代数幾何学シンポジウム記録
1980年度 pp.130-145

On ruled 3-folds

阪大 教養部 満渕 俊樹

近年 Iitaka, Ueno, Viehweg, Kawamata, Fujita らによって高次元代数多様体の分類に関する著しい結果が得られ始めた。そこで我々は $\kappa(X) \leqq 0$ の3次元 or 4次元 compact complex manifolds を調べてみた。

(1.1) 以下 $X, Y, Y_1, Y_2 \cdots$ 等はすべて irreducible reduced compact complex space（単に complex variety と呼ぶ）を示すものとする。Surjective meromorphic maps $f_i : X \to Y_i$, $i=1,2$, に対し次の様な定義を行う。

$Y_1 \overset{X}{*} Y_2$:= meromorphic image of X under the meromorphic map $x \in X \longmapsto (f_1(x), f_2(x)) \in Y_1 \times Y_2$

$f_1 * f_2$:= surjective meromorphic map from X to $Y_1 * Y_2$ defined by $x \longmapsto (f_1(x), f_2(x))$

<u>Proposition</u> (1.1.1)

Assume that $\dim X \leqq 4$. Then

(i) $\kappa(Y_1) \geqq 0$ and $\kappa(Y_2) \geqq 0$ imply $\kappa(Y_1 \overset{X}{*} Y_2) \geqq 0$.

(ii) $\kappa(Y_1) = \dim Y_1$ and $\kappa(Y_2) = \dim Y_2$ imply $\kappa(Y_1 \overset{X}{*} Y_2) = \dim (Y_1 \overset{X}{*} Y_2)$

証明のポイント：(i) $\dim Y_1$, $\dim Y_2$, $\dim Y_1 * Y_2 \leq 4$ なので、case 毎に調べていくことによって、結局 viehweg により示された curve を fibre とする fibration に於ける小平次元の劣加法性に帰着される。 (ii) 一たん (i) がわかれば、Kawamata によって示された base が general type であるような fibration に於ける小平次元の加法性から従う。

__Definition__ (1.1.2) For a complex variety X, we define

$$\mathcal{B}_X := \left\{ (Y, f) \;\middle|\; \begin{array}{l} f: X \to Y \text{ is a surjective meromorphic} \\ \text{map to a complex variety } Y \text{ with } \kappa(Y) \geq 0 \end{array} \right\} \Big/ \sim$$

$$\mathcal{B}'_X := \left\{ (Y, f) \;\middle|\; \begin{array}{l} f: X \to Y \text{ is a surjective meromorphic} \\ \text{map to a complex variety } Y \text{ of general type} \end{array} \right\} \Big/ \sim$$

where $(Y_1, f_1) \sim (Y_2, f_2)$ if there exists a bimeromorphic map $\tau: Y_1 \to Y_2$ such that $f_2 = \tau \circ f_1$.

__Proposition__ (1.1.3)

Assume that $\dim X \leq 4$. Then

(i) There exists a unique element in $\mathcal{B}_X$ (which we denote by $(B(X), \pi_X)$) such that for every $(Y, f) \in \mathcal{B}_X$, there exists a surjective meromorphic map from $B(X)$ to Y which makes the following diagram commutative :

$$\begin{tikzcd} & X \arrow[dl, "\pi_X"'] \arrow[dr, "f"] \arrow[d, phantom, "\circlearrowright"] & \\ B(X) \arrow[rr] & & Y \end{tikzcd}$$

(ii) There exists a unique element in $\mathcal{B}'_X$ (which we denote by $(B'(X), \pi'_X)$) such that for every $(Y, f) \in \mathcal{B}'_X$, there exists a surjective meromorphic map from $B'(X)$ to Y which makes the following diagram commutative :

$$\begin{tikzcd} & X \arrow[dl, "\pi'_X"'] \arrow[dr, "f"] \arrow[d, phantom, "\circlearrowright"] & \\ B'(X) \arrow[rr] & & Y \end{tikzcd}$$

証明のポイント: proposition (1.1.1) の容易な帰結である。

<u>Corollary</u> (1.1.4)

Assume that $\dim X \leqq 4$. Let $f : X \to Y$ be a surjective meromorphic map. Then there exists a surjective meromorphic map $f_b : B(X) \to B(Y)$ (resp. $f'_b : B'(X) \to B'(Y)$), unique up to bimemormorphic equivalence, such that the following diagram commutes :

