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Research Article

Robust Nonlinear H^∞ Control Design via Stable Manifold Method

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This paper proposes a systematic numerical method for designing robust nonlinear H^∞ controllers without a priori lower-dimensional approximation with respect to solutions of the Hamilton-Jacobi equations. The method ensures the solutions are globally calculated with arbitrary accuracy in terms of the stable manifold method that is a solver of Hamilton-Jacobi equations in nonlinear optimal control problems. In this realization, the existence of stabilizing solutions of the Hamilton-Jacobi equations can be derived from some properties of the linearized system and the equivalent Hamiltonian system that is obtained from a transformation of the Hamilton-Jacobi equation. A numerical example is shown to validate the design method.

1. Introduction

Robust controls have been extensively studied to suppress the effects of disturbances or noises on performances of controllers. In particular, the appearance of robust H^∞ control [1] caused the paradigm shift in control theory. The linear H^∞ control has been extended to deal with nonlinear systems [2–4]. The nonlinear H^∞ control design can be described as a problem of solving Hamilton-Jacobi-Isaac equations. However, it is difficult to directly solve Hamilton-Jacobi-Isaac equations as against Riccati equations in the linear case that many practical solving methods have been elaborated. According to the latest reference book [4], there is no systematic numerical approach for solving the Hamilton-Jacobi-Isaac equations at present. Although a lot of efforts have been made [5–13], all the contributions are still valid in a local region around the equilibrium on which low-dimensional approximations of the solutions are valid. Some possible approaches that may yield exact and global solutions are also reviewed in [4].

On the other hand, an effective numerical solver for Hamilton-Jacobi equations in nonlinear optimal control problems that is called the stable manifold method was recently presented [14]. The method has been applied to various control problems [15]. However, their results are basically on pure optimal controls, and robust control designs have not been sufficiently studied in the framework. Optimal controllers without careful thought on robustness might cause instability in systems with disturbances. Thus, the development of robust controls is quite important in nonlinear control design using the stable manifold method.

This paper clarifies the way of implementing robust nonlinear H^∞ control design to the stable manifold method [14]. We believe that our result is the premier report of realizing the nonlinear H^∞ control without a priori lower-dimensional approximation with respect to solutions of the Hamilton-Jacobi equations. The conventional approximate methods based on the Taylor expansion for solving the equations have the critical problem that the valid range of the approximation is unextendable [15]. In our approach, the solutions of the equation can be systematically calculated in a global domain with arbitrary accuracy in terms of the stable manifold method. In our method, we transform the Hamilton-Jacobi-Isaac equation to an equivalent Hamiltonian system under the assumption that there are no cross-product terms in cost functions, and there is no need to restrict the weight on the control to an identity matrix, which is relaxed from the typical simplification on weights. The existence of stabilizing solutions of the Hamilton-Jacobi(-Isaac) equations can be checked by the stabilizability of the linearized system. The numerical scheme of the stable manifold method is based on the separation of the linear part of the Hamiltonian system that is equivalent to the Hamilton-Jacobi(-Isaac) equation from the nonlinear part. The separation can be achieved if a given system is stabilizable, and the transformation for the separation can be systematically given. Hence, we can apply this method to a wide range of nonlinear control systems. The robust performance of the controller can be designed by choosing the design parameter γ that means the upper bound of the worst response, that is, the H^∞ norm of the system defined by $\mathscr{L}^2$-gain.

This paper is organized as follows: Section 2 makes a brief summary of basic definitions of robust nonlinear H^∞ control. Section 3 shows that the nonlinear H^∞ control design can be converted with the stable manifold method. In Section 4, we show the validity of the nonlinear H^∞ controller derived from the stable manifold method by showing a robustness improvement of a controlled vehicle model [16] under disturbances modeled as an artificial effect of side winds and a rough road surface. In this numerical experimentation, we can see that the nonlinear H^∞ controller achieved a higher robust performance than a linear H^∞ controller in the case that a nonlinear optimal regulator fails stabilization under the disturbances.

2. Summary of Robust Nonlinear H^∞ Control

This section makes a brief summary of basic definitions of robust nonlinear H^∞ control.

2.1. Nonlinear H^∞ Control Design. In this paper, we consider the following standard form of control systems as an objective.

Definition 1. Let one consider the following control system defined on a smooth n-dimensional manifold $\mathscr{X} \subseteq \mathbb{R}^n$:

$$\Sigma : \begin{cases} \dot{x} = f(x) + g_1(x) w + g_2(x) u, \\ y = x, \\ z = h(x) + k(x) u, \end{cases} \tag{1}$$

where the vectors $x(t) \in \mathscr{X}$, $u(t) \in \mathscr{U} \subseteq \mathbb{R}^p$, $w(t) \in \mathscr{W}$, $y(t) \in \mathbb{R}^n$, and $z(t) \in \mathbb{R}^q$ denote state variables, control inputs, disturbances, outputs that can be directly measured, and outputs that are controlled, respectively. In (1), one has defined $\mathscr{U}$ and $\mathscr{W}$, respectively, as the set of admissible controls and the set of admissible disturbances, where a function is called admissible if the function is defined on some time interval and it is piecewise continuous. Furthermore, one denotes the initial state at the time t_0 by $x(t_0) = 0$, and the functions $f: \mathscr{X} \to \mathscr{V}(\mathscr{X})$, $g_1: \mathscr{X} \to \mathscr{M}^{n\times r}(\mathscr{X})$, $g_2: \mathscr{X} \to \mathscr{M}^{n\times p}(\mathscr{X})$, $h_1: \mathscr{X} \to \mathbb{R}^s$, and $k: \mathscr{X} \to \mathscr{M}^{p\times m}(\mathscr{X})$ are assumed to be real C^∞-functions of x, where $\mathscr{V}$ is the vector space of all smooth vector fields over $\mathscr{X}$ and $\mathscr{M}^{i\times j}(\mathscr{X})$ is the ring of $(i \times j)$ matrices over $\mathscr{X}$.

