Néron models of Picard varieties

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§ 1. Introduction

Let $R$ be a discrete valuation ring with fraction field $K$, and let $A_K$ be an abelian variety over $K$. Néron showed that $A_K$ can be extended to a smooth and separated group scheme $A$ of finite type over $R$, characterized by the following extension property: for any discrete valuation ring $R'$ étale over $R$, with fraction field $K'$, the restriction map $A(R') \to A_K(K')$ is surjective.

Suppose that $A_K$ is the Jacobian of a proper smooth geometrically connected curve $X_K$. By definition, $A_K$ is the Picard variety of $X_K$. The curve $X_K$ being projective, it can certainly be extended to a proper flat curve $X$ over $R$. One can then ask about the relation between the Néron model $A$ and the Picard functor of $X/R$, if there is any. The answer was given by Raynaud. To get a Néron extension property for étale points...
on the Picard side, it is enough to restrict to those $X$ which are regular. Such models of
the curve $X_K$ do exist, after Abhyankar and Lipman. Then $A$ is the biggest separated
quotient of the schematic closure of $A_K$ in $\text{Pic}_{X/R}$.

Suppose now that $A_K$ is the Picard variety of a proper smooth geometrically con‐
nected scheme $X_K$, of dimension at least two. It is still not known in general whether
$X_K$ admits a proper flat and regular model over $R$.

The purpose of this survey is to sketch the construction of models $X$ of $X_K$, whose
Picard functor does satisfy a Néron extension property for étale points. It is then
possible to reconstruct the Néron model $A$ from the Picard functor $\text{Pic}_{X/R}$. As a
consequence, we obtain that the sections of the identity component $A^0$ of $A$ can be
interpreted as invertible sheaves on $X$ which are algebraically equivalent to zero (when
$R$ is complete with algebraically closed residue field). Note that the converse statement,
namely, the fact that any such sheaf defines a section of $A^0$, is not known in general.
Precisely, we will see that this statement is related to a conjecture of Grothendieck
about the duality theory for Néron models of abelian varieties.

§ 2. Picard varieties

The Picard group of a scheme $X$ is the group of isomorphism classes of invertible
sheaves on $X$, equipped with the tensor product operation. It is denoted by $\text{Pic}(X)$.

**Definition 2.1.** Let $X \to S$ be a morphism of schemes. The Picard functor of
$X$ over $S$ is the fppf sheaf associated to the presheaf

$$(\text{Sch }/S)^\circ \to (\text{Sets})$$

$$T \mapsto \text{Pic}(X \times_S T).$$

It is denoted by $\text{Pic}_{X/S}$.

The Picard functor of a proper scheme $X$ over a field $K$ is representable by a $K$-group
scheme locally of finite type (Murre [26] and Oort [28]). The representing
scheme is still denoted by $\text{Pic}_{X/K}$ and is called the Picard scheme of $X$. The connected
component of the identity section is denoted by $\text{Pic}^0_{X/K}$. It is an open and closed
subgroup scheme of finite type of $\text{Pic}_{X/K}$.

**Theorem 2.2** ([14] 2.1 (ii) and 3.1). Let $X$ be a proper geometrically normal
scheme over a field $K$. Then $\text{Pic}^0_{X/K}$ is proper over $K$, and the reduced subscheme
$\text{Pic}^0_{X/K,\text{red}}$ is a subgroup scheme of $\text{Pic}^0_{X/K}$.

**Definition 2.3.** Let $X$ be a proper geometrically normal scheme over a field $K$.
The abelian variety $\text{Pic}^0_{X/K,\text{red}}$ is the Picard variety of $X$. 
**Definition 2.4.** Let $X$ be a proper scheme over a field $K$. An invertible sheaf $\mathcal{L}$ on $X$ is *algebraically equivalent to zero* if the image of its class under the canonical morphism

$$\text{Pic}(X) \rightarrow \text{Pic}_{X/K}(K)$$

is contained in the Picard variety of $X$. The group of classes of invertible sheaves on $X$ which are algebraically equivalent to zero is denoted by $\text{Pic}^0(X)$.