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi_X"'] \arrow[dr, phantom, "\circlearrowright"] & Y \arrow[d, "\pi_Y"] \\ B(X) \arrow[r, "f_b"] & B(Y) \end{tikzcd} \qquad \left(\text{resp. } \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi_X'"'] \arrow[dr, phantom, "\circlearrowright"] & Y \arrow[d, "\pi_Y'"] \\ B'(X) \arrow[r, "f_b'"] & B'(Y) \end{tikzcd} \right).$$

<u>Corollary</u> (1.1.5)

For a complex variety X of dimension $\leqq 4$, the surjective meromorphic map $\pi_X : X \to B(X)$ canonically induces a bimeromorphic map $(\pi_X)_b' : B'(X) \xrightarrow{\simeq} B'(B(X))$.

<u>Corollary</u> (1.1.6)

Assume that X is a Moishezon variety (or more generally $X \in \mathcal{C}$) with $\dim X \leqq 4$. Let $f : X \to Y$ be a surjective morphism whose general fibre is irreducible with $\varkappa = 0$. Then f_b' is a bimeromorphic map : $B'(X) \xrightarrow{\simeq} B'(Y)$. In particular, if $\varphi : X \to X_0$ is the Iitaka fibration, then φ_b' is a bimeromorphic map : $B'(X) \xrightarrow{\simeq} B'(X_0)$.

<u>Corollary</u> (1.1.7)

Assume that X is a Moishezon variety (or more generally $X \in \mathcal{C}$) with $\dim X \leqq 4$. Then there exists a sequence of surjective meromorphic maps

$$X = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \rightarrow \cdots \xrightarrow{f_{m-1}} X_m \xrightarrow{f_m} X_{m+1}$$

with the following properties:

(1) $\dim X_i > \dim X_{i+1}$, $i = 0, 1, 2, \cdots, m$.

(2) X_{m+1} is of general type.

(3) For each i,

(i) if $\kappa(X_i) = -\infty$, then $X_{i+1} = B(X_i)$ and $f_i = \pi_{X_i}$, i.e., $f_i : X_i \rightarrow X_{i+1}$ is the fibration defined in (i) of (1.1.3).

(ii) if $\kappa(X_i) \geqq 0$, then $f_i : X_i \rightarrow X_{i+1}$ is the Iitaka fibration of X_i. (In particular if $\kappa(X_i) = 0$ for some $0 \leqq i < m$, then $i = m-1$ and X_m is a singleton.)

Such a sequence of surjective meromorphic maps is unique up to bimeromorphic equivalence. Furthermore, for each i, $(0 \leqq i \leqq m)$, $f_m \circ f_{m-1} \circ \cdots \circ f_{i+1} \circ f_i : X_i \rightarrow X_{m+1}$ is the fibration defined in (ii) of (1.1.3), i.e., $X_{m+1} = B'(X_i)$ and $f_m \circ f_{m-1} \circ \cdots \circ f_i = \pi'_{X_i}$.

(1.2) さて以下. dim $\leqq 3$ の complex variety を主に考える。Recall that a complex variety X is of class $\mathcal{C}$ (記号 $X \in \mathcal{C}$) if there exists a morphism from a compact Kähler manifold onto X.

<u>Definition</u> (1.2.1) : a complex variety X is called of Castelnuovo's type (or shortly "of CNT") if there exists no surjective meromorphic map from X to a complex variety Y of $\dim Y > 0$ and $\kappa(Y) \geqq 0$.

<u>Remark</u> (1.2.2) : If $\dim X \leqq 4$, then X is of CNT if and only if $\dim B(X) = 0$.

<u>Proposition</u> (1.2.3) : Let $g : X \to Z$ be a surjective morphism of complex varieties such that (1) Z is of CNT and that (2) general fibres of g are irreducible and of CNT. Then X is of CNT.

証明の方針: $f : X \to Y$ を surjective meromorphic map with $\kappa(Y) \geqq 0$ とすると. 条件 (2) から f は

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & \circlearrowleft & Y \\ & Z \arrow[ur] & \end{tikzcd}$$

と. surjective meromorphic maps の composition の形に factor する。ところが (1) により Y は 1点でなければならない。

Proposition (1.2.4)

Let X be a complex variety of CNT.

