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The cone of effective 1-cycles

by

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Introduction. In this note, our subject is the cone NE(X) of effective 1-cycles on a non-singular projective variety X. We will study how far this convex cone NE(X) is from being polyhedral. If $c_1(X)$, the first Chern class, is ample, then NE(X) is polyhedral (Theorem 1.) In general, NE(X) is close to be "polyhedral" on the half space $\{Z \in N(X) \mid (z.c_1(X)) > 0\}$ (Theorem 3.) Theorem 3 (or even Lemma 5) includes the assertion: K_X, canonical divisor, is numerically effective if X contains no rational curves.

In the next paper, we will consider the application of Theorem 3.

§1. Notation, definitions, and statements.

Let X be a non-singular projective variety of dimension n defined over an algebraically closed field k of characteristic $p \geq 0$, with a very ample divisor H. We will keep these symbols throughout this paper.

By a 1-cycle on X, we understand an element of the free abelian group generated by all the irreducible reduced subvarieties of dimension 1 of X. A 1-cycle $Z = \Sigma\, n_C\, C$ ($n_C \in \mathbb{Z}$) is called effective if $n_C \geq 0$ for all C. If two 1-cycles Z_1 and Z_2 are algebraically equivalent (resp. numerically equivalent) in the usual sense [2], we express it by $Z_1 \approx Z_2$ (resp. $Z_1 \approxeq Z_2$.) Let

$$A(X) = (\{1\text{-cycles on } X\}/\approx) \otimes_{\mathbb{Z}} \mathbb{Q},$$

$$N(X) = (\{1\text{-cycles on } X\}/\approxeq) \otimes_{\mathbb{Z}} \mathbb{R}, \text{ and}$$

$AE(X)$ (resp. $NE(X)$) the smallest convex cone in $A(X)$ (resp. $N(X)$) containing all effective 1-cycles, closed under multiplication by $\mathbb{Q}_+ = \{q \in \mathbb{Q} \mid q \geq 0\}$ (resp. $\mathbb{R}_+ = \{r \in \mathbb{R} \mid r \geq 0\}$.) Via the intersection pairing (.) of 1-cycles and divisors, $N(X)$ is dual to $NS(X) \otimes_{\mathbb{Z}} \mathbb{R}$, where $NS(X)$ is the Neron-Severi group, {divisors of $X\}/\approxeq$. Thus $N(X)$ is a real vector space of finite dimension $\rho(X)$, the rank of $NS(X)$. Let $\| \;\|$ be any norm of $N(X)$. Then $\overline{NE}(X)$, the closure of $NE(X)$ for the metric topology, is dual to the pseudo-ample cone of X (cf. [2]) by Kleiman's criterion for ampleness: a divisor D on X is ample if and only if $(D.Z) > 0$ for all $Z \in \overline{NE}(X) \cap \{Z \in N(X) \mid \|Z\| = 1\}$.

This cone $NE(X)$, which is interesting from various viewpoints, is rational polyhedral if $c_1(X)$, the first Chern class of X, is ample.

<u>Theorem</u> 1. If $c_1(X)$ is ample, then X contains finitely many rational curves $\ell_1, \ell_2, \ldots, \ell_r$ such that $(\ell_i . c_1(X)) \leq n+1$ for all i,

a) $AE(X) = \mathbb{Q}_+[\ell_1] + \ldots + \mathbb{Q}_+[\ell_r]$ if $p > 0$, and

b) $NE(X) = \mathbb{R}_+[\ell_1] + \ldots + \mathbb{R}_+[\ell_r]$ if $p \geq 0$, where

$[Z]$ denotes the class of 1-cycle Z.

To be explicit, a rational curve means an irreducible reduced curve defined over k whose normalization is $\mathbb{P}^1_k$. This theorem enables us to improve our theorem 3 [5].

