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DUALITY OF CUSP SINGULARITIRS

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INTRODUCTION

Arnold introduced the notion of modality of an isolated singularity (roughly the number of moduli) and classified isolated singularities of small modality. Zero-modal hypersurface isolated singularities are Kleinian singularities A_n, D_n, E_6, E_7 and E_8. One-modal (unimodular) hypersurface isolated singularities are simple elliptic singularities $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$, 14 exceptional singularities and cusp singularities $T_{p,q,r}$ with $(1/p)+(1/q)+(1/r)<1$. Moreover he reported that there is a strange duality of the 14 exceptional singularities, which was made clearer later by Pinkham [3]. The purpose of this note is to show that there are similar phenomena for the remaining unimodular singularities. See [5],[6],[7].

§1 THE STRANGE DUALITY OF ARNOLD

We consider the following germs S and S' of isolated singularities at the origins;

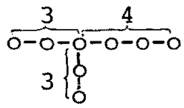
$$S : x^2z + y^3 + z^4 = 0 \quad , \quad S' : x^3 + y^8 + z^2 = 0.$$

S and S' are among the 14 exceptional unimodular singularities. Let $f = x^2z + y^3 + z^4$, $g = x^3 + y^8 + z^2$.

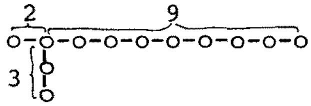
Let $S_t = f^{-1}(t)$, $S'_t = g^{-1}(t)$ $(t \neq 0)$. Then $b_2(S_t) = 10$, $b_2(S'_t) = 14$ and there are bases $e_1, \cdots, e_{10}$ and $f_1, \cdots, f_{14}$ of $H_2(S_t, \mathbb{Z})$ and $H_2(S'_t, \mathbb{Z})$ such that their intersection diagrams are $T_{3,3,4} + H$, $T_{2,3,9} + H$ where

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

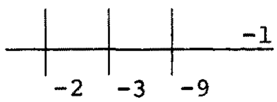
$T_{3,3,4}$:

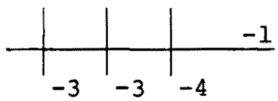


$T_{2,3,9}$:



We call therefore (3,3,4) and (2,3,9) the Gabrielov numbers of S and S' and write Gab(S) = (3,3,4) etc. On the other hand we have resolutions of S and S' with exceptional sets consisting of 4 nonsingular rational curves as below;





where each line denotes a nonsingular rational curve, a negative integer beside it denotes the self intersection number of the curve. We call therefore (2,3,9) and (3, 3,4) the Dolgatchev numbers of S and S' respectively and we write Dolg(S) = (2,3,9) etc. So we have

$$\mathrm{Gab}(S) = \mathrm{Dolg}(S'),\ \mathrm{Dolg}(S) = \mathrm{Gab}(S').$$

For a Dolgatchev triple (p,q,r) of an exceptional singularity U we define $\Delta(U) = pqr-pq-qr-rp$. Then we have

$$\Delta(S) = \Delta(S').$$

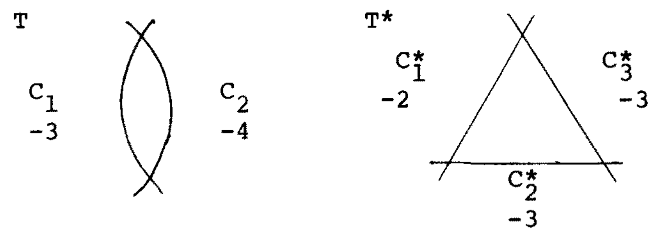
This is part of the strange duality of Arnold.