(i) If $\dim X = 1$, then X is a rational curve.

(ii) Assume that X is smooth and $\dim X = 2$. Then X is either a rational surface or a surface of class VII. (In particular, if $X \in \mathcal{C}$, then X is a rational surface.)

証明の方針： Kodaira の classification による。

Proposition (1.2.5)

Let X be a compact complex manifold of class $\mathcal{C}$ with $\dim X \leq 3$. Then there exists a Zariski open dense subset U of $B(X)$ such that for every $y \in U$, the fibre $\pi_X^{-1}(y)$ is of CNT, i.e., $B(\pi_X^{-1}(y))$ is a singleton.

証明の方針： $\dim$(general fibre of π_X) が 1 か 2 のときを考えればよいが、1 のときは curve を fibre とするような fibration に於ける小平次元の劣加法性 (by Viehweg) により明らか；2 のときは $\dim X = 3$ かつ $\dim B(X) = 1$ と仮定してよいから、Viehweg によって示された小平次元の劣加法性 (C_3) によって general fibre of

π_X は rational surface か ruled surface of genus $g \geq 1$ のいずれかになる。ところが後者の場合は relative albanese map

$$\begin{array}{ccc} X & \longrightarrow & B(X) \\ & \searrow \quad \circlearrowleft \quad \nearrow & \\ & Alb(X/B(X)) & \end{array}$$

が構成され (due to Fujiki).

しかも $Alb(X/B(X))$ は general fibre を genus $g \geq 1$ の curve としてもつような $B(X)$ 上の fibre space となっている。よって $\kappa(Alb(X/B(X))) \geq 0$ だが、これは $B(X)$ の定義に反する。故に general fibre of π_X は rational surface でしかあり得ない。

(1.3) さていよいよ compact complex manifold of dimension 3 with $\kappa = -\infty$ の大雑把なクラスわけを行う。まず Kawai, Ueno, Fujiki の結果から次のことが出てくるのはほぼ明らかである。

<u>Proposition</u> (1.2.6) : Let X be a compact complex manifold with $\kappa(X) = -\infty$, $\dim X = 3$, and $X \in \mathcal{C}$. Then we have one of the following:

i) $\dim B(X) = a(X) = 0$. In this case, X is of CNT and is also simple in Fujiki's sense.

ii) $\dim B(X)=0$ and $X_{alg} = \mathbb{P}^1$. Then for a suitable bimeromorphic model of X, the algebraic reduction $X \to X_{alg}$ has general fibres of one of the following form: (due to Fujiki)

a) relatively minimal Kähler K3-surface

b) complex torus

c) almost homogeneous ruled surface of genus 1.

iii) $\dim B(X)=0$, and $X_{alg} \simeq \mathbb{P}^2$. Then the algebraic reduction $X \to X_{alg}$ is an elliptic fibration.

iv) $\dim B(X)=0$ and $a(X)=3$, i.e., X is a Moishezon manifold and is of CNT.

v) $B(X)$ is a curve of genus $g \geq 1$, and over a Zariski open dense subset of $B(X)$, every fibre $X \to B(X)$ is a rational surface.

vi) $\dim B(X)=2$ and over a Zariski open dense subset of $B(X)$, every fibre of $X \to B(X)$ is a rational curve.

次に X を C として $\kappa(X)=-\infty$ の compact complex manifold を調べてみる。そういった X の構造は C に属する場合と異ってもう少し複雑になる。

Proposition (1.2.7)

Let X be a compact complex manifold of dimension 3 with $\kappa(X) = -\infty$ and $X \notin \mathcal{C}$. Then, replacing X by its appropriate bimeromorphic model, we have one of the following:

(i) X is simple in Fujiki's sense.

(ii) (due to Fujiki) There exists a surface S of class VII_0 with $\kappa(S) = -\infty$ and $a(S) = 0$, and also exists a fibration $\pi : X \to S$ with the following property:

For every surjective meromorphic map $f : X \to Y$ with $\dim Y > 0$, there exists a generically finite meromorphic map $\tilde{f} : S \to Y$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "\pi"'] & Y \arrow[d, phantom, "\circlearrowleft"] \\ & S \arrow[u, "\tilde{f}"'] \end{tikzcd}$$

commutes. (In particular, Y is a surface.)