**Example 2.5.** Let $X$ be a proper curve. As the obstruction to the smoothness of $\text{Pic}_{X/K}$ lives in $H^2(X, \mathcal{O}_X)$ ([14] 2.10 (ii)), the scheme $\text{Pic}_{X/K}$ is smooth, and $\text{Pic}_{X/K, \text{red}} = \text{Pic}_{X/K}^0$. It is the *Jacobian* of $X$, which is an abelian variety if $X$ is smooth.

The *degree* of an invertible sheaf $\mathcal{L}$ on the proper curve $X$ is the difference of the coherent Euler characteristics

$$\chi(\mathcal{L}) - \chi(\mathcal{O}_X).$$

Let $\overline{K}$ be an algebraic closure of $K$ and denote by $\overline{X}_i$ the reduced irreducible components of $X \otimes_K \overline{K}$. Then an invertible sheaf on $X$ is algebraically equivalent to zero if and only if $\mathcal{L}|_{\overline{X}_i}$ is of degree zero for all $i$ ([9] 9.2/13).

**Example 2.6.** Let $X$ be an abelian variety. Denote by

$$m, p_1, p_2 : X \times_K X \rightarrow X$$

the group law on $X$, the first and the second projection. An invertible sheaf on $X$ is *primitive* if the invertible sheaf

$$m^* \mathcal{L} \otimes p_1^* \mathcal{L}^{-1} \otimes p_2^* \mathcal{L}^{-1}$$

on $X \times_K X$ is trivial. In [25] § 13, Mumford constructs an abelian variety which parametrizes the primitive invertible sheaves on $X$. In particular, these sheaves are algebraically equivalent to zero. Conversely, any invertible sheaf on $X$ which is algebraically equivalent to zero is primitive ([25] (vi) page 75). Hence the abelian variety constructed in [25] § 13 coincides with the Picard variety of $X$. As it is realized as a quotient of $X$ by a finite subgroup scheme, it is of the same dimension as $X$. Moreover, the tangent space at the identity section of $\text{Pic}_{X/K}$ is $H^1(X, \mathcal{O}_X)$ ([14] 2.10 (iii)), whose rank over $K$ is the dimension of $X$ ([25] Corollary 2 page 129). It follows that $\text{Pic}_{X/K}$ is smooth, so that $\text{Pic}_{X/K}^0$ is the Picard variety of $X$, called the *dual abelian variety* of $X$. It is denoted by $X'$.

As $X$ admits a section, the identity map in $\text{Pic}_{X/K}(\text{Pic}_{X/K})$ can be represented by an invertible sheaf on $X \times_K \text{Pic}_{X/K}$, which is trivial on the two slices $X \times_K \{0\}$ and $\{0\} \times_K \text{Pic}_{X/K}$ ([9] 8.1.4). Its restriction to the product $X \times_K X'$ is the *birigidified Poincaré sheaf* of $X$. 
§ 3. Néron models of abelian varieties

From now on, $K$ will be the fraction field of a discrete valuation ring $R$, and $k$ will be the residue field of $R$.

**Definition 3.1.** Let $A_K$ be an abelian variety over $K$. A Néron model $A$ of $A_K$ is a smooth and separated scheme of finite type over $R$ with generic fiber $A_K$, such that for all smooth $R$-scheme $Y$, the canonical restriction map

$$\text{Hom}_R(Y, A) \to \text{Hom}_K(Y_K, A_K)$$

is bijective.

A Néron model of $A_K$ is unique up to unique isomorphism, and is a group scheme over $R$ such that the canonical open immersion $A_K \to A$ is a group homomorphism. Moreover, for any étale extension of discrete valuation rings $R \to R'$, any point of $A_K$ with value in the fraction field of $R'$ extends uniquely as an $R'$-point of $A$. In other words, denoting by $R^{\text{sh}}$ the strict henselization of $R$ and by $K^{\text{sh}}$ its fraction field, any $K^{\text{sh}}$-point of $A_K$ extends as an $R^{\text{sh}}$-point of $A$. Conversely:

**Proposition 3.2** ([9] 7.1/1). Let $A_K$ be an abelian variety over $K$ and $A$ be a smooth and separated group scheme of finite type with generic fiber $A_K$. Assume that the restriction map

$$A(R^{\text{sh}}) \to A(K^{\text{sh}})$$

is surjective. Then $A$ is the Néron model of $A_K$.

