RIMS Kôkyûroku Bessatsu
B44 (2013), 25–38

Néron models of Picard varieties

By

Cédric PÉPIN*

Contents

§ 1. Introduction

Let R be a discrete valuation ring with fraction field K, and let A_K be an abelian variety over K. Néron showed that A_K can be extended to a smooth and separated group scheme A of finite type over R, characterized by the following extension property: for any discrete valuation ring R' étale over R, with fraction field K', the restriction map $A(R') \to A_K(K')$ is surjective.

Suppose that A_K is the Jacobian of a proper smooth geometrically connected curve X_K. By definition, A_K is the Picard variety of X_K. The curve X_K being projective, it can certainly be extended to a proper flat curve X over R. One can then ask about the relation between the Néron model A and the Picard functor of X/R, if there is any. The answer was given by Raynaud. To get a Néron extension property for étale points

Received February 23, 2012. Revised November 15, 2012.
2000 Mathematics Subject Classification(s): 14K30, 14A15, 14G20, 11G10.
Key Words: Picard variety, Néron model, Grothendieck's pairing.
Supported by the FWO project ZKC1235-PS-011 G.0415.10
*KU Leuven, Departement Wiskunde, Celestijnenlaan 200B, 3001 Heverlee, Belgium.
e-mail: cedric.pepin@wis.kuleuven.be

on the Picard side, it is enough to restrict to those X which are regular. Such models of the curve X_K do exist, after Abhyankar and Lipman. Then A is the biggest separated quotient of the schematic closure of A_K in $\mathrm{Pic}_{X/R}$.

Suppose now that A_K is the Picard variety of a proper smooth geometrically connected scheme X_K, of dimension at least two. It is still not known in general whether X_K admits a proper flat and *regular* model over R.

The purpose of this survey is to sketch the construction of models X of X_K, whose Picard functor does satisfy a Néron extension property for étale points. It is then possible to reconstruct the Néron model A from the Picard functor $\mathrm{Pic}_{X/R}$. As a consequence, we obtain that the sections of the identity component A^0 of A can be interpreted as invertible sheaves on X which are algebraically equivalent to zero (when R is complete with algebraically closed residue field). Note that the converse statement, namely, the fact that any such sheaf defines a section of A^0, is not known in general. Precisely, we will see that this statement is related to a conjecture of Grothendieck about the duality theory for Néron models of abelian varieties.

§ 2. Picard varieties

The Picard group of a scheme X is the group of isomorphism classes of invertible sheaves on X, equipped with the tensor product operation. It is denoted by $\mathrm{Pic}(X)$.

Definition 2.1. Let $X \to S$ be a morphism of schemes. The *Picard functor of X over S* is the fppf sheaf associated to the presheaf

$$\begin{aligned}(\mathrm{Sch}/S)^\circ &\longrightarrow (\mathrm{Sets})\\ T &\longmapsto \mathrm{Pic}(X \times_S T).\end{aligned}$$

It is denoted by $\mathrm{Pic}_{X/S}$.

The Picard functor of a proper scheme X over a field K is representable by a K-group scheme locally of finite type (Murre [26] and Oort [28]). The representing scheme is still denoted by $\mathrm{Pic}_{X/K}$ and is called the *Picard scheme of X*. The connected component of the identity section is denoted by $\mathrm{Pic}^0_{X/K}$. It is an open and closed subgroup scheme of finite type of $\mathrm{Pic}_{X/K}$.

Theorem 2.2 ([14] 2.1 (ii) and 3.1). *Let X be a proper geometrically normal scheme over a field K. Then $\mathrm{Pic}^0_{X/K}$ is proper over K, and the reduced subscheme $\mathrm{Pic}^0_{X/K,\mathrm{red}}$ is a subgroup scheme of $\mathrm{Pic}^0_{X/K}$.*

Definition 2.3. Let X be a proper geometrically normal scheme over a field K. The abelian variety $\mathrm{Pic}^0_{X/K,\mathrm{red}}$ is the *Picard variety of X*.

Definition 2.4. Let X be a proper scheme over a field K. An invertible sheaf $\mathcal{L}$ on X is *algebraically equivalent to zero* if the image of its class under the canonical morphism

$$\mathrm{Pic}(X) \longrightarrow \mathrm{Pic}_{X/K}(K)$$

is contained in the Picard variety of X. The group of classes of invertible sheaves on X which are algebraically equivalent to zero is denoted by $\mathrm{Pic}^0(X)$.

