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"Precautionary saving with changing income ambiguity"

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Precautionary saving with changing income ambiguity*

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Abstract

We study a two-period saving model where the agent's future income might be ambiguous. Our agent has a version of the smooth ambiguity decision criterion (Klibanoff, Marinacci and Mukerji (2005)), where the agent's perception about ambiguity is described by a second-order belief over first-order risks. We model increasing ambiguity as a spreading-out of the second-order belief. We show that under a "Risk Comonotonicity" condition, our agent saves more when ambiguity in future income increases. We argue that the condition is indispensable for our result.

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1 Introduction and Summary

Does an ambiguity averse agent have a stronger precautionary saving motive when ambiguity in future income increases? We study a saving problem of an ambiguity averse agent

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in the face of ambiguity in future income. The agent has a version of the smooth ambiguity decision criterion that is axiomatized by Klibanoff, Marinacci and Mukerji (2005). When the agent is ambiguity neutral, our problem reduces to the classic one studied by Kimball (1990). Our main result (Proposition 1) is that under an appealing condition, the agent has a stronger precautionary saving motive when the ambiguity in future income increases. The condition roughly says that when "first-order beliefs" about future income change, the expected utility and the expected *marginal* utility from future income move in the opposite directions. Since utility is increasing and marginal utility is decreasing in income, this condition holds intuitively in a variety of contexts. We argue in Section 3 that it is in fact indispensable for our comparative statics result. So our main result is tight in this sense.

The contributions of this paper are twofold. First, we propose a notion of increasing ambiguity for future income, elaborating on the idea in Snow (2010).[1] It relates directly to the informativeness of signals in Blackwell's information theory. Hence, the notion admits various equivalent interpretations and could be useful in a variety of applications. Second, we demonstrate that the notion is plausible at least in our precautionary saving context. When the future looks more ambiguous in our notion, an ambiguity averse agent is shown to saves more as expected.

Our model is similar to, but different from, Berger (2014) and Osaki and Schlesinger (2014), where the agent has the recursive smooth ambiguity decision criterion (Klibanoff, Marinacci and Mukerji (2009)). Notably, Berger (2014) reports an important result that an ambiguity averse agent saves more in the case of ambiguity in future income than the case of no ambiguity, under a condition similar to ours in spirit. The result however is silent if an initially ambiguous future income gets more ambiguous, which our model can handle in a clean way. We speculate that such intermediate cases, though important, are hard to characterize in the recursive smooth ambiguity decision model.

2 Precautionary Saving under Income Ambiguity

An *environment* is a pair (S, Y) of random variables jointly distributed on $\mathbb{R}^2$, where S is a signal and Y an income level. It summarizes an agent's perception about income ambiguity. In Klibanoff, Marinacci and Mukerji (2005)'s terms, each conditional distribution $Y|S = s$, $s \in \mathbb{R}$, is a *first-order belief* about future income, and the distribution of all first-order beliefs

[1] Snow (2010) proposes a notion of increasing ambiguity in a general abstract setup.

$\{Y|S = s, s \in \mathbb{R}\}$ induced by S is his *second-order belief*. We consider the saving problem of an agent who has a sure income $e \in \mathbb{R}_+$ today, an ambiguous income (S, Y) tomorrow, a discounting factor $\beta \in (0, 1)$, and who faces a per unit saving cost $q \in (0, \infty)$. The agent solves

$$\max_z v(e - qz) + \beta \mathbf{E}[\phi(\mathbf{E}[u(z+Y)|S])] \tag{1}$$

where $z \in \mathbb{R}$ is an amount of saving, and v, u, ϕ are increasing and smooth real-valued functions over $\mathbb{R}$ with $v'' < 0$,[2] $u'' \leq 0$, $u''' \geq 0$, $\phi'' \leq 0$, and $\phi''' \geq 0$.

In period 1, the utility from net income is measured by v. In period 2, the utility is calculated first by finding the expected values of u conditional on various first-order beliefs, and then these conditional expected values, after transformed by ϕ, are averaged with respect to the second-order belief. It therefore conforms with the so called smooth ambiguity decision criterion.[3] When ϕ is strictly concave, the agent is strictly ambiguity averse. When ϕ is linear, the agent is ambiguity neutral, and (1) reduces to Kimball (1990)'s classical problem.