Néron models of abelian varieties were constructed in [27].

**Theorem 3.3** (Néron). An abelian variety over $K$ admits a Néron model.


When the abelian variety $A_K$ is given as a Picard variety, we will see in the next section that its Néron model can be constructed using the theory of the Picard functor. Before that, let us illustrate the notion of Néron models by quoting an ad hoc construction for elliptic curves.

**Definition 3.4.** Let $X_K$ be a proper smooth geometrically connected curve over $K$. A regular model of $X_K$ is a proper flat regular scheme $X$ over $R$ with generic fiber $X_K$. It is minimal if for all regular model $Y$ of $X_K$, every birational map $Y \dashrightarrow X$ extends as a morphism $Y \to X$. 
A minimal regular model is unique up to unique isomorphism. The existence of regular models is due to Abhyankar [3] and Lipman [19]. Lipman’s proof is also presented in [2]. Lichtenbaum [18], Shafarevich [33] and Néron [27] proved the existence of minimal ones for curves of genus at least one. It is also presented in [20] 9.3.21.

**Theorem 3.5.** A proper smooth geometrically connected curve over $K$ of genus at least one admits a minimal regular model.

**Theorem 3.6 (Néron).** The Néron model of an elliptic curve over $K$ is realized by the smooth locus of its minimal regular model.

See [27] Chapter III. It is also presented in [9] 1.5 and [20] 10.2.14.

§ 4. Néron models of Picard varieties

§ 4.1. The case of Jacobians

Let $X_K$ be a proper smooth geometrically connected curve over $K$, $J_K := \text{Pic}^0_{X_K/K}$ be its Jacobian and $J$ be the Néron model of $J_K$. Let $X$ be a (not necessarily minimal) regular model $X$ of $X_K$ and $\text{Pic}_{X/R}$ be its Picard functor. Raynaud showed that $J$ can be constructed from $\text{Pic}_{X/R}$ ([31] 8.1.4).

First note that the generic fiber of $\text{Pic}_{X/R}$ is the whole Picard scheme of $X_K$. In order to restrict to a relevant subfunctor with generic fiber $J_K$, one uses the process of schematic closure.

**Definition 4.1.** Let $F$ be an fppf sheaf over the category of $R$-schemes and $G_K$ be a subsheaf of the generic fiber $F_K$. The schematic closure of $G_K$ in $F$ is the fppf sheaf associated to the presheaf $p\overline{G_K}$ defined as follows: for all $R$-scheme $T$, $p\overline{G_K}(T)$ is the set of morphisms $T \to F$ such that there exists a factorization

$$
\begin{array}{ccc}
T & \longrightarrow & Z \\
\downarrow & & \downarrow u \\
F & \longleftarrow & 
\end{array}
$$

with $Z$ a flat $R$-scheme and $u_K \in F_K(Z_K)$ contained in $G_K(Z_K)$.

Let $P$ be the schematic closure of $J_K$ in $\text{Pic}_{X/R}$. By definition, an invertible sheaf $\mathcal{L}$ on $X$ defines an element of $P(R)$ if and only if its restriction to $X_K$ defines an element of $J_K(K)$, that is, $\mathcal{L} \otimes K$ is of degree zero (Example 2.5). In particular, $P$ is not separated as soon as the special fiber $X_k$ is not integral: any integral component of $X_k$ is then a
non-principal divisor, and hence defines a non-zero section of $P$ whose generic fiber is zero. Hence to go from $P$ to a separated sheaf, it is necessary to make these sections equal to zero, that is, to divide $P$ by the schematic closure $E \subset P$ of the zero section of $J_K$.