To the system Σ, we consider the following conditions for simplification.

Assumption 2. (1) $x = 0$ is a unique equilibrium point of the system Σ in (1) when $u = 0$ and $w = 0$.

(2) $f(0) = 0$, $h(0) = 0$, and $k^\top(x)k(x) > 0$ hold.

(3) There exists a unique solution $x(t, t_0, x_0, u)$ on the time interval $[t_0, \infty] \in \mathbb{R}$ that continuously depends on the initial condition x_0.

In robust nonlinear H^∞ control, the effect of the signal w to the reference output z is evaluated by the following inequality that will be related with an $\mathscr{L}^2$-gain in the next definition.

Definition 3. System (1) is said to have an $\mathscr{L}^2$-gain less than or equal to γ from w to z in $\mathscr{X}$ if

$$\|z(t)\|_2^2 \le \gamma^2 \|w(t)\|_2^2 + \beta(x_0), \quad \forall T > t_0 \tag{2}$$

for any $x_0 \in \mathscr{X}$, a fixed u, and some bounded C^0-function $\beta: \mathscr{X} \to \mathbb{R}$ such that $\beta(0) = 0$, where one has defined the $\mathscr{L}^2$-norm

$$\|\nu\|_2 = \left(\int_{t_0}^{T} \|\nu(t)\|^2 \, dt \right)^{1/2} \tag{3}$$

for any $\nu: [t_0, T] \subset \mathbb{R} \to \mathbb{R}^n$, where $\|\cdot\|$ means the Euclidean norm on $\mathbb{R}^n$; that is, $\|\nu(t)\|^2 = \nu^\top(t)\nu(t)$.

According to Definition 3, the usual H^∞ norm in a frequency domain can be interpreted as the following $\mathscr{L}^2$-gain that is the induced norm from $\mathscr{L}^2$ to $\mathscr{L}^2$ in the time domain.

Definition 4. One defines the following H^∞ norm of the system Σ:

$$\|\Sigma\|_{\mathscr{H}^\infty} = \sup_{w \in \mathscr{L}^2 \cap \mathscr{L}_c^\infty \setminus \{0\}} \frac{\|z\|_2}{\|w\|_2}, \quad x(t_0) = 0, \tag{4}$$

where $w \in \mathscr{L}^2 \cap \mathscr{L}_c^\infty \setminus \{0\}$ means that $w \in \mathscr{L}^2$ satisfies $\sup_t |w(t)| \le c$ for some constant c and $w \ne 0$.

Remark 5. In the linear H^∞ control design, the disturbance is defined as a function in $\mathscr{L}^2$. On the other hand, in the nonlinear H^∞ control design, the class of disturbances is limited as $w \in \mathscr{L}^2 \cap \mathscr{L}_c^\infty \setminus \{0\}$, because an asymptotical stability does not always hold in a global domain.

By using the above definitions, we state the main problem that is treated in this paper.

Definition 6 (nonlinear H^∞ control problem). Let $\gamma > 0$ be a constant that is a design parameter with respect to disturbances. Then, find a control input u satisfying $\|\Sigma\|_{\mathscr{H}^\infty} \le \gamma$ for the system Σ in (1).

We will rephrase the above problem as the following minimax optimization problem.

Definition 7 (H^∞ differential game). Consider the cost function

$$J(u,w)=\inf_u \sup_w \int_{t_0}^{T}\left\{\|z(t)\|_2^2-\gamma^2\|w(t)\|_2^2\right\}dt, \quad T>t_0. \tag{5}$$

Then, find the input u that minimizes $J(u,w)$ while the disturbance w maximizes $J(u,w)$ under the constraint described by the system Σ in (1). Furthermore, such solutions (u^*,w^*) must shape a saddle-point equilibrium such that

$$J(u^*,w)\le J(u^*,w^*)\le J(u,w^*) \tag{6}$$

for any disturbance w and any input u that can stabilize the system Σ with the disturbance w^*.

Remark 8. The problem in Definition 7 is not the same problem in Definition 6 in a precise sense; that is, the set of solutions of the problem in Definition 7 is included in that of Definition 6. If the system Σ has a $\mathscr{L}^2$-gain, then the evaluation function J in (5) takes a nonpositive value in the first problem. However, solutions of the second problem are not always nonpositive. Thus, we must check the nonpositiveness separately from solving the second problem.

Remark 9. Finding the worst disturbance w^* is not included in the first problem in Definition 6.