It helps to consider the special case of $v = \phi \circ u$, where the objective function, $\phi \circ u(\cdot) + \beta \mathbf{E}[\phi(\mathbf{E}[u(\cdot)|S])]$, represents an additively time separable preference.[4] In this case, if Y equals to a constant y, then $\mathbf{E}[\phi(\mathbf{E}[u(z+Y)|S])] = \phi \circ u(z+y) = v(z+y)$, so v measures non-random income in each period. If S is a constant, then $\mathbf{E}[\phi(\mathbf{E}[u(z+Y)|S])] = \phi(\mathbf{E}[u(z+Y)])$, which is a monotonic transformation of $\mathbf{E}[u(z+Y)]$. In other words, the preference restricted to the second-period risks is represented by the vNM function u.

Fixing v, u and ϕ as well as e, β and q, we are interested in the comparative statics of optimal saving with respect to the change in environment. We apply a standard technique to establish our results. Throughout, we assume that probability distributions in consideration are well-behaved so that we can apply differentiation under expectation operators. Differentiating (1) with respect to z we get

$$\Psi(z; (S, Y)) := -qv'(e - qz) + \beta \mathbf{E}[\phi'(\mathbf{E}[u(z+Y)|S]) \cdot \mathbf{E}[u'(z+Y)|S]]. \tag{2}$$

Observe that $-qv'(e - qz)$ is decreasing in z, and $\phi'(\mathbf{E}[u(z+Y)|S])$ and $\mathbf{E}[(u'(z+Y))|S]$ are non-increasing in z, so $\Psi(z; (S, Y))$ is decreasing in z. Hence, our precautionary saving

[2] The strict concavity of v is assumed to guarantee the uniqueness of the solution. It simplifies the presentation but is not necessary for our result.

[3] See Klibanoff, Marinacci and Mukerji (2005) for an axiomatization. However, except for the trivial case, our preferences are not recursive in the sense of Klibanoff, Marinacci and Mukerji (2009).

[4] Thus one can extend this type of preference to the sum of an infinite series of discounted utilities, which admits the standard dynamic programming techniques in principle.

problem is a well-defined concave problem. Write $z^*(S,Y)$ for the optimal saving under the environment (S,Y). Then $z^*(S,Y) \le z^*(S',Y')$ if $\Psi(z^*(S,Y);(S',Y')) \ge 0$.

3 Comparative Statics on Environment

3.1 Increasing Background Risks

For an illustrative purpose, we first assume that S, Y and Y' are jointly distributed random variables. We compare the optimal amounts of saving under (S,Y) and (S,Y'), where signals are identically distributed. Suppose that $Y'|S$ is riskier than $Y|S$ with probability one. That is, based on almost all the first-order beliefs, the agent perceives a greater income risk under the latter environment.

When ϕ is linear, Kimball (1990) shows that an ambiguity neutral agent saves more in the face of greater income risks. In our general setup, it is readily seen that an ambiguity averse agent also saves more, i.e., $z^*(S,Y) \le z^*(S,Y')$. Indeed, for each z, $(z+Y')|S$ is riskier than $(z+Y)|S$ with probability one, and since ϕ' is non-decreasing and u is concave, then $\phi'(\mathbf{E}[u(z+Y)|S]) \le \phi'(\mathbf{E}[(u(z+Y'))|S])$ with probability one. Similarly, since u' is convex, $\mathbf{E}[(u'(z+Y))|S] \le \mathbf{E}[(u'(z+Y'))|S]$ with probability one. Hence, $\Psi(z;(S,Y)) \le \Psi(z;(S,Y'))$ holds at each z, and a fortiori at $z^*(S,Y)$.

