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Threefolds whose canonical bundles are not numerically effective
by Shigefumi Mori

In this note, we announce an application of the previous paper [3] with some examples. The proof will be published elsewhere.

§1. Announcement.

We assume that $k$ is an algebraically closed field of characteristic 0 and $X$ is a non-singular projective 3-fold over $k$ whose canonical bundle $K_X$ is not numerically effective. We use the terminology of [3]. By Corollary & [3], $X$ has an extremal ray $R$, which we fix in this section.

Theorem 1. There exists a morphism $\phi : X \to Y$ to a projective variety $Y$, such that (1) $\phi^*\mathcal{O}_X = \mathcal{O}_Y$, and (2) for any irreducible curve $C$ in $X$, $[C] \in R$ if and only if $\dim \phi(C) = 0$. Furthermore, such a $\phi$ is unique up to an isomorphism.

The structure of this $\phi$ is given by the following theorems.

Theorem 2. The extremal ray $R$ is not numerically effective if and only if $\dim Y = 3$. If these conditions are satisfied, then there exists an irreducible divisor $D$ of $X$ such that $X$ is the blowing-up of $Y$ by the ideal defining $\phi(D)$ (given the reduced structure), and we have either

(1) $\phi(D)$ is a non-singular curve and $Y$ is non-singular; $\phi|_D : D \to \phi(D)$ is a $\mathbb{P}^1$-bundle and $(D.\phi^{-1}(n)) = -1$ for any $n \in \phi(D),$

(2) $Q = \phi(D)$ is a point and $Y$ is non-singular; $D \neq \mathbb{P}^2$.
and \( \mathcal{O}_D(D) \cong \mathcal{O}_{\mathbb{P}}(-1) \),

(3) \( Q = \phi(D) \) is an ordinary double point of \( Y \);
\( D \cong \mathbb{P}^1 \times \mathbb{P}^1 \) and \( \mathcal{O}_D(D) \cong \mathcal{O}_{\mathbb{P}}(-1) \otimes \mathcal{O}_{\mathbb{P}}(-1) \), where \( p_1 \) is the \( i \)-th projection,

(4) \( Q = \phi(D) \) is a double point of \( Y \); \( D = \) an irreducible reduced singular quadric surface \( S \) in \( \mathbb{P}^3 \), \( \mathcal{O}_D(D) \cong \mathcal{O}_{S} \otimes \mathcal{O}_{\mathbb{P}}(-1) \), or

(5) \( Q = \phi(D) \) is a quadruple point of \( Y \); \( D \cong \mathbb{P}^2 \), \( \mathcal{O}_D(D) \cong \mathcal{O}_{\mathbb{P}}(-2) \).

Let \( \mathcal{O}_{Y,Q} \) be the local ring of \( Y \) at \( Q \) for cases (3), (4), and (5) in Theorem 2. Then we have

**Theorem 3.**

1. The divisor class group of \( \mathcal{O}_{Y,Q} \) is 0 in cases (3) and (4), and \( \mathbb{Z}/2\mathbb{Z} \) in case (5), and
2. the completion \( \mathcal{O}_{Y,Q}^{\wedge} \) of \( \mathcal{O}_{Y,Q} \) is given by

\[
\mathcal{O}_{Y,Q}^{\wedge} \cong \begin{cases} 
\mathcal{O}_{Y,Q}^{\wedge} \cong k[[x,y,z,u]]/(x^2 + y^2 + z^2 + u^2) & \text{case (3)}, \\
\mathcal{O}_{Y,Q}^{\wedge} \cong k[[x,y,z,u]]/(x^2 + y^2 + z^2 + u^3) & \text{case (4)}, \\
\mathcal{O}_{Y,Q}^{\wedge} \cong k[[x,y,z]]^{(2)} & \text{case (5)},
\end{cases}
\]

where \( k[[x,y,z]]^{(2)} \) is the invariant subring of \( k[[x,y,z]] \) under the action of the involution \( (x,y,z) \mapsto (-x,-y,-z) \).