**Theorem 4.2** (Raynaud). Assume that the map $\text{Br}(K^{\text{sh}}) \rightarrow \text{Br}(X_{K^{\text{sh}}})$ induced by $X_{K^{\text{sh}}}/K^{\text{sh}}$ on the Brauer groups is injective. Then $J = P/E$.

We have denoted by $K^{\text{sh}}$ the fraction field of the strict henselization of $R$. The map $\text{Br}(K^{\text{sh}}) \rightarrow \text{Br}(X_{K^{\text{sh}}})$ is injective for instance if the residue field $k$ is perfect or if $X_K(K^{\text{sh}})$ is non-empty (see [9] page 203 for references).

*Sketch of proof.* We have seen that the fppf sheaf $P/E$ is separated. Its representability by a scheme comes from the following fact: a group object in the category of algebraic spaces which is locally of finite type and separated over $R$ is a scheme (Anantharaman [4]). In general, one cannot apply this result directly because $P$ is not always an algebraic space. However, there exists an algebraic space $(P, Y)$ locally of finite type over $R$ together with an étale epimorphism

$$r : (P, Y) \rightarrow P.$$ 

The space $(P, Y)$ is constructed using the theory of *rigidificators* ([31] §2). Then $P/E$ can be rewritten as $(P, Y)/H$ where $H$ is the schematic closure of the kernel of $r_K$ in $(P, Y)$. Now the above representability result can be applied to conclude that $P/E$ is a separated group scheme locally of finite type over $R$.

The scheme $P/E$ is smooth because the relative dimension of $X$ over $R$ is one: as mentioned in Example 2.5, the obstruction to the (formal) smoothness of $\text{Pic}_{X/R}$ vanishes if the $H^2$ of the fibers of $X/R$ are trivial.

The scheme $P/E$ is of finite type. This can be seen in two different ways. The first one is a consequence of the intersection theory on the regular scheme $X$ ([9] 9.5/11). The second one relies on the finiteness of the Néron-Severi groups of the fibers of $X/R$, and makes use of the existence of the Néron model $J$ of $J_K$ (loc. cit. 9.5/7).

To prove that $J = P/E$, it remains to see that the restriction map $(P/E)(R^{\text{sh}}) \rightarrow (P/E)(K^{\text{sh}})$ is surjective (Proposition 3.2). First, using the Leray spectral sequence for the multiplicative group on $X_{K^{\text{sh}}}$, the injectivity assumption at the level of Brauer groups ensures that $\text{Pic}_{X_K/K}(K^{\text{sh}}) = \text{Pic}(X_{K^{\text{sh}}})$. It follows that

$$(P/E)(R^{\text{sh}}) \rightarrow (P/E)(K^{\text{sh}})$$

is surjective if

$$\text{Pic}(X_{R^{\text{sh}}}) \rightarrow \text{Pic}(X_{K^{\text{sh}}})$$
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is. The latter is true because $X$ is regular. Indeed, as regularity is a local notion with respect to the étale topology, the scheme $X_{\text{R}^\text{sh}}$ is regular too. So if $\mathcal{L}_{K^\text{sh}}$ is an invertible sheaf on $X_{K^\text{sh}}$, which can be interpreted as a divisor on $X_{K^\text{sh}}$, one can consider the associated cycle and take its schematic closure in $X_{\text{R}^\text{sh}}$. Then, because of the regularity of $X_{\text{R}^\text{sh}}$, the resulting 1-codimensional cycle on $X_{\text{R}^\text{sh}}$ is a divisor. The associated invertible sheaf extends $\mathcal{L}_{K^\text{sh}}$ on $X_{\text{R}^\text{sh}}$. □

§ 4.2. Semi-factorial models

In proving Theorem 4.2, the regularity of the model $X$ is used to show that the scheme $P/E$ is of finite type over $R$, via intersection theory on $X$. However we have quoted an alternative argument which is valid even if $X$ is singular. The regularity of $X$ is also used to ensure that the restriction map

$$\text{Pic}(X_{\text{R}^\text{sh}}) \rightarrow \text{Pic}(X_{K^\text{sh}})$$

is surjective. In higher dimension, it is not known at the present time whether a proper smooth scheme over $K$ admits a proper flat regular model over $R$. However, the above surjectivity is a weaker property.

