = (L_{\alpha,n,l})_{l} \in K^\text{sep}[t]_v = \prod_l K^\text{sep}[t - \lambda_l]$ such that $\sigma(L_{\alpha,n}) = \sigma(\alpha) + L_{\alpha,n}/(t - \theta)^n$.

2. Fix elements $\alpha_1, \ldots, \alpha_r \in K$. If $L_{\alpha_1,n,l}, \ldots, L_{\alpha_r,n,l}$ are linearly independent over $K(t)_v$ for some $l \in \mathbb{Z}/d$, then $L_{\alpha_1,n,l'}, \ldots, L_{\alpha_r,n,l'}$ are algebraically independent over $K(t)_v$ for each $l' \in \mathbb{Z}/d$.

**Remark.** In the $\infty$-adic case, an analogous element of $L_{\alpha,n}$ is constructed explicitly, and its value at $t = \theta$ is the $n$-th Carlitz polylogarithm of $\alpha$. Thus we consider $L_{\alpha,n}$ as a $v$-adic formal polylogarithm.

§ 4. Outline of the proof of Theorem 3.1

In this section, we will sketch the proof of Theorem 3.1. We will construct an algebraic group $\Gamma'$ defined over $\mathbb{F}_q(t)_v$ such that the dimension of $\Gamma'$ is equal to the transcendental degree in Theorem 3.1, and there exists an isomorphism $\Gamma' \cong \Gamma$. We continue to use the notations of the previous sections, and factorize $v = \prod_{l \in \mathbb{Z}/d}(t - \lambda_l)$ in $\overline{\mathbb{F}_q}$ so that $\lambda_l^q = \lambda_{l+1}$ for each $l$. Set $F := \mathbb{F}_q(t)_v$, $E := K(t)_v$, $L := K^\text{sep}(t)_v$ and $L_l := K^\text{sep}((t - \lambda_l))$ for each $l$.

Let $X := (X_{ij})$ be an $r \times r$ matrix of independent variables $X_{ij}$, and set $\Delta := \det(X)$. We set $E[X, \Delta^{-1}] := E[X_{11}, X_{12}, \ldots, X_{rr}, \Delta^{-1}]$. We define $E$-algebra homomorphisms $\nu : E[X, \Delta^{-1}] \rightarrow L; X_{ij} \mapsto \Psi_{ij}$ and $\nu_l : E[X, \Delta^{-1}] \rightarrow L_l; X_{ij} \mapsto \Psi_{ijl}$, and set

$$Z := \text{Spec} E[X, \Delta^{-1}]/\text{Ker} \nu$$

and

$$Z_l := \text{Spec} E[X, \Delta^{-1}]/\text{Ker} \nu_l$$

for each $l$. It is clear that the dimension of $Z_l$ is equal to the transcendental degree which we want to calculate. To construct $\Gamma'$, we define matrices $\tilde{\Psi} = (\tilde{\Psi}_{ij})_{i,j} := (\Psi_{ij} \otimes 1)_{i,j}^{-1}(1 \otimes \Psi_{ij})_{i,j} \in \text{GL}_r(L \otimes_E L)$ and $\tilde{\Psi}_{lm} = (\tilde{\Psi}_{ijm})_{i,j} := (\Psi_{ijl} \otimes 1)_{i,j}^{-1}(1 \otimes \Psi_{ijm})_{i,j} \in \text{GL}_r(L_l \otimes_E L_m)$ for each $l, m \in \mathbb{Z}/d$. We define $F$-algebra homomorphisms
\( \mu : F[X, \Delta^{-1}] \to L \otimes_E L; X_{ij} \mapsto \overline{\Psi}_{ij} \) and
\( \mu_{lm} : F[X, \Delta^{-1}] \to L_l \otimes_E L_m; X_{ij} \mapsto \overline{\Psi}_{ijlm} \)
for each \( l \) and \( m \). Set
\[
\Gamma' := \text{Spec } F[X, \Delta^{-1}] / \text{Ker } \mu
\]
and
\[
\Gamma'_{lm} := \text{Spec } F[X, \Delta^{-1}] / \text{Ker } \mu_{lm}.
\]
By a simple calculation, we have \( \text{Ker } \mu_{0,m} = \text{Ker } \mu_{1,m+1} = \cdots \).
Thus we can set \( \Gamma'_{m} := \Gamma'_{0,m} = \Gamma'_{1,m+1} = \cdots \) for each scheme \( Y \) over \( F \), we set \( Y_E := Y \times \text{Spec } F \text{Spec } E \).

\textbf{Proposition 4.1} ([7, Proposition 4.11]).

(1) Let \( \psi : Z \times_E Z \to Z \times_E \text{GL}_{r/E} \) be the morphism of affine \( E \)-schemes defined by \( (u,v) \mapsto (u,u^{-1}v) \). Then \( \psi \) factors through an isomorphism \( \psi' : Z \times_E Z \cong Z \times_E \Gamma'_E \) of affine \( E \)-schemes.

(2) For any \( l \) and \( m \), let \( \psi_{lm} : Z_l \times_E Z_{l+m} \to Z_l \times_E \text{GL}_{r/E} \) be the morphism of affine \( E \)-schemes defined by \( (u,v) \mapsto (u,u^{-1}v) \). Then \( \psi \) factors through an isomorphism \( \psi'_{lm} : Z_l \times_E Z_{l+m} \cong Z_l \times_E \Gamma'_{m,E} \) of affine \( E \)-schemes.

In particular, we have \( \dim \Gamma' = \dim \Gamma'' = \dim Z_l = \text{tr.deg}_E E(\Psi_{11l}, \Psi_{12l}, \ldots, \Psi_{rsl}) \) for each \( l \) and \( m \).

For any object \( N \) in \( C_M \), we can define a \( \Gamma' \)-action on \( V(N) \). This is functorial in \( N \) and we have a functor
\[
\xi : C_M \to \text{Rep}_F(\Gamma')
\]
which is compatible with tensor products, where \( \text{Rep}_F(\Gamma') \) is the category of finite-dimensional \( F \)-representations of \( \Gamma' \). Thus we obtain the morphism \( f : \Gamma' \to \Gamma_v \) of algebraic groups over \( F \) which corresponds to the functor \( \xi \) via the Tannakian duality. We can check easily that the morphism \( f \) is a closed immersion. To prove that \( f \) is an
On $v$-adic periods of $t$-motives

isomorphism, we consider the Galois representation $G_K \to \text{GL}(V(M))$ which is obtained naturally by the definition of $V$. This representation factors as follows:

$$G_K \to \Gamma'(F) \hookrightarrow \Gamma_v(F) \hookrightarrow \text{GL}(V(M)).$$

Since the functor $V$ induces an equivalence of categories $V: C \to \text{Rep}_F(G_K)$, where $\text{Rep}_F(G_K)$ is the category of finite-dimensional continuous $F$-representations of $G_K$ (cf. [6, Appendix]), the set of $F$-valued points $\Gamma_v(F)$ is dense in $\Gamma_v$. Therefore we conclude that the immersion $f : \Gamma' \hookrightarrow \Gamma_v$ is an isomorphism. This is an essentially different point from Papanikolas’ proof for the $\infty$-adic case, in which the Zariski density is not proved and other facts are used to show this isomorphism.

References