Example 2.5. Let X be a proper curve. As the obstruction to the smoothness of $\mathrm{Pic}_{X/K}$ lives in $H^2(X, \mathcal{O}_X)$ ([14] 2.10 (ii)), the scheme $\mathrm{Pic}_{X/K}$ is smooth, and $\mathrm{Pic}^0_{X/K,\mathrm{red}} = \mathrm{Pic}^0_{X/K}$. It is the *Jacobian of* X, which is an abelian variety if X is smooth.

The *degree* of an invertible sheaf $\mathcal{L}$ on the proper curve X is the difference of the coherent Euler characteristics

$$\chi(\mathcal{L}) - \chi(\mathcal{O}_X).$$

Let $\overline{K}$ be an algebraic closure of K and denote by $\overline{X_i}$ the reduced irreducible components of $X \otimes_K \overline{K}$. Then an invertible sheaf on X is algebraically equivalent to zero if and only if $\mathcal{L}|_{\overline{X_i}}$ is of degree zero for all i ([9] 9.2/13).

Example 2.6. Let X be an abelian variety. Denote by

$$m, p_1, p_2 : X \times_K X \longrightarrow X$$

the group law on X, the first and the second projection. An invertible sheaf on X is *primitive* if the invertible sheaf

$$m^*\mathcal{L} \otimes p_1^*\mathcal{L}^{-1} \otimes p_2^*\mathcal{L}^{-1}$$

on $X \times_K X$ is trivial. In [25] § 13, Mumford constructs an abelian variety which parametrizes the primitive invertible sheaves on X. In particular, these sheaves are algebraically equivalent to zero. Conversely, any invertible sheaf on X which is algebraically equivalent to zero is primitive ([25] (vi) page 75). Hence the abelian variety constructed in [25] § 13 coincides with the Picard variety of X. As it is realized as a quotient of X by a finite subgroup scheme, it is of the same dimension as X. Moreover, the tangent space at the identity section of $\mathrm{Pic}_{X/K}$ is $H^1(X, \mathcal{O}_X)$ ([14] 2.10 (iii)), whose rank over K is the dimension of X ([25] Corollary 2 page 129). It follows that $\mathrm{Pic}_{X/K}$ is smooth, so that $\mathrm{Pic}^0_{X/K}$ is the Picard variety of X, called the *dual abelian variety of* X. It is denoted by X'.

As X admits a section, the identity map in $\mathrm{Pic}_{X/K}(\mathrm{Pic}_{X/K})$ can be represented by an invertible sheaf on $X \times_K \mathrm{Pic}_{X/K}$, which is trivial on the two slices $X \times_K \{0\}$ and $\{0\} \times_K \mathrm{Pic}_{X/K}$ ([9] 8.1.4). Its restriction to the product $X \times_K X'$ is the *birigidified Poincaré sheaf of* X.

§ 3. Néron models of abelian varieties

From now on, K will be the fraction field of a discrete valuation ring R, and k will be the residue field of R.

Definition 3.1. Let A_K be an abelian variety over K. A *Néron model* A of A_K is a smooth and separated scheme of finite type over R with generic fiber A_K, such that for all smooth R-scheme Y, the canonical restriction map

$$\mathrm{Hom}_R(Y, A) \longrightarrow \mathrm{Hom}_K(Y_K, A_K)$$

is bijective.

A Néron model of A_K is unique up to unique isomorphism, and is a group scheme over R such that the canonical open immersion $A_K \to A$ is a group homomorphism. Moreover, for any étale extension of discrete valuation rings $R \to R'$, any point of A_K with value in the fraction field of R' extends uniquely as an R'-point of A. In other words, denoting by R^{sh} the strict henselization of R and by K^{sh} its fraction field, any K^{sh}-point of A_K extends as an R^{sh}-point of A. Conversely:

Proposition 3.2 ([9] 7.1/1)**.** *Let A_K be an abelian variety over K and A be a smooth and separated group scheme of finite type with generic fiber A_K. Assume that the restriction map*

$$A(R^{\mathrm{sh}}) \longrightarrow A(K^{\mathrm{sh}})$$

is surjective. Then A is the Néron model of A_K.

Néron models of abelian varieties were constructed in [27].

Theorem 3.3 (Néron)**.** *An abelian variety over K admits a Néron model.*

See [9] for a proof in the language of schemes. See [1], [9] 1.3 and [13] for surveys.

When the abelian variety A_K is given as a Picard variety, we will see in the next section that its Néron model can be constructed using the theory of the Picard functor. Before that, let us illustrate the notion of Néron models by quoting an *ad hoc* construction for elliptic curves.

Definition 3.4. Let X_K be a proper smooth geometrically connected curve over K. A *regular model of* X_K is a proper flat regular scheme X over R with generic fiber X_K. It is *minimal* if for all regular model Y of X_K, every birational map $Y \dashrightarrow X$ extends as a morphism $Y \to X$.