3.2 Risk and Ambiguity Trade Off

Now assume that S, S' and Y are jointly distributed random variables. We compare (S,Y) and (S',Y), i.e., the income distribution is the same, but the signals are different, generating different ambiguity. Recall that when S is a constant, the objective function (1) reduces to $v(e-qz) + \beta\phi(\mathbf{E}[u(z+Y)])$. Thus tomorrow's income is deemed purely risky, not ambiguous at all. At the other extreme, when $S' = Y$ with probability one, (1) reduces to $v(e-qz) + \beta\mathbf{E}[\phi(u(z+Y))]$. So the final income tomorrow is evaluated with a compound function $\phi \circ u$, i.e., it is deemed purely ambiguous rather than risky in our model.

Notice that signal S in the first extreme case is completely uninformative for Y, and signal S' in the second extreme case is perfectly informative. This observation suggests the following criterion for comparison of ambiguous environments, which is essentially equivalent to the definition proposed by Snow (2010):

Definition 1. *An environment* (S', Y) *is* ***no less ambiguous than*** *another environment* (S, Y) *if* S' *is at least as informative as* S *for* Y*, i.e., for each integrable function* f*,* $\mathbf{E}[\mathbf{E}[f(Y)|S']|S] = \mathbf{E}[f(Y)|S]$.

In other words, an environment is more ambiguous if the agent learns more from the signal. Notice that since the agent cannot choose an action contingent on the signal, an additional piece of information is useless per se, and it will even hurt an ambiguity averse agent who cares about first-order beliefs.[5]

The condition above is the same as the informativeness in Blackwell's information theory. Hence, there are well-known equivalent conditions. In the case of discrete random variables, it is equivalent to that for each y and s in the support, $\Pr(Y = y|S = s) = \sum_{s'} \Pr(Y = y|S' = s') \Pr(S' = s'|S = s)$. In general, it says that for each s in the support, the conditional distribution $Y|S = s$ is an average of conditional distributions $E[(Y|S')|S = s]$. That is, the second-order belief over $\{Y|S' = s' : s' \in \mathbb{R}\}$ constitutes a mean preserving spread of that over $\{Y|S = s : s \in \mathbb{R}\}$.[6]

We will show that our agent saves more in the face of the same income risks but more ambiguous environment, under a condition to be stated below. To simplify notation, let

$$W_0(S'; z) := \mathbf{E}[u(z+Y)|S'], \tag{3}$$

$$W_1(S'; z) := \mathbf{E}[u'(z+Y)|S']. \tag{4}$$

By construction, for each z, W_0 and W_1 are random variables measurable with respect to S'. Since u is increasing and u' is non-increasing, it is intuitive that W_0 and W_1 move in the opposite directions with S'. If S' brings good news so that Y tends to be high, then W_0 tends to be high and W_1 tends to be low. But this is not necessarily correct because Y is random conditional on S', and it is something we need to assume.

Definition 2. ***Risk Comonotonicity*** *at* z *is said to be satisfied if* W_0 *and* $-W_1$ *are comonotonic random variables at* z*, i.e., for each pair of realizations* s_1' *and* s_2' *of* S'*,*

$$(W_0(s_1'; z) - W_0(s_2'; z)) \cdot (W_1(s_1'; z) - W_1(s_2'; z)) \leq 0$$

[5] See Grant, Kajii and Polak (1998) for their general discussion on Blackwell's theorem without contingent action choice.

[6] Snow (2010) uses this version, essentially.

Notice that Risk Comonotonicity holds immediately if $Y = S'$ with probability one, i.e., tomorrow's income is purely ambiguous, or if u' is constant, i.e., the agent is risk neutral. Computing $\mathbf{E}[u(z+Y)|S']$ and $\mathbf{E}[u'(z+Y)|S']$ by integration by parts, it can be readily verified that Risk Comonotonicity holds at each z if the distributions in $\{Y|S' = s' : s' \in \mathbb{R}\}$ are ordered by the first-order stochastic dominance as s' changes, or they are ordered by the second-order stochastic dominance.[7]

Proposition 1. *Suppose that (S', Y) is no less ambiguous than (S, Y), and that Risk Comonotonicity holds at $z^*(S, Y)$. Then $z^*(S, Y) \leq z^*(S', Y)$.*