Furthermore, general fibres of π are either elliptic curves or rational curves.

(iii) $X_c = X_{alg}$ = a curve and $\dim B(X) \leqq 1$ and a general fibre of the algebraic reduction $X \to X_{alg}$ is bimeromorphic to one of the following:

a) K3 surface, b) complex torus, c) hyperelliptic surface, d) Enriques surface, e) elliptic surface with trivial canonical bundle, f) surface of class VII_0, g) rational surface ($\neq \mathbb{P}^2$), h) ruled surface of genus 1.

(iv) $B(X_c) = B(X) =$ either a curve of genus ≥ 1 or a singleton, and X_c is a ruled surface, and the $\mathbb{C}$-reduction $X \to X_c$ is an elliptic fibration.

(v) $B(X_c) = X_c =$ a curve, and $B(X)$ is an elliptic surface with odd first Betti number fibred over the curve $B(X)_c$, and a general fibre of $X \to B(X)$ is $\mathbb{P}^1$.

(vi) $B(X_\bullet)_c =$ a curve, and there is a generically finite surjective morphism $f : X \to B(X) \times_{B(X)_c} X_c$ making the following diagram commute:

$$\begin{tikzcd}
 & X \arrow[dl, "\pi_X"'] \arrow[d, "f"] \arrow[dr, "\mathbb{C}\text{-reduction}"] \arrow[dl, phantom, shift left=4, "\circlearrowright"] \arrow[dr, phantom, shift right=4, "\circlearrowright"] & \\
B(X) & B(X) \times_{B(X)_c} X_c \arrow[l, "pr_1"] \arrow[r, "pr_2"'] & X_c
\end{tikzcd}$$

Furthermore X_c is a ruled surface over the curve $B(X)_c$ and $B(X)$ is an elliptic surface fibred over the curve $B(X)_c$.

2. Fibrations associated with holomorphic 2-forms

<u>Proposition</u> (2.1.1) Let X be a projective algebraic manifold with $\dim X = 3$ and $\dim B(X) > 0$. Then $\pi_X : X \to B(X)$ induces

$$H^0(X, S^m(\Omega_X^2)) \cong H^0(B(X), S^m(\Omega_{B(X)}^2)) \quad m=1,2,\cdots$$

$$H^0(X, \Omega_X^1) \cong H^0(B(X), \Omega_{B(X)}^1)$$

Furthermore

i) If $\dim B(X) \geq 2$, then $H^0(X, S^m(\Omega_X^1)) \cong H^0(B(X), S^m(\Omega_{B(X)}^1))$
$m=1,2,\cdots$

ii) $\dim B(X)=1$. Then every $\omega \in H^0(X, S^m(\Omega_X^1))$ is written in the form $\pi_X^*(\omega')$ for some $\omega' \in H^0(B(X), \mathcal{C}(B(X)) \cdot S^m(\Omega_{B(X)}^1))$ s.t. $\int_{B(X)} (\omega' \wedge \overline{\omega'})^{\frac{1}{m}} < +\infty$.

証明の方針: $\dim B(X) > 0$ に注意すると Proposition (1.2.5) により π_X の fibre は rational variety になる。このことから容易にでてくる。

<u>Remark</u> (2.1.2)

上のところで $m \geq 2$ に対しても isomorphism が成りたつ部分があるのは Ueno の指摘による。更に ii) の

$\dim B(X)=1$ の時に $H^0(X, S^m(\Omega_X^1)) \cong H^0(B(X), S^m(\Omega_{B(X)}^1))$ が成りたつことも十分可能性がある。たとえば rational surface を smooth fibre としてもち base space を nonsingular curve としてもつような fibration に於て、singular fibre の所に multiplicity が 1 となる component が含まれていれば同型が成立する。(= Ueno の注意)

<u>Conjecture</u> (2.1.3)

$$\dim B(X)=0 \Rightarrow H^0(X, S^m(\Omega_X^1)) = H^0(X, S^m(\Omega_X^2))=0 \ ? \quad \forall m=1,2,\cdots$$

最後に holomorphic 2-forms から定義される fibration を考えてみる。

X : 3 dim projective algebraic manifold with $\kappa(X) \leq 0$ とする。

$r \underset{\text{defn}}{=}$ rank of the subsheaf of Ω_X^2 generated by the global sections of Ω_X^2

このとき $r = 1$ or 2 or 3.