§2 $T_{3,4,4}$ AND $T_{2,5,6}$

We denote by $T_{p,q,r}$ a germ of an isolated singularity

$$x^p + y^q + z^r - xyz = 0$$

at the origin. Here $1/p + 1/q + 1/r < 1$. We define $\deg(T_{p,q,r}) = p+q+r$, $\operatorname{index}(T_{p,q,r}) = (p-1,q-1,r-1)$, $\Delta(T_{p,q,r}) = pqr-pq-qr-rp$. Let $T = T_{3,4,4}$, $T^* = T_{2,5,6}$. First we resolve the singularities. Their exceptional sets in their minimal resolutions are cycles $C = C_1+C_2$, $C^* = C_1^*+C_2^*+C_3^*$ of nonsingular rational curves with self-intersection numbers described below,



By blowing up the former once we obtain a cycle $C' = C_1'+C_2'+C_3'$ of nonsingular rational curves with $C_1'^2 = -1$, $C_2'^2 = -4$, $C_3'^2 = -5$ where C_2' and C_3' are proper transforms of C_1 and C_2. Now we define cycle$(T) = (1,4,5)$ and cycle$(T^*) = (2,3,3)$. Then the first duality of T and T^* is

$$\mathrm{index}(T) = \mathrm{cycle}(T^*),\ \mathrm{cycle}(T) = \mathrm{index}(T^*).$$

The second is

$$\deg(T) + \deg(T^*) = 24$$

although it is still unclear why this is part of the duality. The third is

$$\Delta(T) = \Delta(T^*).$$

The intersection matrices of C and C* are

$$(C_iC_j) = \begin{pmatrix} -3 & 2 \\ 2 & -4 \end{pmatrix},\quad (C_i^*C_j^*) = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -3 & 1 \\ 1 & 1 & -3 \end{pmatrix}$$

whose determinants are equal to $\Delta(T)$ or $\Delta(T^*)$ up to sign. Next we consider continued fraction expansions. Let $\omega = [[\overline{3,4}]]$. By definition

$$\omega = 3 - \cfrac{1}{4 - \cfrac{1}{3 - \cfrac{1}{4 - \ddots}}} = 3 - \cfrac{1}{4 - \frac{1}{\omega}} = (3+\sqrt{6})/2.$$

Then $1/\omega = [[1,2,\overline{3,2,3}]]$. Since $(2,3,3)$ and $(3,2,3)$ are identified by the cyclic permutation of the irreducible components C_j^*, we may identify $(2,3,3)$ and $(3,2,3)$. Conversely if we start with $\omega^* = [[\overline{3,2,3}]]$ for instance, then we obtain $1/\omega^* = [[1,2,\overline{4,3}]]$. This is the fourth duality of T and T*. Finally we reconsider the exceptional sets in the minimal resolutions. The cycles C and C* are so-called fundamental divisors of the singularities T and T*. So we define $\mathrm{Deg}(T) = -C^2$, $\mathrm{Deg}(T^*) = -(C^*)^2$. Then $\mathrm{Deg}(T) = 3$ and $\mathrm{Deg}(T^*) = 2$. The fifth duality is

$\mathrm{Deg}(T)$ = the number of irreducible components of C^*,

$\mathrm{Deg}(T^*)$ = the number of irreducible components of C.

The duality shown above looks like the strange duality of Arnold very much. In fact (3,4,4) and (2,5,6) are Gabrielov and Dolgatchev numbers of one of the 14 exceptional singularities. By interpreting the above duality suitably we can see a similar kind of duality for $T_{2,3,6}, T_{2,4,4}, T_{3,3,3}$ and $\Pi_{2,2,2,2}$ (in other words $\tilde{E}_8$, $\tilde{E}_7$, $\tilde{E}_6$, $\tilde{D}_5$).

§3 DUALITY THEOREM

Let $\Pi_{p,q,r,s}$ be a germ of an isolated singularity

$$x^p + w^r = yz, \quad y^q + z^s = xw$$

at the origin where p,q,r,s are integers $\geqq 2$, at least one $\geqq 3$. Let $T = \Pi_{p,q,r,s}$. We define $\deg(T) = p+q+r+s$, $\mathrm{index}(T) = (p,q,r,s)$, $\Delta(T) = pqrs - (p+r)(q+s)$. Let C be the exceptional set (the fundamental divisor) of T in the minimal resolution of T. C is a cycle of rational curves. We define $\mathrm{Deg}(T) = -C^2$, length(T) = the number of irreducible components of C. We define $\mathrm{length}(T_{p,q,r})$ in the same way.