Proof. Let $z := z^*(S, Y)$. We want to show that $\Psi(z; (S', Y)) \geq 0$. Write for simplicity $W_i(S') = W_i(S'; z)$, $i = 0, 1$. Since (S', Y) is no less ambiguous than (S, Y), then $\mathbf{E}[\mathbf{E}[u(z+Y)|S']|S] = \mathbf{E}[u(z+Y)|S]$ and $\mathbf{E}[\mathbf{E}[u'(z+Y)|S']|S] = \mathbf{E}[u'(z+Y)|S]$. Thus,

$$\mathbf{E}[W_0(S')|S] = \mathbf{E}[u(z+Y)|S], \tag{5}$$

$$\mathbf{E}[W_1(S')|S] = \mathbf{E}[u'(z+Y)|S]. \tag{6}$$

Since ϕ' is non-increasing, Risk Comonotonicity implies that $\phi'(W_0)$ and W_1 are comonotonic random variables. That is, for each pair of realizations s'_1 and s'_2 of S',

$$(\phi'(W_0(s'_1)) - \phi'(W_0(s'_2))) \cdot (W_1(s'_1) - W_1(s'_2)) \geq 0.$$

Since s'_1 and s'_2 are arbitrary, we can take the expectation of the above, conditional on S, first letting $s'_1 = S'$ and then $s'_2 = S'$. Thus, we have

$$\mathbf{E}[\phi'(W_0(S')) \cdot W_1(S')|S] \geq \mathbf{E}[\phi'(W_0(S'))|S] \cdot \mathbf{E}[W_1(S')|S] \tag{7}$$

with probability one.

Since ϕ' is convex by assumption, then by Jensen's inequality

$$\mathbf{E}[\phi'(W_0(S'))|S] \geq \phi'(\mathbf{E}[W_0(S')|S])$$

with probability one. Since W_1 is a positive random variable, then

$$\mathbf{E}[\phi'(W_0(S'))|S] \cdot \mathbf{E}[W_1(S')|S] - \phi'(\mathbf{E}[W_0(S')|S]) \cdot \mathbf{E}[W_1(S')|S] \geq 0 \tag{8}$$

[7] As a matter of fact, the techniques to establish propositions 1 and 2 in Berger (2014) can be applied almost directly to assure Risk Comonotonicity. So we refer the reader to them for more conditions that guarantee Risk Comonotonicity.

with probability one.

Now since $z = z^*(S, Y)$, in view of (5) and (6), we have

$$-qv'(e - qz) + \beta\mathbf{E}\left[\phi'\left(\mathbf{E}\left[W_0(S')|S\right]\right) \cdot \mathbf{E}\left[W_1(S')|S\right]\right] = 0.$$

Hence,

$$\begin{aligned}
\frac{1}{\beta}\Psi(z; (S', Y)) &= -\frac{q}{\beta}v'(e - qz) + \mathbf{E}\left[\phi'\left(\mathbf{E}\left[u(z+Y)|S'\right]\right) \cdot \mathbf{E}\left[u'(z+Y)|S'\right]\right] \\
&= \mathbf{E}\left[\phi'\left(W_0(S')\right) \cdot W_1(S')\right] - \mathbf{E}\left[\phi'\left(\mathbf{E}\left[W_0(S')|S\right]\right) \cdot \mathbf{E}\left[W_1(S')|S\right]\right] \\
&= \mathbf{E}\left[\mathbf{E}\left[\phi'\left(W_0(S')\right) \cdot W_1(S')|S\right] - \phi'\left(\mathbf{E}\left[W_0(S')|S\right]\right) \cdot \mathbf{E}\left[W_1(S')|S\right]\right] \\
&\geq \mathbf{E}\left[\mathbf{E}\left[\phi'\left(W_0(S')\right)|S\right] \cdot \mathbf{E}\left[W_1(S')|S\right] - \phi'\left(\mathbf{E}\left[W_0(S')|S\right]\right) \cdot \mathbf{E}\left[W_1(S')|S\right]\right] \\
&\geq 0
\end{aligned}$$

as desired, where the third equality holds because $\mathbf{E}[\cdot] = \mathbf{E}[\mathbf{E}[\cdot|S]]$, the first inequality holds by (7), and the last inequality by (8). □