Definition 4.3. Let $X$ be a scheme over $R$. It is semi-factorial over $R$ if the restriction map

$$\text{Pic}(X) \rightarrow \text{Pic}(X_K)$$

is surjective.

Theorem 4.4 ([29] 3.1). A proper geometrically normal scheme over $K$ admits a proper flat normal and semi-factorial model over $R$, which remains semi-factorial after the extension $R \rightarrow R^\text{sh}$.

The first step in the proof of 4.4 is a modification process of a coherent module over a smooth morphism by a blowing-up of the base. The latter comes from the flattening techniques of Raynaud and Gruson [32].

Lemma 4.5. Let $B$ be a noetherian scheme and $Y \rightarrow B$ be a smooth morphism of finite type. Let $\mathcal{M}$ be a coherent $\mathcal{O}_Y$-module which is invertible above a schematically dense open subset $U \subset B$. Then there exists a $U$-admissible blowing-up $B' \rightarrow B$ and an invertible sheaf on $Y \times_B B'$ which coincides with $\mathcal{M} \otimes_B B'$ above $U \times_B B'$.

Sketch of proof of Theorem 4.4. Relying on Lemma 4.5, we will sketch the construction of a semi-factorial model of $X_K$ in the case where the identity map in

$$\text{Pic}_{X_K/K}(\text{Pic}_{X_K/K})$$
can be represented by an invertible sheaf $\mathcal{P}_K$ on $X_K \times_K \text{Pic}_{X_K/K}$ (e.g. if $X_K$ admits a section [9] 8.1.4).

By Nagata’s compactification ([11], [12], [23]), the $K$-scheme $X_K$ admits a proper flat model $X/R$. As the Néron-Severi group of $X_K$ is finitely generated, there exist invertible sheaves $\mathcal{L}_{K,1}, \ldots, \mathcal{L}_{K,r}$ on $X_K$ which generate the image of

$$\text{Pic}(X_K) \rightarrow \text{Pic}_{X_K/K}(K)/\text{Pic}_{X_K/K}^0(K).$$

Each $\mathcal{L}_{K,i}$ can be extended to a coherent module on $X$ ([16] 9.4.8). Let $\Lambda$ be the Néron model of the abelian variety $\text{Pic}_{X_K/K,\text{red}}^0$ and extend $\mathcal{P}_K|_{X_K \times_K \text{Pic}_{X_K/K,\text{red}}^0}$ to a coherent module on $X \times \Lambda$. We obtain in this way a coherent module $\mathcal{M}$ on $X \times R \Lambda$, where

$$\Lambda := \Lambda_0 \coprod_{i=1}^r \text{Spec}(R).$$

Applying Lemma 4.5 with $(Y \to B) = (X \times_R \Lambda \to X)$ and $U = X_K$, we find a blowing-up $X' \to X$ centered in the special fiber of $X/R$ and an invertible sheaf $\widetilde{\mathcal{M}}$ on $X' \times R \Lambda$ which extends $\mathcal{M} \otimes K$.

Let us show that $X'$ is a semi-factorial model of $X_K$ over $R$. Let $\mathcal{L}_K$ be an invertible sheaf on $X_K$. Its image $\lambda_K$ in $\text{Pic}_{X_K/K}(K)$ can be written as

$$\lambda_{K,0} + \sum_{i=1}^r n_i \lambda_{K,i},$$

where $\lambda_{K,0} \in \text{Pic}_{X_K/K}^0(K)$ and for $i = 1, \ldots, r$, $\lambda_{K,i}$ is the image of $\mathcal{L}_{K,i}$ in $\text{Pic}_{X_K/K}(K)$ and $n_i$ is some integer. But there are sections $\lambda_i$ of $\Lambda$ extending the $\lambda_{K,i}$, and the invertible sheaf