<u>Corollary</u> 2. If $c_1(X)$ is ample, then

a) a divisor D on X is ample if D is numerically positive,

b) $\rho(X) = 1$ if every numerically effective divisor is either numerically trivial or ample, and

c) $\rho(X) = 1$ if every non-zero effective divisor is ample and if $p = 0$, where a divisor D is said numerically positive (resp. numerically effective, numerically trivial) if $(D.Z) > 0$ (resp. $(D.Z) \geq 0$, $(D.Z) = 0$) for all irreducible curves Z.

Indeed (a) follows from $NE(X) = \overline{NE}(X)$ by virtue of Kleiman's criterion. If $\rho(X) > 1$, then we can take a divisor D such that $D > 0$ on the interior ($\neq \phi$) of $NE(X)$ and $D = 0$ on $\mathbb{R}_+[\ell_i]$ for some i as a real valued linear function on $N(X)$, which implies that D is numerically effective, and not numerically trivial, or ample. This shows (b), and (c) follows from (b) by Lemma 2, (2) [5].

To study $\overline{NE}(X)$ for general X, we need more definitions. for an arbitrary positive real number ε, let

$A_\varepsilon(X, H) = \{Z \in A(X) \mid (Z.c_1(X)) \leq \varepsilon(Z.H)\}$,

$N_\varepsilon(X, H) = \{Z \in N(X) \mid (Z.c_1(X)) \leq \varepsilon(Z.H)\}$,

$AE_\varepsilon(X, H) = AE(X) \cap A_\varepsilon(X, H)$, and $NE_\varepsilon(X, H) = NE(X) \cap N_\varepsilon(X, H)$. If there is no danger of confusion, $A_\varepsilon(X, H)$, $N_\varepsilon(X, H)$, $AE_\varepsilon(X, H)$, $NE_\varepsilon(X, H)$ will be abbreviated to $A_\varepsilon(X)$, $N_\varepsilon(X)$, $AE_\varepsilon(X)$, $NE_\varepsilon(X)$, respectively.

<u>Theorem</u> 3. For an arbitrary positive ε, there exist a finite number r (≥ 0) of rational curves $\ell_1, \ldots, \ell_r$ in X such that $(\ell_i.c_1(X)) \leq n+1$ for all i,

a) $AE(X) = \mathbb{Q}_+[\ell_1] + \ldots + \mathbb{Q}_+[\ell_r] + AE_\varepsilon(X)$ if $p > 0$, and

b) $\overline{NE}(X) = \mathbb{R}_+[\ell_1] + \ldots + \mathbb{R}_+[\ell_r] + \overline{NE}_\varepsilon(X)$ if $p \geq 0$,

where $\overline{NE}_\varepsilon(X) = \overline{NE}(X) \cap N_\varepsilon(X)$.

Now Theorem 1 follows from Theorem 3. Indeed, if $c_1(X)$ is ample, then $AE_\varepsilon(X) = \overline{NE}_\varepsilon(X) = 0$, when $1/\varepsilon$ is a sufficiently large integer such that $(1/\varepsilon)c_1(X) - H$ is ample. Theorem 3 will be proved in the next section.

§2. Proof of Theorem 3.

We will begin by reformulating Thoerems 4 and 5 in [3].

Theorem 4. For a non-singular projective curve C of genus g over k and a morphism $f : C \longrightarrow X$, there exist a morphism $h : C \longrightarrow X$ and an effective 1-cycle Z with the properties; (a) $(h_*(C).c_1(X)) \leq ng$, (b) an arbitrary irreducible component Z' of Z is a rational curve such that $(Z'.c_1(X)) \leq n+1$, and (c) $f_*(C) \approx h_*(C) + Z$.

