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On ruled 3-folds

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近年，Iitaka, Ueno, Viehweg, Kawamata, Fujitaらによって高次交代数多様体の分類に関する著しい結果が得られ始めました。そこで我々は $X(X) \leq 0$ の 3 次元の 4 次元 compact complex manifolds を調べてみた。

(1.1) 以下 $X, Y, Y_1, Y_2, \ldots$ などはすべて irreducible reduced compact complex space（単に complex variety と呼ぶ）を示すものとする。Surjective meromorphic maps $f_i : X \rightarrow Y_i, i=1, 2$ に対し次の様な定義を行う。

$Y_1 \times Y_2 := $ meromorphic image of $X$ under the meromorphic map $x \in X \rightarrow (f_1(x), f_2(x))$, $x \times Y$ $f_1 \times f_2 :=$ surjective meromorphic map from $X \times Y$ to $Y_1 \times Y_2$

defined by $x \times y \rightarrow (f_1(x), f_2(y))$

Proposition (1.1.1)

Assume that $\dim X \leq 4$. Then

(i) $X(Y_1) \geq 0$ and $X(Y_2) \geq 0$ imply $X(Y_1 \times Y_2) \geq 0$.

(ii) $X(Y_1) = \dim Y_1$ and $X(Y_2) = \dim Y_2$ imply $X(Y_1 \times Y_2) = \dim (Y_1 \times Y_2)$
証明のポイント: (i) \( \dim Y_1, \dim Y_2, \dim Y_1 \times Y_2 \leq d \) で、case 毎に調べていくことによって、結局 \( \text{vichweg} \) により示された curve を fibre とする \( \text{fibration} \) に並ぶ小平面元の加法性に帰着される。 (ii) 一般に (i) がわかれば、Kawamata によって示された case が general type であるよう \( \text{fibration} \) に並ぶ小平面元の加法性から徳う。

**Definition (1.1.2)** For a complex variety \( X \), we define

\[
B_X := \{ (Y, f) | f: X \to Y \text{ is a surjective meromorphic map to a complex variety } Y \text{ with } \dim Y \leq d \}
\]

\[
B' \hat{X} := \{ (Y, f) | f: X \to Y \text{ is a surjective meromorphic map to a complex variety } Y \text{ of general type } \}
\]

where \( (Y_1, f_1) \sim (Y_2, f_2) \) if there exists a birational map \( \tau: Y_1 \to Y_2 \) such that \( f_2 = \tau \circ f_1 \).

**Proposition (1.1.3)**

Assume that \( \dim X \leq 4 \). Then

(i) There exists a unique element in \( B_X \) (which we denote by \( (B(X, \tau_X) \) such that for every \( (Y, f) \in B_X \), there exists a surjective meromorphic map from \( B(X) \) to \( Y \) which
makes the following diagram commutative:

\[
\begin{array}{ccc}
X & \xrightarrow{\pi_x} & \mathcal{C} \\
\downarrow f & & \downarrow \\
\mathcal{B}(X) & \rightarrow & Y
\end{array}
\]

(ii) There exists a unique element in \(\mathcal{B}_x\) (which we denote by \((\mathcal{B}(x), \pi'_x)\)) such that for every \((Y, f) \in \mathcal{B}_x\), there exists a surjective meromorphic map from \(\mathcal{B}(x)\) to \(Y\) which makes the following diagram commutative:

\[
\begin{array}{ccc}
X & \xrightarrow{\pi'_x} & \mathcal{C} \\
\downarrow f & & \downarrow \\
\mathcal{B}'(X) & \rightarrow & Y
\end{array}
\]

証明のポイント: proposition (1.1.1) の容易な帰結である。

Corollary (1.1.4)

Assume that \(\dim X \leq 4\). Let \(f: X \rightarrow Y\) be a surjective meromorphic map. Then there exists a surjective meromorphic map \(f_b: \mathcal{B}(X) \rightarrow \mathcal{B}(Y)\) (resp. \(f'_b: \mathcal{B}'(X) \rightarrow \mathcal{B}'(Y)\)), unique up to bimeromorphic equivalence, such that the following diagram commutes:
Corollary (1.1.5)

For a complex variety $X$ of dimension $\leq 4$, the surjective
meromorphic map $\pi : X \to B(X)$ canonically induces a
meromorphic map $(\pi_X)_b : B(X) \to B'(B(X))$.