A minimal regular model is unique up to unique isomorphism. The existence of regular models is due to Abhyankar [3] and Lipman [19]. Lipman's proof is also presented in [2]. Lichtenbaum [18], Shafarevich [33] and Néron [27] proved the existence of minimal ones for curves of genus at least one. It is also presented in [20] 9.3.21.

Theorem 3.5. *A proper smooth geometrically connected curve over K of genus at least one admits a minimal regular model.*

Theorem 3.6 (Néron)**.** *The Néron model of an elliptic curve over K is realized by the smooth locus of its minimal regular model.*

See [27] Chapter III. It is also presented in [9] 1.5 and [20] 10.2.14.

§ 4. Néron models of Picard varieties

§ 4.1. The case of Jacobians

Let X_K be a proper smooth geometrically connected curve over K, $J_K := \mathrm{Pic}^0_{X_K/K}$ be its Jacobian and J be the Néron model of J_K. Let X be a (not necessarily minimal) regular model X of X_K and $\mathrm{Pic}_{X/R}$ be its Picard functor. Raynaud showed that J can be constructed from $\mathrm{Pic}_{X/R}$ ([31] 8.1.4).

First note that the generic fiber of $\mathrm{Pic}_{X/R}$ is the whole Picard scheme of X_K. In order to restrict to a relevant subfunctor with generic fiber J_K, one uses the process of *schematic closure.*

Definition 4.1. Let F be an fppf sheaf over the category of R-schemes and G_K be a subsheaf of the generic fiber F_K. The *schematic closure of G_K in F* is the fppf sheaf associated to the presheaf $p\overline{G_K}$ defined as follows: for all R-scheme T, $p\overline{G_K}(T)$ is the set of morphisms $T \to F$ such that there exists a factorization

$$\begin{tikzcd} T \arrow[r] \arrow[dr] & Z \arrow[d, "u"] \\ & F \end{tikzcd}$$

with Z a flat R-scheme and $u_K \in F_K(Z_K)$ contained in $G_K(Z_K)$.

Let P be the schematic closure of J_K in $\mathrm{Pic}_{X/R}$. By definition, an invertible sheaf $\mathcal{L}$ on X defines an element of $P(R)$ if and only if its restriction to X_K defines an element of $J_K(K)$, that is, $\mathcal{L} \otimes K$ is of degree zero (Example 2.5). In particular, P is not separated as soon as the special fiber X_k is not integral: any integral component of X_k is then a non-principal divisor, and hence defines a non-zero section of P whose generic fiber is zero. Hence to go from P to a separated sheaf, it is necessary to make these sections equal to zero, that is, to divide P by the schematic closure $E \subset P$ of the zero section of J_K.

Theorem 4.2 (Raynaud). *Assume that the map* $\mathrm{Br}(K^{\mathrm{sh}}) \to \mathrm{Br}(X_{K^{\mathrm{sh}}})$ *induced by* $X_{K^{\mathrm{sh}}}/K^{\mathrm{sh}}$ *on the Brauer groups is injective. Then* $J = P/E$*.*

We have denoted by K^{sh} the fraction field of the strict henselization of R. The map $\mathrm{Br}(K^{\mathrm{sh}}) \to \mathrm{Br}(X_{K^{\mathrm{sh}}})$ is injective for instance if the residue field k is perfect or if $X_K(K^{\mathrm{sh}})$ is non-empty (see [9] page 203 for references).

Sketch of proof. We have seen that the fppf sheaf P/E is separated. Its representability by a scheme comes from the following fact: a group object in the category of algebraic spaces which is locally of finite type and separated over R is a scheme (Anantharaman [4]). In general, one cannot apply this result directly because P is not always an algebraic space. However, there exists an algebraic space (P, Y) locally of finite type over R together with an étale epimorphism

$$r : (P, Y) \longrightarrow P.$$

The space (P, Y) is constructed using the theory of *rigidificators* ([31] §2). Then P/E can be rewritten as $(P, Y)/H$ where H is the schematic closure of the kernel of r_K in (P, Y). Now the above representability result can be applied to conclude that P/E is a separated group scheme locally of finite type over R.

The scheme P/E is smooth because the relative dimension of X over R is one: as mentioned in Example 2.5, the obstruction to the (formal) smoothness of $\mathrm{Pic}_{X/R}$ vanishes if the H^2 of the fibers of X/R are trivial.