2.2. Hamilton-Jacobi-Isaac Equation. Such a two-person zero-sum game as in Definition 7 has a solution if the value function

$$V(x,t)=\inf_u \sup_w \int_{t}^{T}\left\{\|z(\tau)\|_2^2-\gamma^2\|w(\tau)\|_2^2\right\}d\tau \tag{7}$$

is C^1, and V satisfies the dynamic-programming equation

$$-\frac{\partial V}{\partial t}=\inf_u \sup_w \left\{\frac{\partial V}{\partial x}\dot{x}+\|z(t)\|_2^2-\gamma^2\|w(t)\|_2^2\right\}, \quad V(T,x)=0. \tag{8}$$

Now, we consider the infinite-time horizon problem under the conditions $\lim_{T\to\infty}J(u,w)$ remains bounded and the $\mathscr{L}^2$-gain of the system remains finite; that is, we find a time-independent positive-semidefinite function $V\colon \mathscr{X}\to\mathbb{R}$ satisfying the relation

$$\begin{aligned} H(x,p,u,w) &= p^\top\{f(x)+g_1(x)w+g_2(x)u\}+z^\top(t)z(t) \\ &\quad -\gamma^2 w^\top(t)w(t), \\ \inf_u \sup_w H(x,p,u,w) &= 0, \quad V(0)=0 \end{aligned} \tag{9}$$

that is called *the Hamilton-Jacobi-Isaac equation*, where we have defined $p=(\partial V/\partial x)^\top$. From the stationary conditions $\partial H/\partial u=0$ and $\partial H/\partial w=0$, we obtain the following explicit forms of optimal solutions:

$$\begin{aligned} u^* &= -\frac{1}{2}K^{-1}(x)\,\Xi(x,p), \\ w^* &= \frac{1}{2\gamma^2}g_1^\top(x)\,p, \end{aligned} \tag{10}$$

where we have defined $K(x)=k^\top(x)k(x)>0$ and $\Xi(x,p)=g_2^\top(x)p+2k^\top(x)h(x)$. Then, the Hamilton-Jacobi-Isaac equation can be written as

$$\begin{aligned} H(x,p,u,w) &= p^\top f(x)+\frac{1}{4\gamma^2}p^\top g_1(x)g_1^\top(x)p \\ &\quad -\frac{1}{4}\Xi^\top(x,p)K^{-1}(x)\Xi(x,p) \\ &\quad +h^\top(x)h(x)=0. \end{aligned} \tag{11}$$

Indeed, the Hamiltonian H in (11) can be transformed into

$$\begin{aligned} H(x,p,u,w) &= H(x,p,u^*,w^*) \\ &\quad -\gamma^2(w-w^*)^\top(w-w^*) \\ &\quad +(u-u^*)^\top K(x)(u-u^*) \end{aligned} \tag{12}$$

that means the solutions u^* and w^* determine the saddle point of the Hamiltonian.

From the above preliminaries, we can obtain the following fact.

Theorem 10 (see [17]). *If there exists a function $V(x)\in C^1$ such that $H(x,p)=0$, $p=(\partial V/\partial x)^\top$, $V(x)\ge 0$, and $V(0)=0$ for the Hamiltonian $H(x,p)$ in (11), then u^* and w^* in (10) are the solution of the system Σ in (1), and the $\mathscr{L}^2$-gain of the system Σ is less than or equal to γ.*

3. Nonlinear H^∞ Control Design Using Stable Manifold Method

This section derives the way of converting the nonlinear H^∞ control design with the stable manifold method from the viewpoint of the Hamiltonian representation of Hamilton-Jacobi-Isaac equations.

3.1. Stabilizing Solution of Hamilton-Jacobi Equations. Before explaining the implementation of the linear and nonlinear H^∞ control designs to the stable manifold method, we make a brief summary of basic results on the solvability of Hamilton-Jacobi equations.

Assumption 11. We assume that $h(x)^\top k(x)=0$ for all $x\in\mathscr{X}$. For example, in this case, we can write $z=h(x)+k(x)u$ with $h(x)=[h_1(x),0]^\top$, and $k(x)=[0,k_2(x)]^\top$.

Remark 12. In the typical settings [4, 17], the condition $K(x) = k^\top(x)k(x) = I$ that means the unity weighting on the control is introduced to reduce (11) to be a simple quadratic form with respect to g_2 without the weight K^{-1} in addition to the condition in Assumption 11. However, in control designs using the stable manifold method, such a simplification is not necessary.

Proposition 13 (see [17]). *Let one consider the following approximations:*

$$f(x) = Ax + \mathcal{O}\left(|x|^2\right),$$
$$\overline{R}(x) = R + \mathcal{O}(|x|), \tag{13}$$
$$\overline{Q}(x) = \frac{1}{2}x^\top Qx + \mathcal{O}\left(|x|^3\right)$$

in (11), where $\overline{R}(x) := R_2(x) - (1/\gamma^2)g_1(x)g_1^\top(x)$, $\overline{Q}(x) := h^\top(x)h(x)$, *and* A, R, *and* Q *are constant matrixes. If* $V(x)$ *is assumed to be a quadratic form of symmetric matrix* P, *the Hamilton-Jacobi-Isaac equation can be reduced to the Riccati equation:*

$$PA + A^\top P - PRP + Q = 0. \tag{14}$$

Definition 14. A solution of the Riccati equation (14) is called a *stabilizing solution* if $A - RP$ is a stable matrix.