Proof. In the statement, f_* is the cycle-theoretic direct image; $f_*(C) = 0$ if $\dim f(C) = 0$, $[C : f(C)]\, f(C)$ if $\dim f(C) = 1$. We will treat two cases. First we assume $g = 0$. We use induction on $(f_*(C).H)$. If $(f_*(C).c_1(X)) \leq n+1$, then we can set h to be any constant map and $Z = f_*(C)$. If $(f_*(C).c_1(X)) > n+1$, Theorem 4 [3] implies that $f_*(C) \approx Z_1 + Z_2$, where Z_1 and Z_2 are non-zero effective 1-cycles whose components are rational curves. Since $(f_*(C).H) = (Z_1.H) + (Z_2.H)$, we can apply the induction hypothesis to Z_1 and Z_2, and the case $g = 0$ is done. We prove the case $g > 0$ again by induction on $(f_*(C).H)$. If $(f_*(C).c_1(X)) \leq ng$, we can set $h = f$ and $Z = 0$. If $(f_*(C).c_1(X)) > ng$, it follows from the proof of Theorem 5 [3] that $f_*(C) \approx f'_*(C) + U$, where $f' : C \longrightarrow X$ and U is a non-zero effective 1-cycle whose components are rational curves. Since $U \neq 0$, $(f'_*(C).H) < (f_*(C).H)$. Now we have only to apply the induction hypothesis to f' and the result on the case $g = 0$ to each component of U. q.e.d.

Now Theorem 3, (a) is an easy corollary to Theorem 4.

Proof of Theorem 3, (a). Let us consider the set Φ of all the rational curves ℓ in X such that $(\ell.c_1(X)) \leq n+1$ and $[\ell] \notin AE_\varepsilon(X)$. These curves ℓ form a bounded family, i.e. parametrized by a quasi-projective scheme [1, n°221, 4], because $(\ell.H) < (\ell.c_1(X))/\varepsilon < (n+1)/\varepsilon$. Hence there exist finitely many rational curves $\ell_1, \ldots, \ell_r$ which form a complete set of representatives of $\Phi/\approx$. We will show that the convex cone $V = \mathbb{Q}_+[\ell_1] + \ldots + \mathbb{Q}_+[\ell_r] + AE_\varepsilon(X)$ is equal to $AE(X)$. We treat two cases. Let ℓ be a rational curve in X. By Theorem 4, $\ell \approx Z$ for some effective 1-cycle Z whose components Z' are rational curves such that $(Z'.c_1(X)) \leq n+1$. Thus for each component Z' of Z, we have either $Z' \in \Phi$ or $Z' \in AE_\varepsilon(X)$. Hence $[\ell] \in V$, and the rational curve case is done. Let C be a non-singular projective curve of genus $g > 0$ and $f : C \longrightarrow X$ a morphism. Let C_i be the p^{-i}-th power of C and $\pi_i : C_i \longrightarrow C_{i-1}$ the p-th power morphism. We then inductively find morphisms $f_i : C_i \longrightarrow X$ and its image $D_i = f_{i*}(C_i)$ for $i \geq 0$ so that $f_0 = f$, $(D_{i+1}.c_1(X)) \leq ng$, and $p[D_i] - [D_{i+1}] \in V$ *for all* $i \geq 0$. *Indeed, if we apply*

Theorem 4 to $f_i \circ \pi_{i+1} : C_{i+1} \longrightarrow X$, then we get $h = f_{i+1}$ and $h_*(C_{i+1}) = D_{i+1}$ such that $p[D_i] - [D_{i+1}]$ is equivalent to a sum of rational curves which belong to V as we have seen before. Now if $[D_a] \in V$ for some a, then $[D_0] \in V$ because

$$D_0 = \sum_{j=0}^{a-1} p^{-j-1}(pD_j - D_{j+1}) + p^{-a} D_a.$$

If $[D_i] \notin AE_\varepsilon(X)$ for all i, then $(D_i.H) \leq (D_i.c_1(X))/\varepsilon \leq ng/\varepsilon$ for all i. Since $(D_i.H)$ is uniformly bounded, there are numbers a and b such that $D_a \approx D_b$ and $a < b$ [1, n°221]. Then

$$(p^{b-a} - 1)D_a \approx p^{b-a}D_a - D_b = \sum_{i=a}^{b-1} p^{b+1-i}(pD_i - D_{i+1})$$

implies that $[D_a] \in V$, from which follows $[D_0] \in V$. q.e.d.