<u>Proposition</u> (2.2.1) $\quad r=3 \Rightarrow h^{2,0}(X)=3$

(下の simplified proof は Ueno による。)

証明：$h^{2,0}>3$ とする。$\omega_1\wedge\omega_2\wedge\omega_3 \neq 0 \in H^0(X, \det(\Omega_X^2))$ となるように $H^0(X, \Omega_X^2)$ の $\mathbb{C}$-basis $\{\omega_1, \omega_2, \omega_3, \omega_4, \cdots\}$ をとる。Then $\omega_4 = f_1\omega_1 + f_2\omega_2 + f_3\omega_3$ $(\exists f_1, f_2, f_3 \in \mathbb{C}(X))$ ここで f_1 = non-constant rational function と仮定しても一般性を失わない。Then

$$\text{Both}\quad \left.\begin{array}{r} \omega_4\wedge\omega_2\wedge\omega_3 = f_1\,\omega_1\wedge\omega_2\wedge\omega_3 \\ \omega_1\wedge\omega_2\wedge\omega_3 \end{array}\right\} \in H^0(X, \det(\Omega_X^2)) = H^0(X, K_X^{\otimes 2})$$

$\therefore \kappa(X) > 0$ $\therefore$ Contradiction.

<u>Proposition</u> (2.2.2)

Assume $r=1$ and $h^{2,0}(X) \geqq 2$.

$\Rightarrow$ Then $\kappa(X) = -\infty$ and $\dim B(X) = 2$

証明の方針：

$r=1$ なので $\{\omega_1, \omega_2, \cdots, \omega_\nu\}$ を $H^0(X, \Omega_X^2)$ の $\mathbb{C}$-basis とすると

$$\begin{aligned} \Phi : X &\longrightarrow \mathbb{P}^{\nu-1} \\ x &\longmapsto (\omega_1(x) : \omega_2(x) : \cdots : \omega_\nu(x)) \end{aligned}$$

で fibration を作ることができる。（但し X を適当な birational model にとりかえておいて Φ が最初から morphism であると仮定しておいてよい。）

case 1) $\dim \Phi(X) \geq 2$. この時は Bogomolov の Lemma により $\dim \Phi(X) = 2$ が成りたつ。Stein factorization

$$\begin{tikzcd} X \arrow[rr, "\Phi"] \arrow[dr] & \arrow[d, phantom, "\circlearrowright"] & \Phi(X) \\ & B \arrow[ur] & \end{tikzcd}$$

をとったとき $\kappa(X) = -\infty$, $B = B(X)$ となる事が示される。

case 2) $\dim \Phi(X) = 1$. 矢張り Stein factorization

$$\begin{tikzcd} X \arrow[rr, "\Phi"] \arrow[dr] & \arrow[d, phantom, "\circlearrowright"] & \Phi(X) \\ & C \arrow[ur] & \end{tikzcd}$$

をとる。$\kappa(X) \leq 0$ に注意すると variation of Hodge structure の一般論から relative albanese $Alb(X/C)$ を構成すると surface になることがわかる。そして結局 $\kappa(X) = -\infty$, $B(X) = Alb(X/C)$ が成りたつ。

さて以上の如く $r = 1, 3$ の場合はうまくいった。$r = 2$ の場合も同様の fibration (Grassmann variety への map) が構成できるが、例えば Ueno の予想「X : projective algebraic 3-fold with

$\varkappa(X)=0 \Rightarrow h^{2,0}(X)\leq 3$」を証明するには、ひとつだけ大きな gap が残っている。ここでは、これ以上細部にはたち入らないで次の予想（$\varkappa=0$ の場合は Ueno による）をあげておわりにする。

<u>Conjecture</u>: Let X be a 3-dimensional projective alg manifold. Then

1) $\varkappa(X)=0 \Rightarrow h^{2,0}(X)\leq r$?

2) $\dim B(X)=0 \Rightarrow h^{2,0}(X)\leq r$?

（この 2) は前でのべた conjecture (2.1.3) の weak version の特別なものである。）