Corollary (1.1.6)

Assume that $X$ is a Moishezon variety (or more generally
$X \in C$) with $\dim X \leq 4$. Let $f : X \to Y$ be a surjective
morphism whose general fibre is irreducible with $X = 0$.
Then $f_b$ is a meromorphic map : $B(X) \to B'(Y)$. In
particular, if $\mathfrak{g} : X \to X_0$ is the Iitaka fibration, then
$\mathfrak{g}_b$ is a meromorphic map : $B(X) \to B'(X_0)$.

Corollary (1.1.7)

Assume that $X$ is a Moishezon variety (or more generally
$X \in C$) with $\dim X \leq 4$. Then there exists a sequence
of surjective meromorphic maps

\[ X = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \rightarrow \cdots \rightarrow X_m \xrightarrow{f_m} X_{m+1} \]

with the following properties:

1. \( \dim X_i > \dim X_{i+1}, \quad i=0,1,2, \ldots, m. \)
2. \( X_{m+1} \) is of general type.
3. For each \( i, \)
   
   (i) if \( \chi(X_i) = -\infty, \) then \( X_{i+1} = B(X_i) \) and \( f_i = \pi_{X_i}, \) i.e.,
   
   \( f_i : X_i \rightarrow X_{i+1} \) is the fibration defined in (i) of (1.1.3).
   
   (ii) if \( \chi(X_i) \geq 0, \) then \( f_i : X_i \rightarrow X_{i+1} \) is the Iitaka fibration
   
   of \( X_i. \) (In particular, if \( \chi(X_i) = 0 \) for some \( 0 \leq i < m, \)
   
   then \( i = m-1 \) and \( X_m \) is a singleton.)

Such a sequence of surjective meromorphic maps is unique up
to birational equivalence. Furthermore, for each \( i (0 \leq i \leq m), \)

\( f_m \circ f_{m-1} \circ \cdots \circ f_{i+1} \circ f_i : X_i \rightarrow X_{m+1} \)
is the fibration defined in

(ii) of (1.1.3), i.e., \( X_{m+1} = B(X_i) \) and \( f_m \circ f_{m-1} \circ \cdots \circ f_i = \pi_{X_i}. \)

(1.2) 以下、\( \dim X \leq 3 \) の複素多様体 \( X \) と考える。

Recall that a complex variety \( X \) is of class \( \mathcal{C} \) (記号
\( X \in \mathcal{C} \)) if there exists a morphism from a compact Kähler
manifold onto \( X. \)
Definition (1.2.1): A complex variety $X$ is called of Castelnuovo type (or shortly “of CNT”) if there exists no surjective meromorphic map from $X$ to a complex variety $Y$ of $\dim Y > 0$ and $\chi(Y) \neq 0$.

Remark (1.2.2): If $\dim X \leq 4$, then $X$ is of CNT if and only if $\dim B(X) = 0$.

Proposition (1.2.3): Let $g : X \to Z$ be a surjective morphism of complex varieties such that (1) $Z$ is of CNT and that (2) general fibres of $g$ are irreducible and of CNT. Then $X$ is of CNT.

証明の要点: $f : X \to Y$ is a surjective meromorphic map with $\chi(Y) \neq 0$, so $g = f$ satisfies the conditions (1) and (2). Therefore, $X$ is of CNT.
Proposition (1.2.4)

Let $X$ be a complex variety of $\text{CNT}$.

(i) If $\dim X = 1$, then $X$ is a rational curve.

(ii) Assume that $X$ is smooth and $\dim X = 2$. Then $X$ is either a rational surface or a surface of class $\text{III}$. (In particular, if $X \in \mathcal{C}$, then $X$ is a rational surface.)