$$\widetilde{\mathcal{M}}|_{1 \times_R \lambda_0} \otimes_{i=1}^r (\widetilde{\mathcal{M}}|_{1 \times_R \lambda_i})^{\otimes n_i}$$

extends

$$\mathcal{P}_K|_{1 \times_K \lambda_{K,0}} \otimes_{i=1}^r \mathcal{L}_{K,i}^{\otimes n_i} \simeq \mathcal{P}_K|_{1 \times_K \lambda_{K,0}} \otimes_{i=1}^r (\mathcal{P}_K|_{1 \times_K \lambda_{K,i}})^{\otimes n_i} \simeq \mathcal{P}_K|_{1 \times_K \lambda_K} \simeq \mathcal{L}_K$$

on $X'$.

Using semi-factorial models instead of regular ones, the proof of Theorem 4.2 remains valid in higher dimension, except for the fact that the Picard functor $\text{Pic}_{X/R}$ is formally smooth, which is no longer true in general. Thus we have to include another step in the process of constructing the Néron model of the Picard variety of $X_K$ from $\text{Pic}_{X/R}$.

**Definition 4.6.** Let $G$ be a group scheme over $R$ which is locally of finite type. A group smoothening of $G$ is a morphism $\tilde{G} \to G$ of $R$-group schemes, with $\tilde{G}$ smooth, satisfying the following universal property: any $R$-morphism from a smooth $R$-scheme to $G$ admits a unique factorization through $\tilde{G} \to G$. 
A group smoothening of $G$ exists and is unique up to unique isomorphism ([9] 7.1/4)\(^1\).

**Theorem 4.7.** Let $X_K$ be a proper geometrically normal and geometrically connected scheme over $K$, and $A_K := \text{Pic}_{X_K/K,\text{red}}^0$ be its Picard variety. Let $X$ be a proper flat model of $X_K$ which is semi-factorial over $R^\text{sh}$. Denote by $P$ the schematic closure of $A_K$ in $\text{Pic}_{X/R}$ and by $E$ that of the unit section. Assume that the map $\text{Br}(K^\text{sh}) \to \text{Br}(X_{K^\text{sh}})$ induced by $X_{K^\text{sh}}/K^\text{sh}$ on the Brauer groups is injective. Then $P/E$ is a scheme and its group smoothening realizes the Néron model of $A_K$.

### §4.3. Identity components

**Definition 4.8.** Let $G$ be a commutative group functor over the category of $R$-schemes, whose fibers are representable by schemes locally of finite type. The *identity component of $G$* is the subfunctor $G^0$ defined as follows. For all $R$-scheme $T$, $G^0(T)$ is the set of morphisms $T \to G$ whose two fibers $T_K \to G_K$ and $T_k \to G_k$ factor through the identity components of $G_K$ and $G_k$ respectively.

Let us examine the relationship between the identity components of the Picard functor and of the Néron model.

**Theorem 4.9** (Raynaud). In the situation of Theorem 4.2, the canonical map

$$\text{Pic}_{X/R}^0 \to J^0$$

is an epimorphism of fppf sheaves. It is an isomorphism if the gcd of the geometric multiplicities of the irreducible components of $X_k$ is 1.

See [31] 4.2.1) and 8.2.1.

**Theorem 4.10** ([9] 9.6/1). In the situation of Theorem 4.2, assume that $R$ is complete and $k$ algebraically closed. Then the canonical map

$$\text{Pic}_{X/R}^0(R) \to J^0(R)$$

is surjective.

In the situation of Theorem 4.7, there is no canonical map from $\text{Pic}_{X/R}^0$ to $A$, because of the defect of smoothness of $\text{Pic}_{X/R}$. Hence, to get analogous statements, it

\(^1\)In loc. cit., the group scheme $G$ is assumed to be of finite type over $R$. However, the result remains true without the quasi-compactness assumption. Indeed, as $G$ is a group scheme, its defect of smoothness is the same at any $R^\text{sh}$-section; in particular, this defect is bounded, so that the same proof works if $G$ is only locally of finite type.
is necessary to smooth the Picard functor. However, as the latter is not representable in general, the group smoothening process cannot be applied directly to $\text{Pic}_{X/R}$. We thus have to replace the Picard functor by the rigidified one, which is an algebraic space, for which the group smoothening does make sense. See [29] 10.3 and 10.5. As a corollary, we obtain some information on the algebraic equivalence on $X/R$.