The scheme P/E is of finite type. This can be seen in two different ways. The first one is a consequence of the intersection theory on the regular scheme X ([9] 9.5/11). The second one relies on the finiteness of the Néron-Severi groups of the fibers of X/R, and makes use of the existence of the Néron model J of J_K (*loc. cit.* 9.5/7).

To prove that $J = P/E$, it remains to see that the restriction map $(P/E)(R^{\mathrm{sh}}) \to (P/E)(K^{\mathrm{sh}})$ is surjective (Proposition 3.2). First, using the Leray spectral sequence for the multiplicative group on $X_{K^{\mathrm{sh}}}$, the injectivity assumption at the level of Brauer groups ensures that $\mathrm{Pic}_{X_K/K}(K^{\mathrm{sh}}) = \mathrm{Pic}(X_{K^{\mathrm{sh}}})$. It follows that

$$(P/E)(R^{\mathrm{sh}}) \to (P/E)(K^{\mathrm{sh}})$$

is surjective if

$$\mathrm{Pic}(X_{R^{\mathrm{sh}}}) \longrightarrow \mathrm{Pic}(X_{K^{\mathrm{sh}}})$$

is. The latter is true because X is *regular*. Indeed, as regularity is a local notion with respect to the étale topology, the scheme $X_{R^{\mathrm{sh}}}$ is regular too. So if $\mathcal{L}_{K^{\mathrm{sh}}}$ is an invertible sheaf on $X_{K^{\mathrm{sh}}}$, which can be interpreted as a divisor on $X_{K^{\mathrm{sh}}}$, one can consider the associated cycle and take its schematic closure in $X_{R^{\mathrm{sh}}}$. Then, because of the regularity of $X_{R^{\mathrm{sh}}}$, the resulting 1-codimensional cycle on $X_{R^{\mathrm{sh}}}$ is a divisor. The associated invertible sheaf extends $\mathcal{L}_{K^{\mathrm{sh}}}$ on $X_{R^{\mathrm{sh}}}$. □

§ 4.2. Semi-factorial models

In proving Theorem 4.2, the regularity of the model X is used to show that the scheme P/E is of finite type over R, via intersection theory on X. However we have quoted an alternative argument which is valid even if X is singular. The regularity of X is also used to ensure that the restriction map

$$\mathrm{Pic}(X_{R^{\mathrm{sh}}}) \longrightarrow \mathrm{Pic}(X_{K^{\mathrm{sh}}})$$

is surjective. In higher dimension, it is not known at the present time whether a proper smooth scheme over K admits a proper flat regular model over R. However, the above surjectivity is a weaker property.

Definition 4.3. Let X be a scheme over R. It is *semi-factorial over R* if the restriction map

$$\mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X_K)$$

is surjective.

Theorem 4.4 ([29] 3.1)**.** *A proper geometrically normal scheme over K admits a proper flat normal and semi-factorial model over R, which remains semi-factorial after the extension $R \to R^{\mathrm{sh}}$.*

The first step in the proof of 4.4 is a modification process of a coherent module over a smooth morphism by a blowing-up of *the base*. The latter comes from the flattening techniques of Raynaud and Gruson [32].

Lemma 4.5. *Let B be a noetherian scheme and $Y \to B$ be a smooth morphism of finite type. Let $\mathcal{M}$ be a coherent $\mathcal{O}_Y$-module which is invertible above a schematically dense open subset $U \subset B$. Then there exists a U-admissible blowing-up $B' \to B$ and an invertible sheaf on $Y \times_B B'$ which coincides with $\mathcal{M} \otimes_B B'$ above $U \times_B B'$.*

Sketch of proof of Theorem 4.4. Relying on Lemma 4.5, we will sketch the construction of a semi-factorial model of X_K in the case where the identity map in

$$\mathrm{Pic}_{X_K/K}(\mathrm{Pic}_{X_K/K})$$

can be represented by an invertible sheaf $\mathcal{P}_K$ on $X_K \times_K \operatorname{Pic}_{X_K/K}$ (e.g. if X_K admits a section [9] 8.1.4).