証明の方針: Kodairaのclassificationによる。

Proposition (1.2.5)

Let $X$ be a compact complex manifold of class $\mathcal{C}$ with $\dim X \leq 3$, then there exists a Zariski open dense subset $U$ of $\mathcal{B}(X)$ such that for every $y \in U$, the fibre $\pi_X^{-1}(y)$ is of $\text{CNT}$, i.e., $\mathcal{B}(\pi_X^{-1}(y))$ is a singleton.

証明の方針: $\dim (\text{general fibre of } \pi_X) \leq 1$ か否か、このときを考えればよいが。1のときは Curve を fibreとするような fibration に帰ける小平次元の容加性（by Viehweg）により明らか；2のときは $\dim X = 3$ で $\dim \mathcal{B}(X) = 1$ と仮定してよいから，Viehwegによって示された小平次元の容加性（$\mathcal{C}_3$）によって general fibre of
$\mathbb{R}_x$ is a rational surface of ruled surface of genus $g \geq 1$ in a similar manner. The case in which $\text{rel}$ is relative Albanese map $\mathbb{R}_x \to B(X)$ on $\text{Alb}(X/B(X))$.

Similarly $\text{Alb}(X/B(X))$ is a general fibre $g \geq 1$ curve as well as $B(X)$ the fibre space. Therefore $\mathbb{R}(\text{Alb}(X/B(X))) = 0$ and $\text{Alb}(X/B(X))$ is not defined. Hence general fibre of $\mathbb{R}_x$ is a rational surface differently.

(1.3) We define $\mathcal{G}_x$ compact complex manifold of dimension 3 with $x = -\infty$ of framework and classification are executed. Mainly Kawai, Ueno, Fujiki's results, from the results coming to the form is almost straightforward.

**Proposition (1.2.6):** Let $X$ be a compact complex manifold with $X(X) = -\infty$, $\dim X = 3$, and $X \in \mathcal{G}_x$. Then we have one of the following:

1) $\dim B(X) = a(X) = 0$. In this case, $X$ is of CNT and is also simple in Fujiki's sense.
ii) \( \dim B(X) = 0 \) and \( X_{\text{alg}} = \mathbb{P}^1 \), then for a suitable

time-warp model of \( X \), the algebraic reduction \( X \to X_{\text{alg}} \)

has a general fibres of one of the following form: (due to Fujiki)

\begin{enumerate}
  \item relatively minimal Kähler K3-surface
  \item complex torus
  \item almost homogeneous ruled surface of genus 1.
\end{enumerate}

iii) \( \dim B(X) = 0 \), and \( X_{\text{alg}} = \mathbb{P}^2 \). Then the algebraic reduction

\( X \to X_{\text{alg}} \) is an elliptic fibration.

iv) \( \dim B(X) = 0 \) and \( q(X) = 3 \), i.e., \( X \) is a non-kähler

manifold \text{ and is of CNT.}

v) \( B(X) \) is a curve of genus \( g \geq 1 \), and over a Zariski

open dense subset of \( B(X) \), every fibre \( X \to B(X) \) is a

rational surface.

vi) \( \dim B(X) = 2 \) and over a Zariski open dense subset of

\( B(X) \), every fibre of \( X \to B(X) \) is a rational curve.

次に \( X \) として \( X(\mathbb{C}) = -\infty \) の compact complex manifold

を調べてみる。そういった \( X \) の構造は \( Y \) に属する場合

と違ってもう少し複雑になる。

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Proposition (1.2.7)

Let \( X \) be a compact complex manifold of dimension 3 with \( \kappa(X) = -\infty \) and \( X \not\equiv \mathbb{C} \). Then, replacing \( X \) by its appropriate birational model, we have one of the following:

(i) \( X \) is simple in Fujiki's sense.

(ii) (due to Fujiki) There exists a surface \( S \) of class \( M_0 \) with \( \kappa(S) = -\infty \) and \( \kappa(S) = 0 \), and also exists a fibration \( \pi : X \to S \) with the following property:

For every surjective meromorphic map \( f : X \to Y \) with \( \dim Y > 0 \), there exists a generically finite meromorphic map \( \tilde{f} : S \to Y \) such that the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\pi \downarrow & & \uparrow \tilde{f} \\
S & & \\
\end{array}
\]

commutes. (In particular, \( Y \) is a surface.)