THEOREM 1. Let S be the set of all $T_{p,q,r}$ and $\Pi_{p,q,r,s}$ with length less than 5. Then there is a bijection i of S onto itself such that for any T of S

0) $i(i(T)) = T$,

1) $\mathrm{index}(T) = \mathrm{cycle}(i(T))$, $\mathrm{cycle}(T) = \mathrm{index}(i(T))$,

2) $\deg(T) + \deg(i(T)) = 24$,

3) $\Delta(T) = \Delta(i(T))$,

4) an assertion about continued fraction expansions,

5) $\mathrm{Deg}(T) = \mathrm{length}(i(T))$, $\mathrm{length}(T) = \mathrm{Deg}(i(T))$.

By suitable extensions of the above definitions we obtain Duality Theorem of cusp singularities in the general case. We notice that $\#(S) = 38$ and $i(T_{p,q,r}) = T_{s,t,u}$ iff (p,q,r) and (s,t,u) are Gabrielov and Dolgatchev numbers of one of the exceptional singularities.

§4 INOUE-HIRZEBRUCH SURFACES

Let K be a real quadratic field with $(\ )'$ the conjugation, M a complete module in K, i.e. a free module in K of rank two. Let $U^+(M) = \{\alpha \in K;\ \alpha M = M,\ \alpha > 0,\ \alpha' > 0\}$, V be a subgroup of $U^+(M)$ of finite index. It is known that $U^+(M)$ is infinite cyclic. Let H be the upper half plane $\{z \in \mathbb{C};\ \mathrm{Im}(z) > 0\}$. Define the actions of M and $U^+(M)$ on $\mathbb{C} \times H$ by

$$m : (z_1, z_2) \to (z_1 + m, z_2 + m')$$
$$\alpha : (z_1, z_2) \to (\alpha z_1, \alpha' z_2) \ .$$

Let $G(M,V)$ be the group generated by the actions of M and V on $\mathbb{C} \times H$ as above. The action of $G(M,V)$ on $\mathbb{C} \times H$ is free and properly discontinuous so that we have a quotient complex space $X'(M,V) := \mathbb{C}\times H/G(M,V)$. By adding to $X'(M,V)$ an ideal point ∞ called a cusp and endowing the union of ∞ and $X'(M,V)$ with a suitable topology and a suitable structure as a ringed space, we obtain a normal complex space $X(M,V)$. Let ω be a real quadratic irrationality with $\omega > 1 > \omega' > 0$. Let $1/\omega = [[f_1,\cdots,f_h,\overline{e_1,\cdots,e_k}]]$, and set $\omega^* = [[\overline{e_1,\cdots,e_k}]]$.

LEMMA 1. There exists β in K such that $\beta\beta' = -1$, $\beta(\mathbb{Z} + \mathbb{Z}\omega) = \mathbb{Z} + \mathbb{Z}\omega^*$.

Let $M = \mathbb{Z} + \mathbb{Z}\omega$, $N = \mathbb{Z} + \mathbb{Z}\omega^*$. Then $U^+(M) = U^+(N)$. Let V be a subgroup of $U^+(M)$ of finite index. Let (z_1,z_2) and (w_1,w_2) be the coordinates of $X(M,V)$ and $X(N,V)$ with cusps deleted respectively. Then by identifying them by the relation $w_1 = \beta z_1$, $w_2 = \beta' z_2$, we can form a compact complex space $Y = Y(M,V)$ with cusp singularities.

THEOREM 2 (Inoue [2]). The minimal model $S(M,V)$ of $Y(M,V)$ has $b_1 = 1$, $b_2 > 0$ and no meromorphic functions except constants.