Theorem 15 (see [17]). *Consider the Hamilton-Jacobi equation* $H(x, p) = p^\top f(x) - (1/2)p^\top R_2(x)p + q(x) = 0$ *in nonlinear optimal control problems, where* $p = (\partial V/\partial x)^\top$ *and* $R_2(x) = g_2(x)K^{-1}(x)g_2^\top(x)$. *If the Riccati equation derived from the Hamilton-Jacobi equation has a stabilizing solution, then there exists a stabilizing solution* $V(x)$ *of the Hamilton-Jacobi equation such that* $f(x) - R_2(x)p(x)$ *is asymptotically stable.*

3.2. Calculation of Stabilizing Solutions via Stable Manifold Method. In this section, we clarify H^∞ control design procedures in stable manifold method. The objective of the stable manifold method [14] is to calculate a stable manifold of stabilizing solutions of the Hamilton-Jacobi equation by using the following iterative numerical scheme:

(1) Transform the equivalent Hamiltonian system of the Hamilton-Jacobi equation as

$$\begin{bmatrix} \dot{x}' \\ \dot{p}' \end{bmatrix} = \begin{bmatrix} F & 0 \\ 0 & -F^T \end{bmatrix} \begin{bmatrix} x' \\ p' \end{bmatrix} + \begin{bmatrix} n_s\left(t, x', p'\right) \\ n_u\left(t, x', p'\right) \end{bmatrix} \tag{15}$$

by the coordinate transformation

$$\begin{bmatrix} x' \\ p' \end{bmatrix} = \begin{bmatrix} I & S \\ P & PS + I \end{bmatrix}^{-1} \begin{bmatrix} x \\ p \end{bmatrix}, \tag{16}$$

where S is the matrix that is a solution of Lyapunov equation $FS + SF^T = F$ and $F = A - RP$.

(2) Calculate sequences $\{x_k'(t, \xi)\}$ and $\{p_k'(t, \xi)\}$ determined by

$$\begin{aligned} x_{k+1}'(t, \xi) &= e^{Ft}\xi \\ &\quad + \int_0^t e^{F(t-s)} n_s\left(s, x_k'(s, \xi), p_k'(s, \xi)\right) ds, \\ p_{k+1}'(t, \xi) &= -\int_t^\infty e^{-F^T(t-s)} n_u\left(s, x_k'(s, \xi), p_k'(s, \xi)\right) ds \end{aligned} \tag{17}$$

for a certain parameter $\xi \in \mathbb{R}^n$, where $x_0'(t, \xi) = e^{Ft}\xi$ and $p_0'(t, \xi) = 0$.

(3) By iteratively applying (17), extend a solution along an initial vector ξ in a plain surface spanned by P under the condition that the Hamiltonian of the right side of (11) is sufficiently close to zero.

(4) If a solution passes through a desired initial state of control systems, then the iteration is finished. If not, back to procedure (2) and try with other ξ.

We can actually transform the Hamilton-Jacobi-Isaac equation (11) into the following Hamiltonian system.

Lemma 16. *Under Assumption 11, (11) can be transformed into the equivalent Hamiltonian system:*

$$\begin{aligned} \dot{x} &= \frac{\partial H^\top}{\partial p} = f(x) - \frac{1}{2}\left(R_2(x) - \frac{1}{\gamma^2}g_1(x)g_1^\top(x)\right)p, \\ \dot{p} &= -\frac{\partial H^\top}{\partial x} \\ &= -\frac{\partial f}{\partial x}^\top(x)p - \frac{1}{2\gamma^2}p^\top\frac{\partial g_1}{\partial x}(x)g_1^\top(x)p \\ &\quad + \frac{1}{4}\left(\frac{\partial}{\partial x}p^\top R_2(x)p\right)^\top - 2\frac{\partial h^\top}{\partial x}(x)h(x), \end{aligned} \tag{18}$$

where we have defined $R_2(x) = g_2(x)K^{-1}(x)g_2^\top(x)$.

From the facts discussed in the previous section, we can obtain the condition for the applicability of the stable manifold method.

Theorem 17. *Let us consider a nonlinear* H^∞ *control problem for system (1). For the Riccati equation (14) corresponding to the Hamilton-Jacobi-Isaac equation (11) of the problem under the approximation (13), if the Hamiltonian matrix*

$$H = \begin{bmatrix} A & -R \\ -Q & -A^\top \end{bmatrix} \tag{19}$$

does not have eigenvalues on the imaginary axis and (A, R) *is stabilizable, then we can calculate the stabilizing solution of the Hamilton-Jacobi-Isaac equation by using the stable manifold method.*

Proof. A stable manifold can be described by $p = (\partial V/\partial x)^\top$, and such a function $V(x)$ exists if the Hamiltonian matrix of the Riccati equation corresponding to the Hamilton-Jacobi-Isaac equation does not have eigenvalues on the imaginary axis [17]. Indeed, this fact is used in the proof of Theorem 15. If the linearized system (A, R) is stabilizable and $R \geq 0$ or $R \leq 0$, there exists the stabilizing solution of the Riccati equation [17]. Now, we assumed that $K > 0$; then $K^{-1} > 0$; that is, $R \geq 0$ or $R \leq 0$, because R is the linear part of $\overline{R}(x) := R_2(x) - (1/\gamma^2) g_1(x) g_1^\top(x)$, where $R_2(x) = g_2(x) K^{-1}(x) g_2^\top(x)$. Hence, there also exists a stabilizing solution $V(x)$ of the Hamilton-Jacobi-Isaac equation according to Theorem 15. Consequently, in such a case, we can directly find p derived from the stabilizing solution $V(x)$ by the stable manifold method. The Hamiltonian system representation in (15) can be given by the system in Lemma 16 and the linearization in (13). □

4. Numerical Example

We will check the validity of the nonlinear H^∞ control design via the stable manifold method by showing a robustness improvement of a controlled vehicle model [16].