The remaining cases are treated by

**Theorem 4.**

If \( R \) is numerically effective, then \( Y \) is non-singular, \( \rho(X) = \rho(Y) + 1 \), and we have either

1. \( \dim Y = 2 \), and for an arbitrary geometric point \( \eta \) of \( Y \), the scheme-theoretic fiber \( X_\eta \) is isomorphic to a conic of \( \mathbb{P}^2_{k(\eta)} \), where \( k(\eta) \) is the field of \( \eta \) (i.e. \( X_\eta \) is isomorphic to either a smooth conic, a reducible conic, or a double line,
(2) $\dim Y = 1$, and for an arbitrary geometric point $\eta$ of $Y$, $X_\eta$ is an irreducible reduced surface such that $w_{X_\eta}^{-1}$ is ample, or

(3) $\dim Y = 0$, and $X$ is a Fano 3-fold, (these 3-folds are classified by Iskovski [2].)

§2. Exceptional divisors.

The most interesting part of section 1 is Theorem 2. Examples for Theorem 2 can be given by considering birational morphisms.

**Theorem 5.** Let $\pi : X \rightarrow Z$ be a birational morphism (which is not an isomorphism) of non-singular projective 3-folds. Then $X$ contains an extremal rational curve $\mathcal{E}$ such that (1) $\dim \pi(\mathcal{E}) = 0$ and (2) $\mathcal{E}$ is not numerically effective. Hence the exceptional set of $\pi$ contains a divisor described in Theorem 2.

**Examples 6.** Let $Z$ be a non-singular projective 3-fold.

(1): Let $C_1$ and $C_2$ be non-singular projective curves in $Z$ intersecting transversally at 2 points $P_1$ and $P_2$. If we operate Hironaka's twisted blowing-up to $C_1$ and $C_2$ (e.g. blowing up $C_1$ first near $P_1$ and $C_2$ first near $P_2$), then the "blowing up" $\pi : X \rightarrow Z$ does not have a divisor described in Theorem 2.
However, this does not contradict our theorems, because our $X$ is not projective.

(2): Let $C$ be an irreducible projective curve in $Z$ with one ordinary double point $P$ as singularities. If we blow up $C$, then the blown-up variety $Y$ has one ordinary double point $Q$ lying over $P$ as singularities. If we resolve the singularity by blowing up $Q$ and get a smooth 3-fold $X$, $\pi: X \to Z$ and $\phi: X \to Y$, then $D = \phi^{-1}(Q)$ is the divisor described in case (3) of Theorem 2.

We remark that we can not start with an arbitrary ordinary double point because of Theorem 3, (1).

(3): Let $C$ be an irreducible projective curve in $Z$ with one ordinary cusp $P$ as singularities. If we blow up $C$, the blown-up variety $Y$ has one double point $Q$ lying over $P$ as singularities which falls in case (4) of Theorem 2. If we blow up $Q$ to get a smooth 3-fold $X$, $\phi: X \to Y$ and $\pi: X \to Z$, then $D = \phi^{-1}(Q)$ is the divisor described in case (4).
(4): Let $C_1$, $C_2$, and $C_3$ be non-singular projective curves in $Z$ intersecting transversally at one point $P$. If we operate projective Hironaka's modification in [1], we get a smooth 3-fold $X$ and $\pi : X \to Z$, and $D$, the divisorial part of $\pi^{-1}(P)$, is the divisor given in case 5 of Theorem 2.

We will finish this section by proving Theorem 5. The proof consists of a few easy lemmas. We keep the notation of Theorem 5 till the end of this section.

**Lemma 7.** $\pi_* : N(X) \to N(Z)$ has the property $\pi_* \operatorname{NE}(X) \subseteq \operatorname{NE}(Z)$.

Indeed, we have $\pi_* \operatorname{NE}(X) \subseteq \operatorname{NE}(Z)$ by the definition of $\pi_*$, which implies Lemma 7 by continuity of $\pi_*$. 

**Lemma 8.** There is an effective 1-cycle $C$ on $X$ such that $\pi_* C = 0$ and $(C.c_1(X)) > 0$.

**Proof.** Let $E$ be the effective divisor on $X$ such that $\operatorname{Supp} E$ is the exceptional set of $\pi$ and $K_X = \pi^* K_Z + E$. We treat two cases.