By Nagata's compactification ([11], [12], [23]), the K-scheme X_K admits a proper flat model X/R. As the Néron-Severi group of X_K is finitely generated, there exist invertible sheaves $\mathcal{L}_{K,1}, \dots, \mathcal{L}_{K,r}$ on X_K which generate the image of

$$\operatorname{Pic}(X_K) \longrightarrow \operatorname{Pic}_{X_K/K}(K)/\operatorname{Pic}^0_{X_K/K}(K).$$

Each $\mathcal{L}_{K,i}$ can be extended to a coherent module on X ([16] 9.4.8). Let Λ_0 be the Néron model of the abelian variety $\operatorname{Pic}^0_{X_K/K,\mathrm{red}}$ and extend $\mathcal{P}_K|_{X_K \times_K \operatorname{Pic}^0_{X_K/K,\mathrm{red}}}$ to a coherent module on $X \times_R \Lambda_0$. We obtain in this way a coherent module $\mathcal{M}$ on $X \times_R \Lambda$, where

$$\Lambda := \Lambda_0 \coprod_{i=1}^{r} \operatorname{Spec}(R).$$

Applying Lemma 4.5 with $(Y \to B) = (X \times_R \Lambda \to X)$ and $U = X_K$, we find a blowing-up $X' \to X$ centered in the special fiber of X/R and an invertible sheaf $\widetilde{\mathcal{M}}$ on $X' \times_R \Lambda$ which extends $\mathcal{M} \otimes K$.

Let us show that X' is a semi-factorial model of X_K over R. Let $\mathcal{L}_K$ be an invertible sheaf on X_K. Its image λ_K in $\operatorname{Pic}_{X_K/K}(K)$ can be written as

$$\lambda_{K,0} + \sum_{i=1}^{r} n_i \lambda_{K,i}$$

where $\lambda_{K,0} \in \operatorname{Pic}^0_{X_K/K}(K)$ and for $i = 1, \dots, r$, $\lambda_{K,i}$ is the image of $\mathcal{L}_{K,i}$ in $\operatorname{Pic}_{X_K/K}(K)$ and n_i is some integer. But there are sections λ_i of Λ extending the $\lambda_{K,i}$, and the invertible sheaf

$$\widetilde{\mathcal{M}}|_{1 \times_R \lambda_0} \otimes_{i=1}^{r} (\widetilde{\mathcal{M}}|_{1 \times_R \lambda_i})^{\otimes n_i}$$

extends

$$\mathcal{P}_K|_{1 \times_K \lambda_{K,0}} \otimes_{i=1}^{r} \mathcal{L}_{K,i}^{\otimes n_i} \simeq \mathcal{P}_K|_{1 \times_K \lambda_{K,0}} \otimes_{i=1}^{r} (\mathcal{P}_K|_{1 \times_K \lambda_{K,i}})^{\otimes n_i} \simeq \mathcal{P}_K|_{1 \times_K \lambda_K} \simeq \mathcal{L}_K$$

on X'. □

Using semi-factorial models instead of regular ones, the proof of Theorem 4.2 remains valid in higher dimension, except for the fact that the Picard functor $\operatorname{Pic}_{X/R}$ is formally smooth, which is no longer true in general. Thus we have to include another step in the process of constructing the Néron model of the Picard variety of X_K from $\operatorname{Pic}_{X/R}$.

Definition 4.6. Let G be a group scheme over R which is locally of finite type. A *group smoothening of* G is a morphism $\widetilde{G} \to G$ of R-group schemes, with $\widetilde{G}$ smooth, satisfying the following universal property: any R-morphism from a smooth R-scheme to G admits a unique factorization through $\widetilde{G} \to G$.

A group smoothening of G exists and is unique up to unique isomorphism ([9] 7.1/4)[1].

Theorem 4.7. *Let X_K be a proper geometrically normal and geometrically connected scheme over K, and $A_K := \mathrm{Pic}^0_{X_K/K,\mathrm{red}}$ be its Picard variety. Let X be a proper flat model of X_K which is semi-factorial over R^{sh}. Denote by P the schematic closure of A_K in $\mathrm{Pic}_{X/R}$ and by E that of the unit section. Assume that the map $\mathrm{Br}(K^{\mathrm{sh}}) \to \mathrm{Br}(X_{K^{\mathrm{sh}}})$ induced by $X_{K^{\mathrm{sh}}}/K^{\mathrm{sh}}$ on the Brauer groups is injective. Then P/E is a scheme and its group smoothening realizes the Néron model of A_K.*

§ 4.3. Identity components

Definition 4.8. Let G be a commutative group functor over the category of R-schemes, whose fibers are representable by schemes locally of finite type. The *identity component of* G is the subfunctor G^0 defined as follows. For all R-scheme T, $G^0(T)$ is the set of morphisms $T \to G$ whose two fibers $T_K \to G_K$ and $T_k \to G_k$ factor through the identity components of G_K and G_k respectively.

Let us examine the relationship between the identity components of the Picard functor and of the Néron model.

Theorem 4.9 (Raynaud). *In the situation of Theorem 4.2, the canonical map*

$$\mathrm{Pic}^0_{X/R} \longrightarrow J^0$$

is an epimorphism of fppf sheaves. It is an isomorphism if the gcd of the geometric multiplicities of the irreducible components of X_k is 1.