4.1. Control Model. We assume that the left side and right side wheels of a vehicle have the same property, and the vehicle should be stabilized to some direction under a constant speed. Then, the equivalent 2-wheel model with respect to yawing without rolling and pitching motions is given as follows:

$$\begin{aligned} x &= [\beta \;\; r \;\; \theta \;\; \delta \;\; Y]^\top, \\ w &= [w_1 \;\; w_2]^\top, \end{aligned} \tag{20}$$

$$f(x) = \left[-\frac{\sin\beta}{mV_0} F_x + \frac{\cos\beta}{mV_0} F_y - r \quad \frac{2l_f}{I} C_f \cos\delta - \frac{2l_r}{I} C_r \quad r \quad 0 \quad V_0 \sin(\beta + \theta) \right]^\top, \tag{21}$$

$$\begin{aligned} g_1(x) &= \begin{bmatrix} \dfrac{\cos(\beta+\theta)}{(mV_0)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}^\top, \\ g_2(x) &= [0 \;\; 0 \;\; 0 \;\; 1 \;\; 0]^\top, \end{aligned} \tag{22}$$

where the control input u is the steering angle speed, the state vector x consists of the slip angle β at center of gravity (COG), the yaw rate r, the direction θ, the steering angle δ, and the lateral position Y of the vehicle, and note that the vertical position is ignored under the assumption of motions around a constant speed. Furthermore, the translational forces F_x and F_y and the cornering force of each wheel Y_i are written as follows:

$$\begin{aligned} F_x &= 2Y_f \sin(\beta_f + \delta) + 2Y_r \sin\beta_r, \\ F_y &= 2Y_f \cos(\beta_f + \delta) + 2Y_r \cos\beta_r, \\ Y_i &= C_i \cos\beta_i \end{aligned} \tag{23}$$

for $i = \{f, r\}$ that means the front and the rear wheels, respectively, where β_i is the slip angle of wheels, C_i is the lateral force of wheels, and C_i and β are related by

$$C_i = \mu N_i \sin\left[a \tan^{-1}\left\{b\beta_i - c\left(b\beta_i - \tan^{-1}(b\beta_i)\right)\right\}\right], \tag{24}$$

where $a = 1.23$, $b = 3.25$, and $c = -6.00$ are experimental parameters, $\mu = 0.2$ is a friction constant between road surface and tire, and $N_f = 5.48$ and $N_r = 4.21$ are vertical loads of each wheel. In (22), the following physical parameters are used: the constant speed $V_0 = 17.7$, the mass $m = 990$, the moment of inertia $I = 683$, the distance from front axle to COG $l_f = 1.0$, and the distance from rear axle to COG $l_r = 1.3$.

4.2. Disturbance Models. We applied the following disturbance to the model during simulations:

$$\begin{aligned} w_1 &= \begin{cases} 0.47 & (0 \leq t \leq 0.5) \\ 0 & (\text{otherwise}), \end{cases} \\ w_2 &= \begin{cases} \dfrac{1}{40} \displaystyle\sum_{k=60}^{100} \sin(kt) & (2 \leq t \leq 5) \\ 0 & (\text{otherwise}) \end{cases} \end{aligned} \tag{25}$$

that mean artificial effects of side winds and rough road surfaces (see Figure 1). However, the particular information of these disturbances defined by the above relations is not used in the design of H^∞ controllers, but we only determine the upper bound of the disturbance, that is, γ as a design parameter.

4.3. Additional Calculation. According to Theorem 15, we must check an obtained function $V(x)$ is nonnegative. Because the stable manifold method gives the pair of the variables (x, p) as a solution, we must calculate V from p obtained from the simulation by

$$V(x(t)) = \int_0^\infty p^\top \dot{x} \, dt. \tag{26}$$

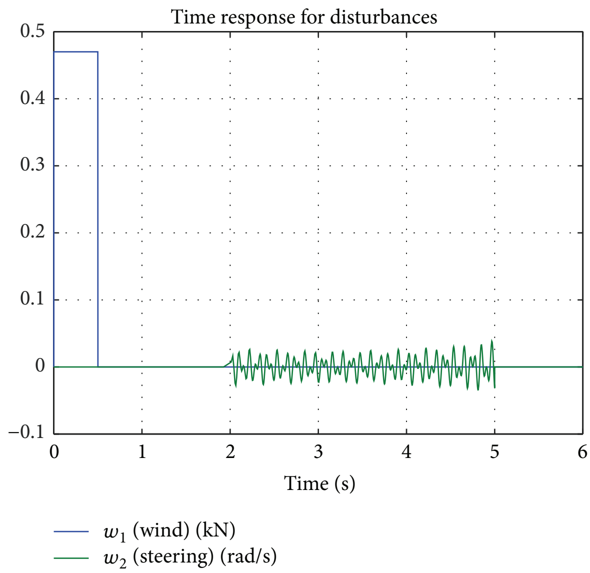

FIGURE 1: Disturbances.

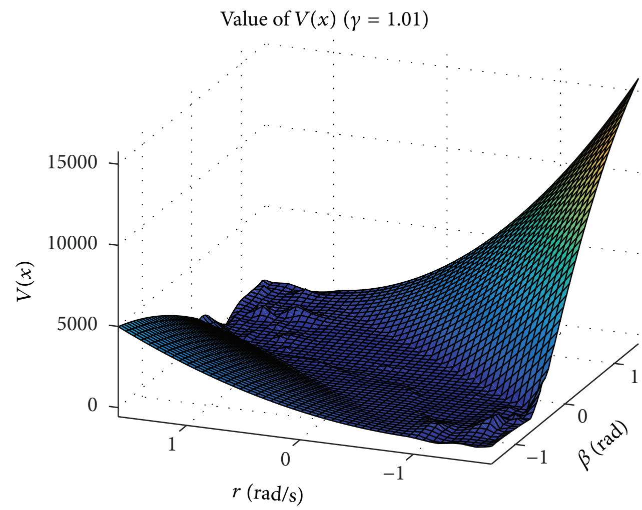

FIGURE 3: Stabilizing solution (view 2).