We call $S(M,V)$ an Inoue-Hirzebruch surface (associated with (M,V)) and $Y(M,V)$ a singular Inoue-Hirzebruch surface (with two cusps). Let p and q be the cusps of

$X(M,V)$ and $X(N,V)$ and we denote by the same p and q the cusps of $Y = Y(M,V)$.

We notice that any of $T_{p,q,r}$ and $\Pi_{p,q,r,s}$ is isomorphic to (Y,p) for some M and V. If $T(\in S)$ is isomorphic to the germ of Y at p (Y,p), then $i(T)$ is isomorphic to (Y,q). And then $\Delta(T) = \#$(the torsion part of $H_1(\mathbb{R}\times H/G(M,V),\mathbb{Z}))$ where $\mathbb{R}\times H/G(M,V)$ is a subset of $X(M,V)$ by the natural inclusion of $\mathbb{R}\times H$ into $\mathbb{C}\times H$. Since it is a subset of $X(N,V)$ too, this explains THEOREM 1 3). The relation between M and N is well described by the following

LEMMA 2 (Kenji Ueno) There exists a totally positive γ such that $N = \gamma(M^*)'$ where $M^* = \{x\in K;\ tr(xy) \in \mathbb{Z}$ for any y in $M\}$, $(M^*)' = \{x';\ x \in M^*\}$. In particular $X(N,V)$ is isomorphic to $X((M^*)',V)$.

THEOREM 3. Assume that (Y,p) and (Y,q) belong to S. Then $Def(Y)$ (:= the deformation functor of Y) is non-obstructed and $Def(Y) = Def(Y,p)\times Def(Y,q)$, Y is smoothable by flat deformation. Any smooth deformation of Y is a minimal K3 surface.

THEOREM 4. Assume that (Y,p) and (Y,q) belong to S. Let Z be Y with q resolved (i.e. with q replaced by a cycle C^* of rational curves). Then Z is smoothable by flat deformation with C^* preserved. Any smooth deformation Z_t of Z with C^* preserved is the projective plane $\mathbb{P}^2$ blown up along finitely many points lying on a rational cubic curve with a node and K_{Z_t} (:= the canonical line bundle of Z_t) = $-C^*$. Moreover $H(Y,p)$:= $\{a \in H_2(Z_t, \mathbb{Z}); aC_j^* = 0$ for any irreducible component C_j^* of $C^*\}$ has a $\mathbb{Z}$-base in $R(Y,p) := \{a \in H(Y,p): a^2 = -2\}$ whose intersection diagram (Dynkin diagram) is $T_{p,q,r}$ or $\Pi_{p,q,r,s}$ corresponding to the type of the singularity (Y,p).

The above two theorems were studied by J. Wahl and E. Looijenga too, but in more detail.

By <u>an elliptic deformation</u> $\mathcal{Z}_t$ (or $\mathcal{U}_t$) of Z (or (Y,p)) we mean a fibre of $\pi : \mathcal{Z} \to D$ (or $f: \mathcal{U} \to D$) such that $\mathcal{Z}_0 = Z$ (or $\mathcal{U}_0 = (Y,p)$) and $h^1(\tilde{\mathcal{Z}}_t, \mathcal{O}_{\tilde{\mathcal{Z}}_t}) = 1$ (or $h^1(\tilde{\mathcal{U}}_t, \mathcal{O}_{\tilde{\mathcal{U}}_t}) = 1$) where $\tilde{\mathcal{Z}}_t$ (or $\tilde{\mathcal{U}}_t$) is the nonsingular model of $\mathcal{Z}_t$ (or $\mathcal{U}_t$).

<u>THEOREM 5</u> There exists a proper flat family $f : \mathfrak{X} \to B$ such that $\mathfrak{X}_0 = Z$ and f is versal for both elliptic deformations of Z and elliptic deformations of (Y,p). Nonsingular models of $\mathfrak{X}_t$ are surfaces with $b_1 = 1$ and global spherical shells.