Remark 1. *Risk Comonotonicity is used to establish (7): $\phi'(W_0(S'))$ and $W_1(S')$ are positively correlated conditional on S. Since positive correlation is weaker than comonotonicity, we could strengthen Proposition 1 using (7) as an assumption. But as we shall discuss in the next section, some sort of comonotonicity is indispensable for a robust comparative statics result.*

Remark 2. *A similar comparative statics analysis can be carried out for the recursive case, where the relevant first-order effect corresponding to (2) is:*

$$-qv'(e - qz) + \frac{\beta\mathbf{E}\left[\phi'\left(\mathbf{E}\left[u(z+Y)|S\right]\right) \cdot \mathbf{E}\left[u'(z+Y)|S\right]\right]}{\phi'\left\{\phi^{-1}\left(E\left[\phi\left(E\left[\mathbf{E}\left[u(z+Y)|S\right]\right]\right)\right]\right)\right\}}$$

Notice that the denominator of the fraction cancels out if S is constant, i.e., there is no ambiguity, and this is the property Berger (2014) takes advantage of. However, if S is not constant, i.e., both (S, Y) and (S', Y) are ambiguous environments, the comparison of the first-order effects appears to be complicated, and there does not seem to be an analogous result as Proposition 1 in this setup.

3.3 Tightness of Comonotonicity Assumption

We shall argue that Risk Comonotonicity is indispensable for a robust comparative statics result to hold. That is, if comonotonicity fails at some z, then the saving implication is reserved in some problem.

Let q, u and v be given as assumed. Suppose further that $u'' < 0$, and for ease of exposition that $u > 0$ so that when we construct ϕ later, we only needs to define it for positive numbers. Moreover, assume that $\lim_{x\to 0} v'(x) = \infty$ and $\lim_{x\to\infty} v'(x) = 0$ so that an optimal consumption level in the first period is positive.

Let (S', Y) be an environment where S' takes values from $\{s'_0, ..., s'_n\}$, s'_i with probability $p_i > 0$, $i = 0, ..., n$, and $\sum p_i = 1$. Suppose that for some $z > 0$, $\mathbf{E}[u(z+Y)|S']$ and $-\mathbf{E}[u'(z+Y)|S']$ are not comonotonic, say $\mathbf{E}[u(z+Y)|S' = s'_0] > \mathbf{E}[u(z+Y)|S' = s'_1]$ and $\mathbf{E}[u'(z+Y)|S' = s'_0] > \mathbf{E}[u'(z+Y)|S' = s'_1]$. For simplicity, let $w_{0j} := \mathbf{E}[u(z+Y)|S' = s'_j]$ and $w_{1j} := \mathbf{E}[u'(z+Y)|S' = s'_j]$, $j = 0, 1$. Thus, $w_{i1} < w_{i0}$ for $i = 0, 1$.

We shall construct ϕ, e and (S, Y) such that (S', Y) is no less ambiguous than (S, Y), but the optimal saving under (S, Y) is more than that under (S', Y).

Let random variables S, S', Y be generated as follows. First draw a number from $\{s_1, ..., s_n\}$, s_1 with probability $p_0 + p_1$, s_i with probability p_i, $i = 2, ..., n$, and set S to be the drawn number. If s_1 is drawn, choose s'_0 with probability $\frac{p_0}{p_0+p_1}$, and s'_1 with probability $\frac{p_1}{p_0+p_1}$ and set S' to be the chosen number. If s_i, $i \neq 1$, is drawn, set $S' = s'_i$. Finally, choose Y according to the conditional probability distribution given S'. Clearly, the joint distribution of S' and Y is the same as in the given environment (S', Y). By construction, (S', Y) is no less ambiguous than (S, Y).