To prove a result in characteristic 0, we prove a variant of Theorem 3, (b) which is actually equivalent to Theorem 3, (b).

Lemma 5. Let Z be an effective 1-cycle on X such that $(Z.c_1(X)) > 0$, and M an arbitrary ample divisor on X. Then there exists a rational curve Z' such that

$$\frac{n+1}{(M.Z')} \geq \frac{(c_1(X).Z')}{(M.Z')} \geq \frac{(c_1(X).Z)}{(M.Z)} .$$

Proof. If we can prove the lemma in characteristic $p > 0$, we can prove the lemma in characteristic 0 by using the arguments on schemes over Spec $\mathbb{Z}$ because the inequality in the theorem gives an upper bound of $(M.Z')$; $(M.Z') \leq (n+1)(M.Z)/(c_1(X).Z)$ which is independent of p (see the proof of Theorem 6 in [3].) Hence assuming that $p > 0$, we can apply Theorem 3, (a). We choose ε so that $1/\varepsilon$ is a natural number and

$(1/\varepsilon)M - 2(M.Z)H$ is ample. Then there exist non-negative rational numbers $a_1, \ldots, a_r$ and $Y \in NE_\varepsilon(X)$ such that $[Z] = \Sigma a_i[\ell_i] + Y$. Since $Y \in NE_\varepsilon(X)$ and $(M.Y) \geq 2\varepsilon(M.Z)(H.Y)$, we see $(c_1(X).Y) \leq \varepsilon(H.Y) \leq (M.Y)/2(M.Z)$. Thus

$$\frac{(c_1(X).Z)}{(M.Z)} \leq \frac{\Sigma a_i(c_1(X).\ell_i) + (M.Y)/2(M.Z)}{\Sigma a_i(M.\ell_i) + (M.Y)}$$

and since $a_i \geq 0$ and $(M.Y) \geq 0$, we have

$$\frac{(c_1(X).Z)}{(M.Z)} \leq \mathrm{Max}\,\{\mathrm{Max}_i \frac{(c_1(X).\ell_i)}{(M.\ell_i)}, \frac{1}{2(M.Z)}\}.$$

Since $(c_1(X).Z) \geq 1$, we can take $Z' = \ell_i$ for some i.

q.e.d.

Let us prove Theorem 3, (b). As in the proof of Theorem 3, (a), the set Φ of rational curves ℓ in X such that $(\ell.c_1(X)) \leq n+1$ and $[\ell] \not\in NE\,(X)$ is bounded, and $\Phi/\approx$ has a complete set of representatives $\ell_1, \ldots, \ell_r$. we claim

<u>Lemma</u> 6. The cone $V = \mathbb{R}_+[\ell_1] + \ldots + \mathbb{R}_+[\ell_r] + \overline{NE}_\varepsilon(X)$ is closed in $N(X) \simeq \mathbb{R}^{\rho(X)}$.

<u>Proof</u>. Let $Z \in N(X)$ be a limit of $Z(i) = a(i, 1)\ell_1 + \ldots + a(i, r)\ell_r + Y(i)$ $(i \geq 1)$, where $a(i, j) \in \mathbb{R}_+$ and $Y(i) \in \overline{NE}\,(X)$. Then the sequence $(Z(i).H)$ is bounded because $(Z(i).H) \longrightarrow (Z.H)$ as $i \longrightarrow \infty$. Since $a(i, j) \leq (Z(i).H)/(\ell_j.H)$ and $(Y(i).H) \leq (Z(i).H)$, the numbers $a(i, j)$, $(Y(i).H)$, and hence $\|Y(i)\|$ have a uniform upper bound by Kleiman's criterion. Thus there exists a subsequence $Z(n_i)$ such that $a(n_i, j)$ and $Y(n_i)$ converge

as $i \longrightarrow \infty$, whence $Z \in V$. q.e.d.