**Definition 4.11.** Let $X$ be a proper $R$-scheme. An invertible sheaf on $X$ is *algebraically equivalent to zero (relative to $R$)* if the image of its class under the canonical morphism

$$\text{Pic}(X) \to \text{Pic}_{X/R}(R)$$

is contained in the subgroup $\text{Pic}_{X/R}^0(R)$. The group of classes of invertible sheaves on $X$ which are algebraically equivalent to zero is denoted by $\text{Pic}^0(X)$.

In other words, an invertible $\mathcal{O}_X$-module is algebraically equivalent to zero if its restrictions to the fibers $X_K$ and $X_k$ are (Definition 2.4).

**Theorem 4.12 ([29] 10.9).** In the situation of Theorem 4.7, assume that $R$ is complete and $k$ algebraically closed. Then the image of the restriction map

$$A^0(R) \to A_K(K)$$

is contained in the image of the restriction map

$$\text{Pic}^0(X) \to \text{Pic}^0(X_K) \to A_K(K).$$

There is a particular situation where the inverse inclusion holds.

**§ 4.4. A conjecture of Grothendieck**

Let $A_K$ be an abelian variety over $K$ and $A'_K$ be its Picard variety, that is, its dual abelian variety (Example 2.6). Let $A$ be the Néron model of $A_K$ and $A'$ be that of $A'_K$. When $A$ is semi-abelian, Künemann showed in [17] that there exists a canonical projective flat regular $R$-scheme containing $A$ as a dense open subscheme. In general, a variant of Theorem 4.4 provides a projective flat normal $R$-scheme $\overline{A}$ containing $A$ as a dense open subscheme, such that the restriction map

$$\text{Pic}(\overline{A}) \to \text{Pic}(A)$$

is surjective, and remains surjective after the extension $R \to R^{sh}$ ([29] 6.2). In particular, Theorem 4.7 applies with $X = \overline{A}$ to get a construction of $A'$ from $\text{Pic}_{\overline{A}/R}$. Moreover, when $R$ is complete and $k$ algebraically closed, Theorem 4.12 asserts that there is a
canonical commutative diagram

\[
\begin{array}{c}
A'_{K}(K) \rightarrow & \sim & \text{Pic}^{0}(A_{K}) \\
\uparrow & & \uparrow \\
(A')^{0}(R) & \rightarrow & \text{Pic}^{0}(\overline{A})
\end{array}
\]

(the injectivity of \(\text{Pic}^{0}(\overline{A}) \rightarrow \text{Pic}^{0}(A_{K})\) comes from the fact that \(\overline{A}\) admits a section, \([31] 6.4.1.3\)). In particular, the bottom map is injective. The question of its surjectivity is related to a conjecture of Grothendieck about the Néron models \(A\) and \(A'\).

Let \(\mathcal{P}_{K}\) be the birigidified Poincaré sheaf on \(A_{K} \times_{K} A'_{K}\). Considering \(\mathcal{P}_{K}\) as a line bundle, and removing its zero section, one gets a \(\mathbb{G}_{m,K}\)-torsor on \(A_{K} \times_{K} A'_{K}\), still denoted by \(\mathcal{P}_{K}\). The torsor \(\mathcal{P}_{K}\) is endowed with a richer structure, coming from the fact that there is no non-trivial homomorphism from an abelian scheme to the multiplicative group and from the Theorem of the Square for abelian schemes. The resulting structure is the one of a biextension of \((A_{K}, A'_{K})\) by \(\mathbb{G}_{m,K}\) \(([15] \text{VII 2.9.5})\). The latter means that \(\mathcal{P}_{K}\) admits two partial group scheme structures, namely one over each of the factors of the product \(A_{K} \times_{K} A'_{K}\), and that for each of these structures it is an extension of \(A_{K} \times_{K} A'_{K}\) by the multiplicative group, in a compatible manner \((\text{loc. cit. 2.1})\).