For simplicity we assume $\mathrm{Deg}(Y,p) \geq 5$.

<u>THEOREM 5</u> (CONTINUED) Define the "Dynkin diagram"

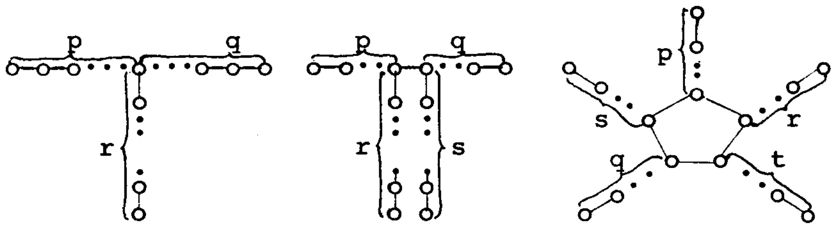
of Z or (Y,p) as follows,

$$T_{p,q,r} \qquad \text{if index}(Y,p) = (p-1,q-1,r-1)$$

$$\Pi_{p,q,r,s} \qquad \text{if index}(Y,p) = (p,q,r,s)$$

$$W_{p,q,r,s,t} \qquad \text{if index}(Y,p) = (p,q,r,s,t)$$

where index(Y,p) := cycle(Y,q) which is the sequence of (-1) times selfintersection numbers of the exceptional rational curves. (See [6].) Then the singularities of elliptic deformations $\mathcal{X}_t$ of Z are in one to one correspondence with proper subdiagrams containing one of $T_{2,3,6}, T_{2,4,4}, T_{3,3,3}, \Pi_{2,2,2,2,2}$ and $W_{1,1,1,1,1}$ (in other words $\tilde{E}_8, \tilde{E}_7, \tilde{E}_6, \tilde{D}_5, \tilde{A}_4$). In particular the singularities of $\mathcal{X}_t$ are simple elliptic singularities, cusp singularities or rational double singularities A_k.



$T_{p,q,r}$ $\qquad$ $\Pi_{p,q,r,s}$ $\qquad$ $W_{p,q,r,s,t}$

By THEOREM 4 there exists a proper flat family $f : \mathcal{Y} \to D$ such that $\mathcal{Y}_0 = Z$ (= a singular Inoue-Hirzebruch surface with one cusp) and $\mathcal{Y}_t$ ($t \neq 0$) is a rational surface. We notice that Z is by no means an algebraic surface. It is also remarkable to notice

THEOREM 6 (T. Oda [8]) There exists a proper flat family $f : \mathfrak{X}^* \to D$ such that $\mathfrak{X}^*_0$ = a rational surface with a double curve and $\mathfrak{X}^*_t$ $(t \neq 0)$ is a nonsingular Inoue-Hirzebruch surface.

§5 COHN'S SUPPORT POLYGONS

Let M be a complete module in a real quadratic field K. We embed M into $\mathbb{R}^2$ by the mapping $x \to (x, x')$. By this mapping we identify M as a subset of $\mathbb{R}^2$. We define $M^+ := \{x \in M;\ x > 0,\ x' > 0\}$, $M^- := \{x \in M;\ x > 0,\ x' > 0\}$ which we view as subsets of $\mathbb{R}^2$. We let $\Sigma^+(M)$ and $\Sigma^-(M)$ be the convex hulls of M^+ and M^- respectively. Then $\Sigma^{\pm}(M)$ is a convex set bounded by infinitely many line segments connecting two points of $M^{\pm}$. Let $\partial\Sigma^{\pm}(M)$ be the boundary of $\Sigma^{\pm}(M)$. We number $\partial\Sigma^{\pm}(M) \cap M$ consecutively. If $M = \mathbb{Z} + \mathbb{Z}\omega$ and ω is a totally positive quadratic irrationality with $\omega > 1 > \omega' > 0$ (i.e. reduced), then we may assume $\partial\Sigma^+(M) \cap M = \{n_j;\ j \in \mathbb{Z}\}$, $\partial\Sigma^-(M) \cap M = \{n^*_j;\ j \in \mathbb{Z}\}$, $n_0 = 1$, $n_1 = \omega$, $n^*_0 = (\omega - 1)/\omega^*$, $n^*_{-1} = \omega - 1$. $U^+(M)$ acts on $M^{\pm}$ therefore on $\partial\Sigma^{\pm}(M) \cap M$. $\#(\partial\Sigma^{\pm}(M) \cap M \bmod U^+(M))$ is finite. There exist positive integers a_j and a^*_j (≥ 2) such that