Next we construct ϕ. Let η be a smooth, positive, decreasing, convex and integrable function such that

$$\frac{p_0}{p_0+p_1}\eta(w_{00}) + \frac{p_1}{p_0+p_1}\eta(w_{01}) - \eta(\frac{p_0}{p_0+p_1}w_{00} + \frac{p_1}{p_0+p_1}w_{01}) < \frac{p_0 p_1[\eta(w_{01}) - \eta(w_{00})](w_{10} - w_{11})}{(p_0+p_1)(p_0 w_{10} + p_1 w_{11})}.$$

This is possible since the left hand side can be made arbitrarily small by making η flat, keeping the right hand side unchanged. Let $\phi(t) := \int_0^t \eta(s)\,ds$, $t > 0$. Clearly, $\phi' > 0$, $\phi'' = \eta' < 0$ and $\phi''' = \eta'' > 0$, as required.

Finally, let e be such that

$$-qv'(e - qz) + \mathbf{E}\left[\phi'(\mathbf{E}[u(z+Y)|S]) \cdot \mathbf{E}[u'(z+Y)|S]\right] = 0,$$

so that z is optimal under (S, Y).

It remains to demonstrate that the optimal saving under (S', Y) is less than z, and it suffices to show that (2) is negative at (S', Y). Notice that $\mathbf{E}[u(z+Y)|S = s_1] = \frac{p_0}{p_0+p_1}w_{00} + \frac{p_1}{p_0+p_1}w_{01}$, $\mathbf{E}[u'(z+Y)|S = s_1] = \frac{p_0}{p_0+p_1}w_{10} + \frac{p_1}{p_0+p_1}w_{11}$, and that the expectations conditional on $S = s_i$ and on $S' = s'_i$ coincide for $i = 2, ..., n$. We therefore have:

$$
\begin{aligned}
&\mathbf{E}\left[\phi'\left(\mathbf{E}\left[u\left(z+Y\right)|S'\right]\right)\cdot\mathbf{E}\left[u'\left(z+Y\right)|S'\right]\right]-\mathbf{E}\left[\phi'\left(\mathbf{E}\left[u\left(z+Y\right)|S\right]\right)\cdot\mathbf{E}\left[u'\left(z+Y\right)|S\right]\right]\\
=&p_0\phi'(w_{00})w_{10}+p_1\phi'(w_{01})w_{11}\\
&-(p_0+p_1)\phi'(\frac{p_0}{p_0+p_1}w_{00}+\frac{p_1}{p_0+p_1}w_{01})\cdot(\frac{p_0}{p_0+p_1}w_{10}+\frac{p_1}{p_0+p_1}w_{11})\\
=&(p_0+p_1)[\frac{p_0\phi'(w_{00})w_{10}}{p_0+p_1}+\frac{p_1\phi'(w_{01})w_{11}}{p_0+p_1}\\
&-(\frac{p_0\phi'(w_{00})}{p_0+p_1}+\frac{p_1\phi'(w_{01})}{p_0+p_1})\cdot(\frac{p_0}{p_0+p_1}w_{10}+\frac{p_1}{p_0+p_1}w_{11})\\
&+(\frac{p_0\phi'(w_{00})}{p_0+p_1}+\frac{p_1\phi'(w_{01})}{p_0+p_1})\cdot(\frac{p_0}{p_0+p_1}w_{10}+\frac{p_1}{p_0+p_1}w_{11})\\
&-\phi'(\frac{p_0}{p_0+p_1}w_{00}+\frac{p_1}{p_0+p_1}w_{01})\cdot(\frac{p_0}{p_0+p_1}w_{10}+\frac{p_1}{p_0+p_1}w_{11})]\\
=&-\frac{p_0p_1}{p_0+p_1}[\eta(w_{01})-\eta(w_{00})]\cdot(w_{10}-w_{11})\\
&+[\frac{p_0}{p_0+p_1}\eta(w_{00})+\frac{p_1}{p_0+p_1}\eta(w_{01})-\eta(\frac{p_0}{p_0+p_1}w_{00}+\frac{p_1}{p_0+p_1}w_{01})]\cdot(p_0w_{10}+p_1w_{11})\\
<&0,
\end{aligned}
$$

where the last inequality holds by the construction of η. This proves that (2) is negative at (S', Y).

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