Going back to the proof of Theorem 3, (b), we will assume that $V \neq \overline{NE}(X)$ and show that this leads to a contradiction. By the ampleness of H, $\overline{NE}(X) \cap \{Y \in N(X) \mid (Y.H) = 1\}$ is compact. Hence by the separation theorem for convex sets, there is an element $M \in NS(X) \otimes_{\mathbb{Z}} \mathbb{R}$ with the properties, (a) $M \geq 0$ on $\overline{NE}(X)$ and $M(Z) = 0$ for some non-zero Z in $\overline{NE}(X)$, and (b) $M > 0$ on $V - \{0\}$ considered as a real valued function on $N(X)$. By the above compactness, there exist a sequence $\{M_j\}_{j \geq 0}$ of ample divisors and a sequence $\{m_j\}_{j \geq 0}$ of natural numbers such that M is the limit of M_j/m_j in $NS(X) \otimes_{\mathbb{Z}} \mathbb{R}$ as $j \longrightarrow \infty$. Let Z (given in the condition (a)) be the limit of $[Z_j]/n_j$, where Z_j is an effective 1-cycle and n_j a natural number. Since $V_1 = V \cap \{Y \in N(X) \mid (Y.H) = 1\}$ is compact, $(c_1(X).Y)/(M_j/m_j.Y)$ converge uniformly to $(c_1(X).Y)/(M.Y)$ when $j \longrightarrow \infty$ as functions on V_1. Hence $(c_1(X).Y)/(M_j/m_j.Y)$ $(j \geq 0,\ Y \in V - \{0\})$ are uniformly bounded. We have $(c_1(X).Z_j)/(M_j/m_j.Z_j) \longrightarrow +\infty$ as $j \longrightarrow \infty$ because $(M.Z) = 0$ and $(c_1(X).Z) > 0$ $(Z \notin V.)$ Hence for a sufficiently large j, we have

$$\frac{(c_1(X).Z_j)}{(M_j.Z_j)} > \frac{(c_1(X).Y)}{(M_j.Y)} \quad \text{for all } Y \in V - \{0\}.$$

If we apply Lemma 5 to these Z_j and M_j (note that $(c_j(X).Z_j) > 0$,) we get a rational curve ℓ such that

$$\frac{n+1}{(M_j.\ell)} \geq \frac{(c_1(X).\ell)}{(M_j.\ell)} \geq \frac{(c_1(X).Z_j)}{(M_j.Z_j)} .$$

This inequality (together with the above) means that $\ell \notin V$

and $(c_1(X).\ell) \leq n+1$. Since $V \supseteq \overline{NE}_\varepsilon(X)$, we have $\ell \notin \overline{NE}_\varepsilon(X)$ and $\ell \in \Phi$. This implies $[\ell] \in V$, which is a contradiction. Thus Theorem 3, (b) is proved.

§3. Concluding remarks.

A half line $R = \mathbb{R}_+[Z]$ in $N(X)$ is called an <u>extremal ray</u> if (1) $(Z.c_1(X)) > 0$, and (2) Z_1 and Z_2 in $\overline{NE}(X)$ belong to R if $Z_1 + Z_2 \in R$. A rational curve ℓ in X is an <u>extremal rational curve</u> if $(\ell.c_1(X)) \leq n+1$ and $\mathbb{R}_+[\ell]$ is an extremal ray.

It is not hard to restate Theorem 3 without using H and ε (cf. [4].) Here we simply state an immediate corollary.

<u>Corollary</u> 7. X has an extremal rational curve if and only if K_X is not numerically effective.

Only if part is obvious. If K_X is not numerically effective, then $\overline{NE}(X) \neq \overline{NE}_\varepsilon(X, H)$. Then at least one of ℓ_i's in Theorem 3 is an extremal rational curve.

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