See [31] 4.2.1 1) and 8.2.1.

Theorem 4.10 ([9] 9.6/1). *In the situation of Theorem 4.2, assume that R is complete and k algebraically closed. Then the canonical map*

$$\mathrm{Pic}^0_{X/R}(R) \longrightarrow J^0(R)$$

is surjective.

In the situation of Theorem 4.7, there is no canonical map from $\mathrm{Pic}^0_{X/R}$ to A, because of the defect of smoothness of $\mathrm{Pic}_{X/R}$. Hence, to get analogous statements, it

[1] In *loc. cit.*, the group scheme G is assumed to be of finite type over R. However, the result remains true without the quasi-compactness assumption. Indeed, as G is a *group* scheme, its defect of smoothness is the same at any R^{sh}-section; in particular, this defect is bounded, so that the same proof works if G is only *locally* of finite type.

is necessary to smooth the Picard functor. However, as the latter is not representable in general, the group smoothening process cannot be applied directly to $\mathrm{Pic}_{X/R}$. We thus have to replace the Picard functor by the rigidified one, which is an algebraic space, for which the group smoothening does make sense. See [29] 10.3 and 10.5. As a corollary, we obtain some information on the algebraic equivalence *on* X/R.

Definition 4.11. Let X be a proper R-scheme. An invertible sheaf on X is *algebraically equivalent to zero (relative to* R*)* if the image of its class under the canonical morphism

$$\mathrm{Pic}(X) \longrightarrow \mathrm{Pic}_{X/R}(R)$$

is contained in the subgroup $\mathrm{Pic}^0_{X/R}(R)$. The group of classes of invertible sheaves on X which are algebraically equivalent to zero is denoted by $\mathrm{Pic}^0(X)$.

In other words, an invertible $\mathcal{O}_X$-module is algebraically equivalent to zero if its restrictions to the fibers X_K and X_k are (Definition 2.4).

Theorem 4.12 ([29] 10.9)**.** *In the situation of Theorem 4.7, assume that* R *is complete and* k *algebraically closed. Then the image of the restriction map*

$$A^0(R) \longrightarrow A_K(K)$$

is contained in the image of the restriction map

$$\mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}^0(X_K) \longrightarrow A_K(K).$$

There is a particular situation where the inverse inclusion holds.

§ 4.4. A conjecture of Grothendieck

Let A_K be an abelian variety over K and A'_K be its Picard variety, that is, its dual abelian variety (Example 2.6). Let A be the Néron model of A_K and A' be that of A'_K. When A is semi-abelian, Künnemann showed in [17] that there exists a canonical projective flat regular R-scheme containing A as a dense open subscheme. In general, a variant of Theorem 4.4 provides a projective flat normal R-scheme $\overline{A}$ containing A as a dense open subscheme, such that the restriction map

$$\mathrm{Pic}(\overline{A}) \longrightarrow \mathrm{Pic}(A)$$

is surjective, and remains surjective after the extension $R \to R^{\mathrm{sh}}$ ([29] 6.2). In particular, Theorem 4.7 applies with $X = \overline{A}$ to get a construction of A' from $\mathrm{Pic}_{\overline{A}/R}$. Moreover, when R is complete and k algebraically closed, Theorem 4.12 asserts that there is a canonical commutative diagram

$$\begin{tikzcd} A'_K(K) \arrow[r, "\sim"] & \mathrm{Pic}^0(A_K) \\ (A')^0(R) \arrow[r] \arrow[u, hook] & \mathrm{Pic}^0(\overline{A}) \arrow[u, hook] \end{tikzcd}$$

(the injectivity of $\mathrm{Pic}^0(\overline{A}) \to \mathrm{Pic}^0(A_K)$ comes from the fact that $\overline{A}$ admits a section, [31] 6.4.1 3)). In particular, the bottom map is injective. The question of its surjectivity is related to a conjecture of Grothendieck about the Néron models A and A'.

Let $\mathcal{P}_K$ be the birigidified Poincaré sheaf on $A_K \times_K A'_K$. Considering $\mathcal{P}_K$ as a line bundle, and removing its zero section, one gets a $\mathbb{G}_{m,K}$-torsor on $A_K \times_K A'_K$, still denoted by $\mathcal{P}_K$. The torsor $\mathcal{P}_K$ is endowed with a richer structure, coming from the fact that there is no non-trivial homomorphism from an abelian scheme to the multiplicative group and from the Theorem of the Square for abelian schemes. The resulting structure is the one of a *biextension of* (A_K, A'_K) *by* $\mathbb{G}_{m,K}$ ([15] VII 2.9.5). The latter means that $\mathcal{P}_K$ admits two partial group scheme structures, namely one over each of the factors of the product $A_K \times_K A'_K$, and that for each of these structures it is an extension of $A_K \times_K A'_K$ by the multiplicative group, in a compatible manner (*loc. cit.* 2.1).