Case 1: $\dim \pi(\operatorname{Supp} E) = 1$.

By Bertini's theorem, there is a smooth hyperplanesec...
of \( Z \) such that \( \pi^{-1}(L) \) is irreducible and non-singular. Then we have

\[
(K_{\pi}.E_{\pi}^{-1}(L)) = (\pi^{*}K_{Z}.E_{\pi}^{-1}(L)) + (E^{2}_{\pi} \cdot E_{\pi}^{-1}(L))
= (E^{*}(K_{Z}).E_{\pi}^{-1}(L)) + (\mathcal{O}_{\pi}^{-1}(L)(E)^{2})_{\pi}^{-1}(L)
= (\pi^{*}E_{Z}.L) + (\mathcal{O}_{\pi}^{-1}(L)(E)^{2})_{\pi}^{-1}(L).
\]

Since \( \pi^{*}E = 0 \), we have \( (\pi^{*}E_{Z}.L) = 0 \). We have

\[
(\mathcal{O}_{\pi}^{-1}(L)(E)^{2})_{\pi}^{-1}(L) < 0 \text{ because } C = E \cdot E_{\pi}^{-1}(L) \neq 0, \text{ since}
\]

\( \dim \pi(\text{Supp } E) = 1 \) is an exceptional divisor of \( \pi^{-1}(L) \rightarrow L \).

Hence \( \pi^{*}C = 0 \) and \( (K_{\pi}.C) < 0 \).

Case 2: \( \dim \pi(\text{Supp } E) = 0 \).

Let \( M \) be a smooth hyperplanesecion of \( X \), hence \( M \) and \( E \) intersect properly and \( M \cdot E \neq 0 \). Then we have

\[
(K_{\pi}.E_{M}) = (\pi^{*}K_{Z}.E_{M}) + (E^{2}_{M})
= (K_{Z}.\pi^{*}(E_{M})) + (\mathcal{O}_{M}(E)^{2})_{M}.
\]

Now \( \pi^{*}(E_{M}) = 0 \) because \( \dim \pi(\text{Supp } E) = 0 \), and \( (\mathcal{O}_{M}(E)^{2})_{M} < 0 \) because \( E \cdot M \) is an exceptional divisor of \( M \rightarrow \pi(M) \). Hence \( C = E \cdot M \) has the required property. q.e.d.

Lemma 9. There is an extremal rational curve \( \epsilon \) on \( X \) such that \( \pi^{*}\epsilon = 0 \).

Proof. Let \( H \) be an arbitrary ample divisor on \( X \) and \( \epsilon \) a small enough positive number so that \([C] \) given in Lemma 8 does not belong to \( \overline{NE}_{\epsilon}(X, H) \). By Theorem 3 in [3], \([C] \) is written as

\[
[C] = \sum_{i=1}^{r} a_{i} [\epsilon_{i}] + V,
\]
where \( a_i > 0, \ell_1 \) are extremal rational curves for all \( i \), and \( V \in \overline{\text{NE}}_e(X) \). Hence \( \pi_*[\ell_1] + \pi_*V = 0 \) and \( \pi_*[\ell_1], \pi_*V \in \overline{\text{NE}}(Z) \) by Lemma 7. Since \( Z \) is projective, we have \( a_i \pi_*[\ell_1] = 0 \) for all \( i \) and \( \pi_*V = 0 \) by Kleiman's criterion of projectivity: \( \overline{\text{NE}}(Z) \cap \{-\overline{\text{NE}}(Z)\} = \{0\} \). Since \([C] \notin \overline{\text{NE}}_e(X)\), there is at least one \( j \) such that \( a_j \neq 0 \). Then \( \ell_j \) has the required property.

q.e.d.

**Lemma 10.** The curve \( \ell \) in Lemma 9 is not numerically effective.

If \( E \) is the effective divisor on \( X \) given in the proof of Lemma 8, then

\[
(\ell.E) = (\ell.K_X) - (\ell.\pi^*K_Z) \]

\[
= (\ell.K_X) - (\pi_*\ell_K_Z) < 0.
\]

Thus Theorem 5 is proved, and it is easy to check the assertions in Examples 6.
References