Furthermore, a general fibre of \( \pi \) are either elliptic curves or rational curves.

(iii) \( X_c = X_{\text{alg}} = \text{a curve} \) and \( \dim B(x) \leq 1 \) and a general fibre of the algebraic reduction \( X \to X_{\text{alg}} \) is \( \mathbb{C} \).
of higher bidegree is isomorphic to one of the following:

a) K3 surface,  b) complex torus,  c) hyperelliptic surface,

d) Enriques surface,  e) elliptic surface with trivial canonical bundle,  f) surface of class $\text{M}_0$,  g) rational surface ($\equiv \mathbb{P}^2$),  h) ruled surface of genus 1.

(iv) $B(X_c) = B(X) = \text{either a curve of genus } \geq 1 \text{ or a singleton,}$

and $X_c$ is a ruled surface, and the $C$-reduction $X \rightarrow X_c$ is an elliptic fibration.

(v) $B(X_c) = X_c = \text{a curve}$, and $B(X)$ is an elliptic surface with

odd first Betti number fibred over the curve $B(X)_c$, and a general fibre of $X \rightarrow B(X)$ is $\mathbb{P}^1$.

(vi) $B(X)_c = \text{a curve}$, and there is a generically finite

surjective morphism $f : X \rightarrow B(X \times X)_c$ making the following diagram commute:

Furthermore $X_c$ is a ruled surface over the curve $B(X)_c$ and $B(X)$ is an elliptic surface fibred over the curve $B(X)_c$. 

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2. Fibraions associated with holomorphic 2-forms

Proposition (2.14) Let \( X \) be a projective algebraic manifold with

\[
\dim X = 3 \quad \text{and} \quad \dim B(X) > 0.
\]

Then \( \mathcal{B}_x : X \to B(X) \) induces

\[
H^0(X, s^m(Q^2_X)) \cong H^0(B(X), s^m(Q^2_{B(X)})) \quad m=1,2,\ldots \\
H^0(X, Q^1_X) \cong H^0(B(X), Q^1_{B(X)}).
\]

Furthermore

i) If \( \dim B(X)=2 \), then \( H^0(X, s^m(Q^2_X)) \cong H^0(B(X), s^m(Q^2_{B(X)})) \)

\[
m=1,2,\ldots.
\]

ii) \( \dim B(X)=1 \). Then every \( \omega \in H^0(X, s^m(Q^2_X)) \) is written

in the form \( \Pi^*_X(\omega') \) for some \( \omega' \in H^0(B(X), s^m(Q^2_{B(X)})) \)

i.e.

\[
\int_{B(X)} \omega' \cdot \overline{\omega'} \frac{1}{m} < +\infty.
\]

証明の主言: \( \dim B(X)=2 \) に注意すること Proposition (1.2.5)

により \( \Pi^*_X \) の fibre は rational variety になる。このことから容易に証明できる。

Remark (2.1.2)

上の定理 \( m \geq 2 \) に対しては isomorphism が成り立つ

部分があるのは Ueno の引用による。更に ii) は
dim \text{BC}(X) = 1 の時 に と \text{H}^0(\text{BC}(X), S^m(Q_{\text{BC}(X)})) となる
なりつつ ことも十分可能性がある。 した ば
rational surface を smooth fibre とし て 3 base space と nonsingular curve としても うような fibration に つ
って singular fibre の所に multiplicity の 1 となる component が含まれて い れば 同型が成立する。
(= Iwao の注意)

Conjecture (2.1.3)

\[ \dim \text{BC}(X) = 0 \quad \Rightarrow \quad \text{H}^0(X, S^m(Q_X)) = \text{H}^0(X, S^m(Q^2_X)) = 0 \]

で

最後に holomorphic 2-forms から定義される fibration を考えている。

\[ X : 3 \dim \text{projective algebraic manifold with } \chi(X) \leq 0 \]