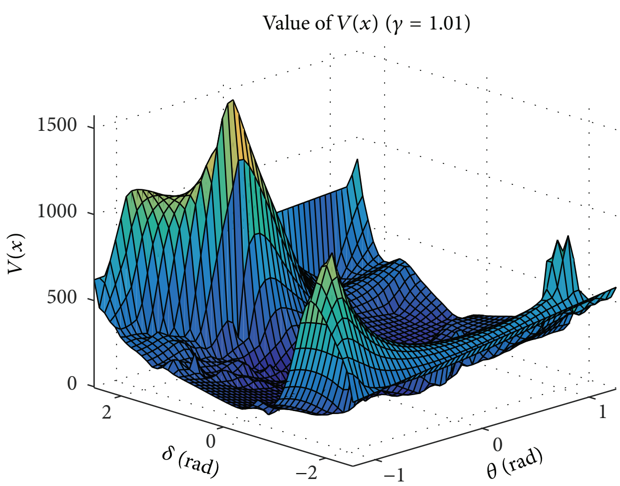

FIGURE 2: Stabilizing solution (view 1).

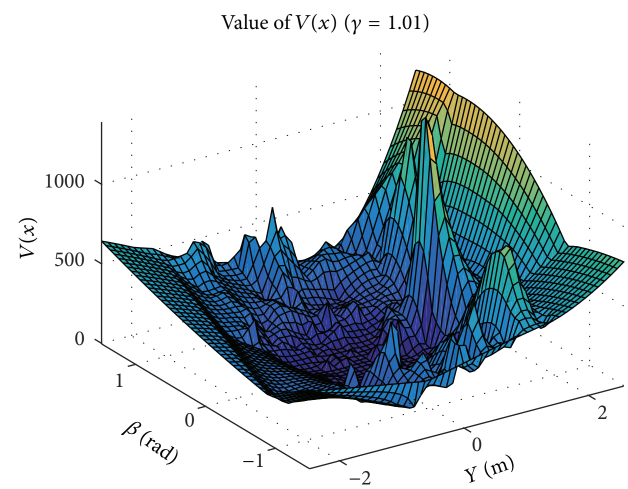

FIGURE 4: Stabilizing solution (view 3).

4.4. Numerical Results. We carried out the simulation using the stable manifold method for the model with $\gamma = 1.01$. Figures 2–4 show the three projections of the stabilizing solution $V(x)$ calculated by (26), where please note that $V(x)$ is defined on the fifth-dimensional space of x. We can see that $V(x)$ is nonnegative.

Figures 5–10 show the time plots of the state variables controlled by the linear and nonlinear H^∞ controllers. The convergence performance of the time responses was improved by the nonlinear H^∞ controller. Then, the values of the objective functions of the linear and nonlinear controls were $J = -0.0809$ and $J = -0.0818$, respectively. Indeed, from these figures, we can see that the amplitude of the control input generated by the nonlinear H^∞ controller is smaller than that of the linear H^∞ controller.

On the other hand, we did the simulation for the same model controlled by a nonlinear optimal regulator that does

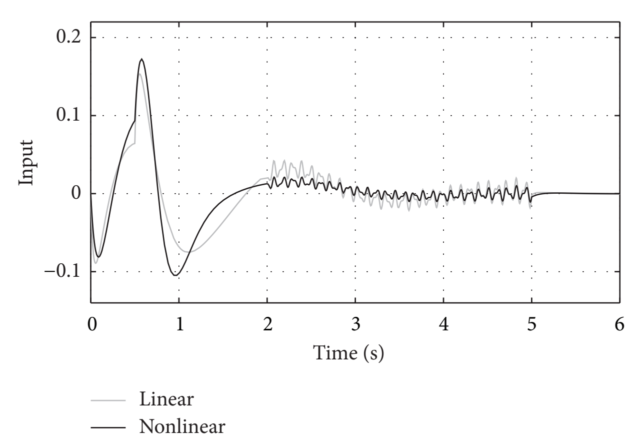

FIGURE 5: Time response of inputs.

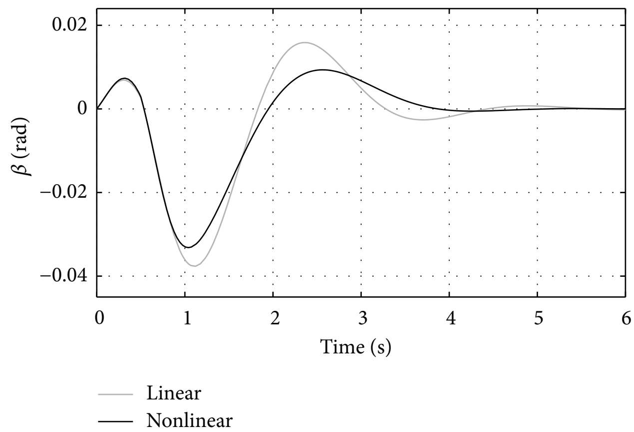

Figure 6: Time response of slip angles.