Grothendieck studied the question of the extension of \(\mathcal{P}_{K}\) over \(R\) as a biextension of \((A, A')\) by \(\mathbb{G}_{m,R}\). Precisely, he constructed the obstruction to the existence of such an extension. This obstruction lives on the group of connected components of the special fibers of \(A\) and \(A'\). The latter are the étale \(k\)-group schemes \(\Phi_{A} := A_{k}/A_{k}^{0}\) and \(\Phi_{A'} := A'_{k}/(A'_{k})^{0}\), and the obstruction is a pairing

\[\langle \ , \ \rangle : \Phi_{A} \times_{k} \Phi_{A'} \rightarrow \mathbb{Q}/\mathbb{Z}\]

canonically defined from \(\mathcal{P}_{K}\) \(([15] \text{IX 1.2.1})\).

**Conjecture 4.13** (Grothendieck). *The pairing \(\langle \ , \ \rangle\) is perfect.*

In particular, as soon as \(\Phi_{A}\) is non-zero, the obstruction \(\langle \ , \ \rangle\) should not vanish, and the Poincaré biextension \(\mathcal{P}_{K}\) should not extend to the Néron models \(A\) and \(A'\). However, the duality between \(A_{K}\) and \(A'_{K}\) should be reflected at the level of the component groups \(\Phi_{A}\) and \(\Phi_{A'}\).

Let us indicate the cases where the conjecture is proved. First, Grothendieck studied the restriction of the pairing to the \(\ell\)-parts of the component groups, with \(\ell\) prime to the characteristic of \(k\), and he also investigated the semi-stable reduction case; see \([15] \text{IX 11.3 and 11.4}\). See \([7]\) and \([34]\) for full proofs. In \([15] \text{IX 1.3}\), Grothendieck also mentions an unpublished work of Artin and Mazur in the case of the Jacobian of
a proper smooth curve. Next, Bégueri proved the conjecture in the mixed characteristic case with perfect residue field ([6]), and McCallum in the case where $k$ is finite ([24]). Then Bosch proved the conjecture for abelian varieties with potentially multiplicative reduction, again for perfect residue fields ([10]). Bertapelle and Bosch provided counter-examples to the conjecture when the residue field $k$ is not perfect ([5]). In the case where $A_K$ is the Jacobian of a proper smooth geometrically connected curve $X_K$, Bosch and Lorenzini proved the conjecture when $X_K$ admits a point in an unramified extension of $K$ ([8]; see also [22] and [30] for slight generalizations). They also provide new counter-examples in the case where $k$ is not perfect. Finally, Loerke proved the conjecture for abelian varieties of small dimension ([21]). The equal characteristic case with infinite residue field remains open in general.

Here is a consequence of the perfectness of Grothendieck’s pairing $\langle,\rangle$.

**Theorem 4.14** ([30] 3.2.1). Assume $R$ complete and $k$ algebraically closed. Let $\overline{A}$ be a proper flat normal $R$-scheme containing $A$ as a dense open subscheme, and such that the restriction map $\text{Pic}(\overline{A}) \to \text{Pic}(A)$ is surjective. If Conjecture 4.13 attached to the abelian variety $A_K$ is true, then the canonical map

$$(A')^0(R) \to \text{Pic}^0(\overline{A})$$

is bijective.

There are two steps in the proof of 4.14. First, Bosch and Lorenzini showed that Grothendieck’s pairing is a specialization of Néron’s local height pairing attached to $A_K$ ([8] 4.4). Second, one describes Néron’s pairing in terms of intersection multiplicities on the semi-factorial compactification $\overline{A}$. One can then interpret the perfectness of Grothendieck’s pairing as a condition on the algebraic equivalence on $\overline{A}$, and when the latter holds, the map $(A')^0(R) \to \text{Pic}^0(\overline{A})$ is surjective.

**References**

(2012), 1315–1348.