$$n_{j-1} + n_{j+1} = a_j n_j,\quad n^*_{j-1} + n^*_{j+1} = a^*_j n^*_j \quad (j \in \mathbb{Z})$$

Let $\mathrm{Dec}^+ = \{\{0\},\ \mathbb{R}_+ n_j,\ \mathbb{R}_+ n_{j-1} + \mathbb{R}_+ n_j \quad (j \in \mathbb{Z})\}$

$\mathrm{Dec}^- = \{\{0\},\ \mathbb{R}_+ n^*_j,\ \mathbb{R}_+ n^*_{j-1} + \mathbb{R}_+ n^*_j \quad (j \in \mathbb{Z})\}$.

Then evidently Dec^+ and Dec^- are cone decompositions of $\mathbb{R}_+ \times \mathbb{R}_+$ and $\mathbb{R}_+ \times \mathbb{R}_-$ respectively. By the general theory of torus embeddings we can construct complex algebraic varieties locally of finite type $\mathrm{Temb}(\mathrm{Dec}^+)$ and $\mathrm{Temb}(\mathrm{Dec}^-)$. The groups $U^+(M)$ and V act upon both of them freely and properly discontinuously. The quotient surfaces $\mathrm{Temb}(\mathrm{Dec}^{\pm})/V$ are naturally minimal resolutions of (Y,p) and (Y,q) ([8]) where $Y = Y(M,V)$. By THEOREM 1 (or by definition in the general case) $\mathrm{index}(Y,p) = (a_j^* ; j=1, \cdots, s)$ (= the representatives of a_j^* mod V) and $\mathrm{index}(Y,q) = (a_j : j=1,\cdots,t)$ (= the representatives of a_j mod V) if $s \geqq 3$ or $t \geqq 3$ respectively.

§ 6 FOURIER-JACOBI SERIES

Let $X'(M,V)$ be the natural image of $H \times H$ in $X(M,V)$, $X^0(M,V)$ the union of $X'(M,V)$ and the unique cusp of $X(M,V)$. Clearly $X^0(M,V)$ is an open neighborhood of the cusp ∞. For a totally positive m in M^* we can define a convergent power series $F_m(z_1,z_2)$ on $X^0(M,V)$ by

$$F_m(z_1,z_2) = \sum_{v \in V} \exp(2\pi i(vmz_1 + V'm'z_2)).$$

Let n_j^* $(j=1,\cdots,s)$ be the representatives of $\partial\Sigma^-(M) \cap M$ mod V. We notice that $m \equiv m^*$ mod V implies $F_m = F_{m^*}$. On the other hand THEOREM 1 says $s = \mathrm{Deg}((X(M,V),\infty))$. Let ω be a totally positive reduced quadratic irrationality (i.e. $\omega > 1 > \omega' > 0$), $M = \mathbb{Z} + \mathbb{Z}\omega$. We define a $\mathbb{Z}$ homomorphism f of K onto K by $f(x) = (x/(\omega-\omega'))'$. This f induces a bijection of M^- with $(M^*)^+$ where $M^* = M'/(\omega-\omega')$.

THEOREM 7-1 Assume $s \geq 3$. Then $(X(M,V),\infty)$ is embedded into $\mathbb{C}^s$ by $F_{f(n_j^*)}$ $(j=1,\cdots,s)$.