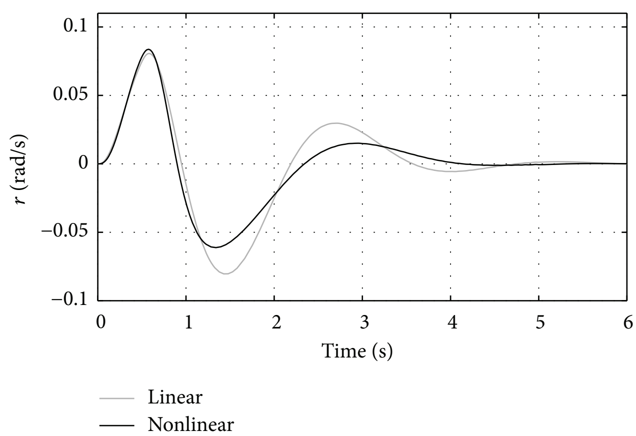

Figure 7: Time response of yaw rates.

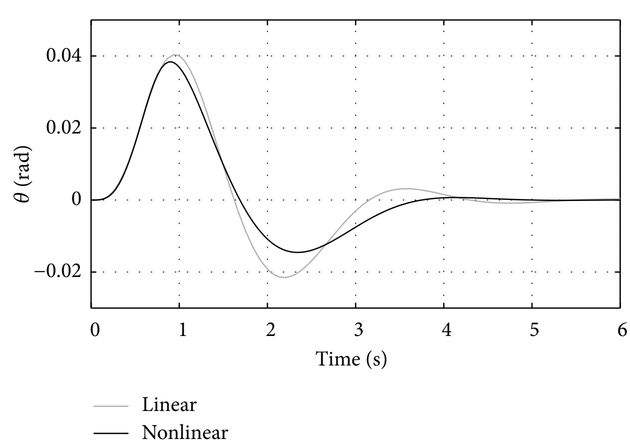

Figure 8: Time response of directions.

not have any guarantee with respect to robustness. Figure 12 shows the time plot of the state variables by the nonlinear optimal regulator with the unit weight 1 to control inputs. However, the trajectory diverged; that is, the system became unstable. Note that Figure 11 is the plot in which the time

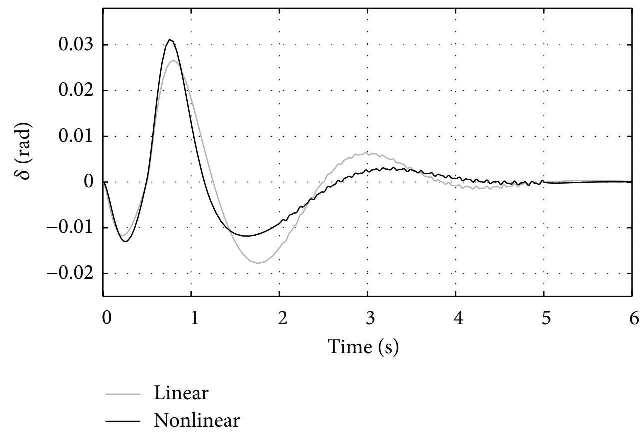

Figure 9: Time response of steering angles.

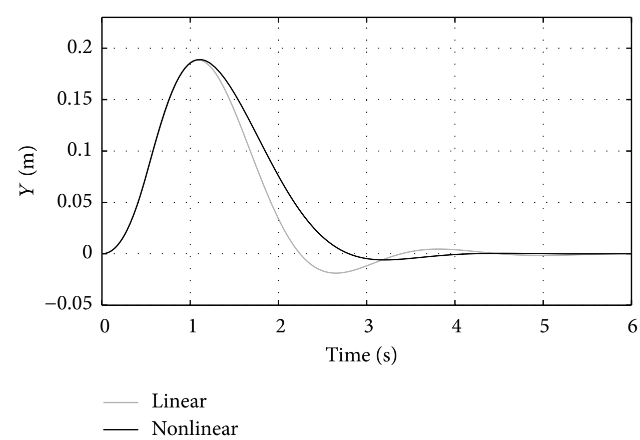

Figure 10: Time response of lateral positions.

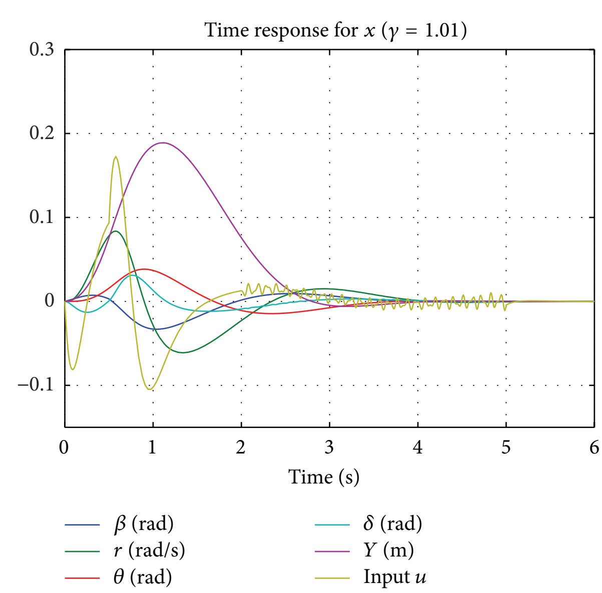

Figure 11: Time responses of nonlinear H^{∞} control.