とする。

\[ r \text{ def. rank of the subsheaf of } S^2_X \text{ generated by the global sections of } S^2_X \]

\[ \chi \leq 0 \text{ のとき } r = 1 \text{ or } 2 \text{ or } 3. \]

Proposition (2.2.1) \[ r = 3 \quad \Rightarrow \quad \chi^{2,0}(X) = 3 \]

( \text{ F 0 simplified proof is Iwao’s } \text{ result. } )
証明: $\mathbb{R}^2 \supset 0$ とする。$\omega_1 \wedge \omega_2 \wedge \omega_3 = 0 \in H^0(x, \mathcal{O}_x(\mathbb{R}^2))$
とおせば $H^0(x, \mathcal{O}_x(\mathbb{R}^2))$ の基底 $\{\omega_1, \omega_2, \omega_3, \omega_4, \ldots\}$
とする。$\omega_4 = f_1 \omega_1 + f_2 \omega_2 + f_3 \omega_3$ かつ $f_1, f_2, f_3 \in C(x)$
で$f_1 = \text{non-constant rational function}$ とき定めても
一般性を失わない。$\therefore$
Both $\omega_1 \wedge \omega_2 \wedge \omega_3 = f_1 \omega_1 \wedge \omega_2 \wedge \omega_3$
$\omega_1 \wedge \omega_2 \wedge \omega_3 
\in H^0(x, \mathcal{O}_x(\mathbb{R}^2)) = H^0(x, \mathcal{O}_x)$(1)
$x(x) > 0$ に矛盾。

Proposition (2.2)
Assume $r=1$ and $\mathbb{R}^2 \supset 1$.
$\Rightarrow \therefore x(x) = -\infty$ and dim $B(x) = 2$
証明の方法:
$r=1$ のので $\{\omega_1, \omega_2, \ldots, \omega_n\} \in H^0(x, \mathcal{O}_x)$ の基底とすると
$\exists: X \rightarrow \mathbb{P}^{r-1}$
$X \rightarrow (\omega_1(x): \omega_2(x): \cdots : \omega_n(x))$
で fibration を作ることができる。（但し $X$ を適當
準 spin rational model としておいて至極最初か
ら morphism であると仮定しておいてよい。）

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case 1) $\dim \overline{\nu}(X) \geq 2$,  この時は Bogomolov の Lemma により $\dim \overline{\nu}(X) = 2$ が成り立つ。Stein factorization

$$
\begin{array}{c}
X \xrightarrow{\pi} \overline{\nu}(X) \\
\downarrow \quad \quad \quad \downarrow \\
\qquad B
\end{array}
$$

とするととき $\nu(X) = -\infty$, $B = B(X)$ となる事が示される。

case 2) $\dim \overline{\nu}(X) = 1$, 矢張り Stein factorization

$$
\begin{array}{c}
X \xrightarrow{\pi} \overline{\nu}(X) \\
\downarrow \quad \quad \quad \downarrow \\
\qquad C
\end{array}
$$

とると，$\nu(X) \leq 0$ に注意すると Variation of Hodge structure の一般論から relative albanese $\text{Alb}(X/C)$ を構成すると surface になることがわかる。そして結局 $\nu(X) = -\infty$, $B(X) = \text{Alb}(X/C)$ が成り立つ。

さて以上の如く $r=1, 3$ の場合はうまくいかなかった。$r=2$ の場合も同様の fibration (Grassmann variety への map) で構成できることは、例えば "

We use the notation $X$: projective algebraic 3-fold with

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$\chi(x) = 0 \Rightarrow \chi^2(x) \leq 3'$ を証明するには、ひとつだけ大きな gap が残っている。ここでは、それ以上細部には入らないで、次に予想 ( $\chi = 0$ の場合は Ueno による) をあげておわりにする。

Conjecture: Let $X$ be a 3-dimensional projective alg. manifold, then

1) $\chi(x) = 0 \Rightarrow \chi^2(x) \leq r$
2) $\dim B(x) = 0 \Rightarrow \chi^2(x) \leq r$

(この 2 は前で述べた conjecture (2.1.3) の weak version の特例なものである。)