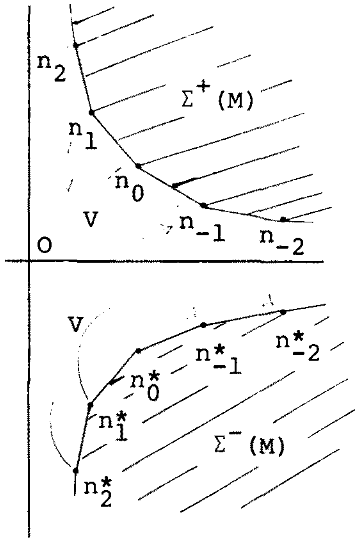
THEOREM 7-2 Assume $s = 2$. Then $(X(M,V),\infty)$ is embedded into $\mathbb{C}^3$ by $F_{f(n_j^*)}$ $(j=-1/2,0,1)$ where $n^*_{-1/2} = n^*_{-1} + n^*_0$.

THEOREM 7-3 Assume $s = 1$. Then $(X(M,V),\infty)$ is embedded into $\mathbb{C}^3$ by $F_{f(n_j^*)}$ $(j=-1/4,-1/2,-1)$ where $n^*_{-1/2} = n^*_{-1} + n^*_0$, $n^*_{-1/4} = n^*_{-1/2} + n^*_0$.

THEOREM 7 was proved also by Ueno.

The above choices of n_j^* in the cases s = 1 and 2 match the definitions of cycle(T) which seem to be rather artificial. Let us check this by the example in §2.

Let $\omega = [[\overline{3,4}]]$, $\omega^* = [[\overline{3,2,3}]]$, $M = \mathbb{Z} + \mathbb{Z}\omega$, $N = \mathbb{Z} + \mathbb{Z}\omega^*$, $V = U^+(M)$. Then $(X(M,V),\infty) \cong T_{3,4,4}$ and $(X(N,V),\infty) \cong T_{2,5,6}$. Temb($Dec^+$) and Temb($Dec^-$) are minimal resolutions of $(X(M,V),\infty)$ and $(X(N,V),\infty)$ respectively. Then the support polygon is as follows.



	representatives
$n_{-1} + n_1 = 3n_0$	n_0, n_1
$n_0 + n_2 = 4n_1$	
$n_1 + n_3 = 3n_2$	
....	
$n^*_{-1} + n^*_1 = 3n^*_0$	
$n^*_0 + n^*_2 = 2n^*_1$	n^*_0, n^*_1, n^*_2
$n^*_1 + n^*_3 = 3n^*_2$	
$n^*_2 + n^*_4 = 3n^*_3$	
....	

Let $n_{2k-(1/2)} = n_{2k-1} + n_{2k}$. Then we have

$$n_{-1} + n_0 = n_{-1/2},\ n_{-1/2} + n_1 = 4n_0,\ n_0 + n_{3/2} = 5n_1.$$

Recall cycle$(T_{3,4,4}) = (1,4,5)$ and this was defined by blowing up once. By the general theory of torus embeddings any equivariant blowing-up of Temb(Dec^+) corresponds to the subdivision of Dec^+. Let $f_j = F_{f(n^*_j)}$ $(j=0,1,2)$, $g_j = F_{((\omega^*-1)n_j/(\omega^*-\omega^{*\prime}))}$, $(j=-1/2,0,1)$.

Then we can show that

$$f_0^4+f_1^3+f_2^4-f_0f_1f_2 = \text{formal power series of } f_0,f_1,f_2$$

(terms of higher degree in some sense)

$$g_{-1/2}^2+g_0^5+g_1^6-g_{-1/2}g_0g_1 = \text{formal power series of } g_{-1/2},g_0,g_1$$

(terms of higher degree in some sense).

We notice that $(a_0^*, a_1^*, a_2^*) = (3,2,3)$ and $(a_0, a_1) = (3,4)$ so the triple defined newly is $(1,4,5)$. Similar facts are seen for all $T_{p,q,r}$ and $\Pi_{p,q,r,s}$. For the detail see [7].

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