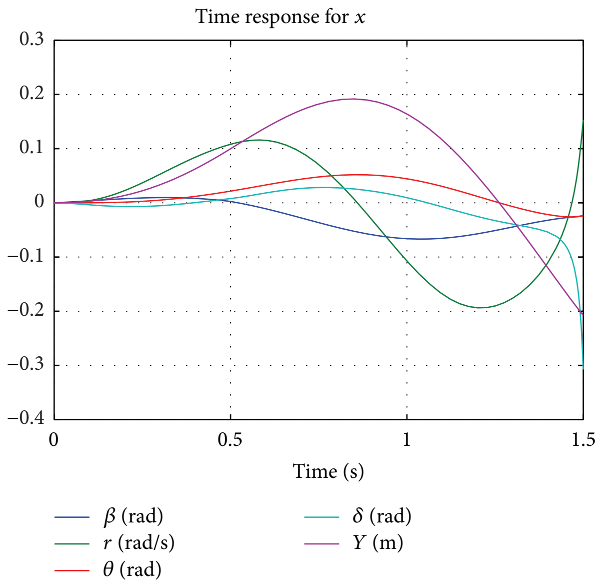

FIGURE 12: Time responses of nonlinear optimal regulator.

responses of the nonlinear H^{∞} controller in Figures 5–10 are collected.

Consequently, we can confirm that the nonlinear H^{∞} controller achieved a higher robust performance in this case.

5. Conclusion

We proposed the way of integrating robust nonlinear H^{∞} control design to the stable manifold method. Furthermore, the numerical experimentation was shown for checking the validity of the robust control for the vehicle model with disturbances. The stable manifold method does not require the information of analytical solutions, and we only have to prepare the description of nonlinear systems. Hence, we expect this method to be applied to a lot of control objects that could not be considered due to theoretical difficulties before.

At present, we realized only the full-state feedback case. The output feedback case can be considered as a challenging future work.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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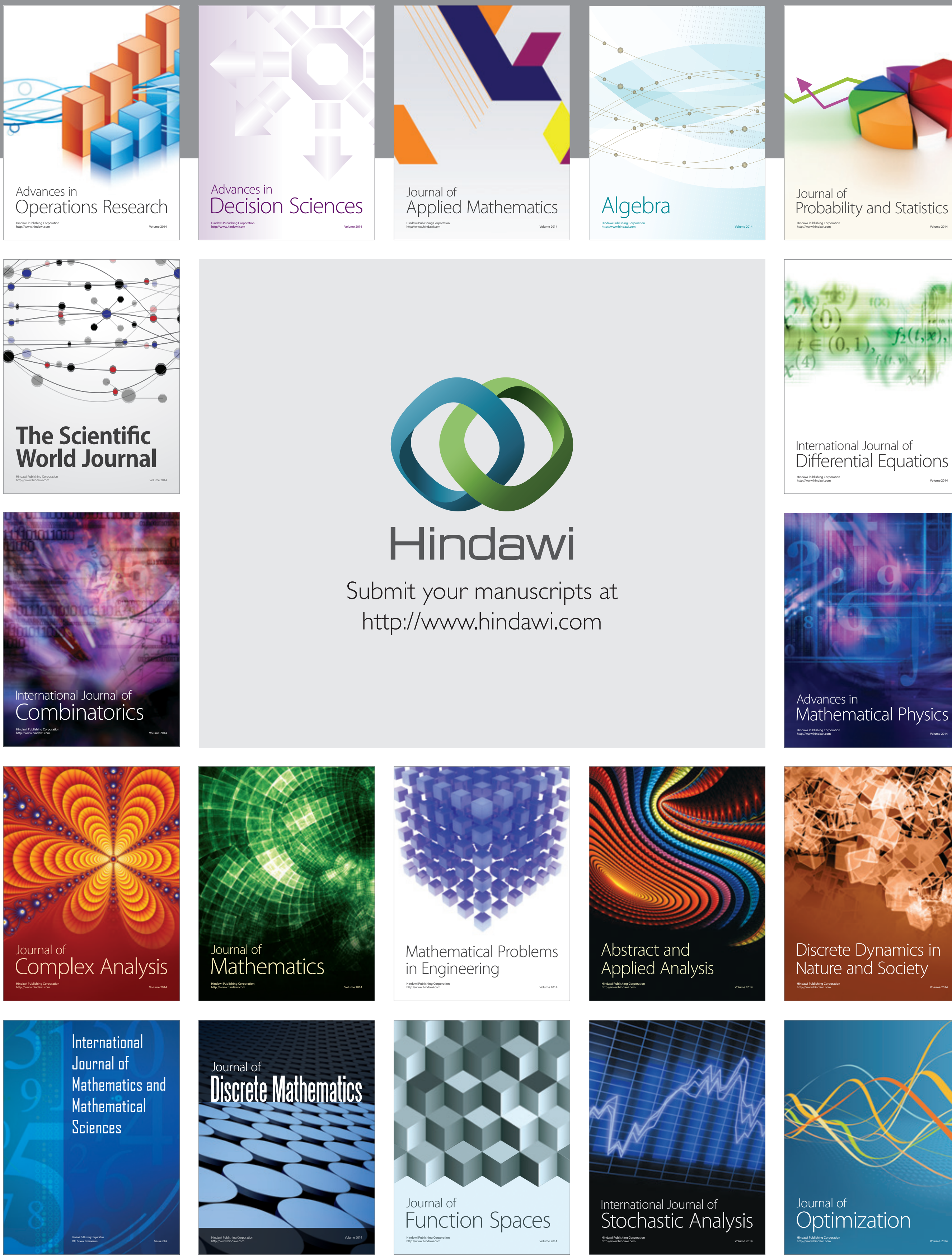
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