Grothendieck studied the question of the extension of $\mathcal{P}_K$ over R as a biextension of (A, A') by $\mathbb{G}_{m,R}$. Precisely, he constructed the obstruction to the existence of such an extension. This obstruction lives on the *group of connected components* of the special fibers of A and A'. The latter are the étale k-group schemes $\Phi_A := A_k/A_k^0$ and $\Phi_{A'} := A'_k/(A'_k)^0$, and the obstruction is a pairing

$$\langle\ ,\ \rangle : \Phi_A \times_k \Phi_{A'} \longrightarrow \mathbb{Q}/\mathbb{Z}$$

canonically defined from $\mathcal{P}_K$ ([15] IX 1.2.1).

Conjecture 4.13 (Grothendieck). *The pairing* $\langle\ ,\ \rangle$ *is perfect.*

In particular, as soon as Φ_A is non-zero, the obstruction $\langle\ ,\ \rangle$ should not vanish, and the Poincaré biextension $\mathcal{P}_K$ should *not* extend to the Néron models A and A'. However, the duality between A_K and A'_K should be reflected at the level of the component groups Φ_A and $\Phi_{A'}$.

Let us indicate the cases where the conjecture is proved. First, Grothendieck studied the restriction of the pairing to the ℓ-parts of the component groups, with ℓ prime to the characteristic of k, and he also investigated the semi-stable reduction case; see [15] IX 11.3 and 11.4. See [7] and [34] for full proofs. In [15] IX 1.3, Grothendieck also mentions an unpublished work of Artin and Mazur in the case of the Jacobian of a proper smooth curve. Next, Bégueri proved the conjecture in the mixed characteristic case with perfect residue field ([6]), and McCallum in the case where k is finite ([24]). Then Bosch proved the conjecture for abelian varieties with potentially multiplicative reduction, again for perfect residue fields ([10]). Bertapelle and Bosch provided counter-examples to the conjecture when the residue field k is not perfect ([5]). In the case where A_K is the Jacobian of a proper smooth geometrically connected curve X_K, Bosch and Lorenzini proved the conjecture when X_K admits a point in an unramified extension of K ([8]; see also [22] and [30] for slight generalizations). They also provide new counter-examples in the case where k is not perfect. Finally, Loerke proved the conjecture for abelian varieties of small dimension ([21]). The equal characteristic case with infinite residue field remains open in general.

Here is a consequence of the perfectness of Grothendieck's pairing $\langle\ ,\ \rangle$.

Theorem 4.14 ([30] 3.2.1). *Assume R complete and k algebraically closed. Let $\overline{A}$ be a proper flat normal R-scheme containing A as a dense open subscheme, and such that the restriction map $\mathrm{Pic}(\overline{A}) \to \mathrm{Pic}(A)$ is surjective. If Conjecture 4.13 attached to the abelian variety A_K is true, then the canonical map*

$$(A')^0(R) \longrightarrow \mathrm{Pic}^0(\overline{A})$$

is bijective.

There are two steps in the proof of 4.14. First, Bosch and Lorenzini showed that Grothendieck's pairing is a specialization of Néron's local height pairing attached to A_K ([8] 4.4). Second, one describes Néron's pairing in terms of intersection multiplicities on the semi-factorial compactification $\overline{A}$. One can then interpret the perfectness of Grothendieck's pairing as a condition on the algebraic equivalence on $\overline{A}$, and when the latter holds, the map $(A')^0(R) \to \mathrm{Pic}^0(\overline{A})$ is surjective.

References

[1] M. Artin, Néron Models, Arithmetic Geometry, G. Cornell, J. Silverman, Eds, Springer, 1986.

[2] M. Artin, Lipman's Proof of Resolution of Singularities for Surfaces, Arithmetic Geometry, G. Cornell, J. Silverman, Eds, Springer-Verlag, 1986.

[3] S. S. Abhyankar, Resolution of singularities of arithmetic surfaces, Proc. Conf. on Arithmetic Algebraic Geometry, Purdue, 1963.

[4] S. Anantharaman, Schémas en groupes, espaces homogènes et espaces algébriques sur une base de dimension 1, Bull. Soc. Math. France, **33** (1973), 5–79.

[5] A. Bertapelle and S. Bosch, Weil restriction and Grothendieck's duality conjecture, J. Algebraic Geom., **9** (2000), 155–164.

[6] L. Bégueri, Dualité sur un corps local à corps résiduel algébriquement clos, Mém. Soc. Math. France, **108** (1980)

[7] A. Bertapelle, On perfectness of Grothendieck's pairing for ℓ-parts of component groups, J. Reine Angew. Math., **538** (2001), 223–236.

[8] S. Bosch and D. Lorenzini, Grothendieck's pairing on components groups of Jacobians, Invent. Math., **148** (2002), 353–396.

[9] S. Bosch, W. Lütkebohmert and M. Raynaud, Néron Models, Ergeb. Math. Grenzgeb., 3. Folge, **21**, Springer-Verlag, 1990.

[10] S. Bosch, Component groups of abelian varieties and Grothendieck's duality conjecture, Ann. Inst. Fourier, **47** (1997), 1257–1287.

[11] B. Conrad, Deligne's notes on Nagata's compactifications, J. Ramanujan Math. Soc., **22** (2007), 205–257.

[12] P. Deligne, Le théorème de plongement de Nagata, Kyoto J. Math., **50** (2010), 661-670.

[13] B. Edixhoven and M. Romagny, Group schemes out of birational group laws, Néron models, to appear in the proceedings of *École d'été: schémas en groupes, une introduction au séminaire SGA 3 de Demazure-Grothendieck*, Luminy, 2011.

[14] A. Grothendieck, Technique de descente et théorèmes d'existence en géométrie algébrique. VI. Les schémas de Picard : propriétés générales, Séminaire Bourbaki, Exp. 236, 1961-1962, Soc. Math. France, Paris (1995), 221–243.

[15] A. Grothendieck, P. Deligne and N. Katz, Groupes de monodromie en géométrie algébrique, Séminaire de Gémétrie Algébrique 7, Lecture Notes in Math., **288**, Springer (1972).

[16] A. Grothendieck and J. Dieudonné, Éléments de Gémétrie Algébrique, Ch. 1, Le langage des schémas, Publ. Math. Inst. Hautes Études Sci., **4** (1960).

[17] K. Künnemann, Projective regular models for abelian varieties, semistable reduction and the height pairing, Duke Math. J., **95** (1998), 161–212.

[18] S. Lichtenbaum, Curves over discrete valuation rings, Amer. J. Math., **90** (1968), 380–405.

[19] J. Lipman, Desingularization of two dimensional schemes, Ann. of Math., **107** (1978), 151–207.

[20] Q. Liu, Algebraic geometry and arithmetic curves, Oxford Grad. Texts Math., **6**, Oxford Univ. Press, paperback new edition (2006).

[21] K. Loerke, Reduction of abelian varieties and Grothendieck's pairing, *Arxiv* 0909.4425v1.pdf.

[22] D. Lorenzini, Grothendieck's pairing for Jacobians and base change, J. Number Theory, **128** (2008), 1448–1457.

[23] W. Lütkebohmert, On compactification of schemes, Manuscripta Math., **80** (1993), 95–111.

[24] W. McCallum, Duality Theorems for Néron models, Duke Math. J., **53** (1986), 1093–1124.

[25] D. Mumford, *Abelian varieties*, Tata Institute of Fundamental Research, Bombay, Oxford University Press, 1974.

[26] J. P. Murre, On contravariant functors from the category of preschemes over a field into the category of abelian groups, Publ. Math. Inst. Hautes Études Sci., **23** (1964), 5–43.

[27] A. Néron, Modèles minimaux des variétés abéliennes, Publ. Math. Inst.Hautes Étueds Sci., **21** (1964), 5–128.

[28] F. Oort, Sur le schéma de Picard, Bull. Soc. Math. France, **90** (1966), 119–127.

[29] C. Pépin, Modèles semi-factoriels et modèles de Néron, Math. Ann., **355** (2013), 147–185.

[30] C. Pépin, Néron's pairing and relative algebraic equivalence, Algebra Number Theory, **6-7**

(2012), 1315–1348.

[31] M. Raynaud, Spécialisation du foncteur de Picard, Publ. Math. Inst. Hautes Études Sci., **38** (1970), 27–76.

[32] M. Raynaud and L. Gruson, Critères de platitude et de projectivité, Invent. Math., **13** (1971), 1–89.

[33] I. R. Shafarevich, Lectures on minimal models and birational transformations of two dimensional schemes, Tata Inst. Fund. Res. Stud. Math., **37**, Bombay, 1966.

[34] A. Werner, On Grothendieck's pairing on component groups in the semistable reduction case, J. Reine Angew. Math., **436** (1997), 205–215.