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ALGEBRAIC VARIETIES OF GENERAL TYPE

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Introduction.

In order to have any kind of classification theory for algebraic varieties, we need to consider in detail the case of varieties of general type. This in turn leads us to investigating the pluricanonical systems on such varieties. For surfaces, we have a number of vital results concerning these linear systems; in particular, we know that a canonical model exists, and that the 5-canonical system defines a birational morphism onto its canonical model. It is this that enables us for instance to construct a coarse moduli space for surfaces of general type.

It is therefore natural to ask whether any of these results on pluricanonical systems generalize to higher dimensions. In general the answer would appear to be No, but it would also seem (especially in dimension 3) that there are redeeming features. The present paper therefore considers these questions for dimensions ≥ 3, pointing out the similarities and the differences compared with the case of surfaces.

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Part I : Preliminaries, and the case of surfaces.

Let V be a smooth, compact algebraic variety of dimension d defined over $k = \mathbb{C}$. If L is an invertible sheaf on V, then L corresponds to a divisor class Cl(D), and the complete linear system $|D|$ is given by the zeros of the sections $s \in H^0(V,L)$.

If $H^0(V,L) \neq 0$, choose a basis $s_0, \ldots, s_N$, and define a rational map $\phi_D = \phi_L : V \dashrightarrow \mathbb{P}^N$ by

$$x \longmapsto (s_0(x);\ldots;s_N(x)).$$

This is a morphism outside the base locus of $|D|$.

Definition 1.1. Let $\mathbb{N}(V,D)$ denote the set of positive integers n such that $H^0(V,nD) \neq 0$. We define the *D-dimension* $\kappa(V,D)$ *of V* by

$$\kappa(V,D) = \begin{cases} -\infty \text{ if } \mathbb{N}(V,D) = \emptyset \\ \max\limits_{n \in \mathbb{N}(V,D)} \{\phi_{nD}(V)\} \text{ otherwise.} \end{cases}$$

On V we have the canonical sheaf $\omega_V = \Omega_V^d$ of regular d-forms on V, with a corresponding divisor K_V. The *Kodaira dimension* $\kappa(V)$ *of V* is then defined to be $\kappa(V,K_V)$. We say that V is of *general type* if $\kappa(V) = d$; i.e. V is of general type if and only if ϕ_{mK_V} is generically finite for some positive m.

Example 1.2. In the case of curves, V is of general type if and only if the genus $g(V) \geq 2$. Note that ϕ_K itself is generically finite, and is an isomorphism if and only if V is not hyperelliptic. The map ϕ_{2K} is an isomorphism if and only if $g(V) > 2$, and ϕ_{3K} is always an isomorphism.

Now recall the following result of Iitaka (see [5] or [12]).

Theorem 1.3. Let V be a variety of dimension d and Kodaira dimension κ. Then there exists a fibre space $f : \tilde{V} \to W$ such that

$\tilde{V} \sim V$ (birationally equivalent),

$W \sim \phi_{nK}(V)$ for sufficiently large $n \in \mathbb{N}(V,K_V)$, and

$\kappa(\tilde{V}_w) = 0$ for the general fibre $\tilde{V}_w$ of $f : \tilde{V} \to W$.

(By definition, a fibre space is a proper surjective morphism of smooth varieties with connected fibres). □

Therefore, we see that in order to obtain a classification theory for algebraic varieties, we need in particular to study the cases when $\kappa(V) = d$, 0 and $-\infty$. In this paper we concentrate on the first case.

From (1.3), we see that if V is of general type (i.e. ϕ_{mK} is generically finite for some m), then ϕ_{nK} is birational for n sufficiently large. We can however be more precise than this.

Proposition 1.4. If ϕ_{mK} is generically finite and not a double cover of $\mathbb{P}^d$ (say $P_m(V) > d+1$), then $\phi_{(dm+1)K}$ is birational.

Proof. See [13] or [14]. □

Remarks 1.5.

(a) We need to exclude the case when ϕ_{mK} is a double cover of $\mathbb{P}^d$; consider for example the case of a curve of genus 2.

(b) There exist surfaces with ϕ_{2K} generically finite (and $P_2 > 3$), but ϕ_{4K} not birational ; e.g. any surface whose minimal model has $K^2 = 1$ and $p_g = 2$ (see [8], Proposition 4.7).

(c) There exist threefolds with ϕ_{2K} generically finite (and $P_2 > 4$), but ϕ_{6K} not birational (see [13]).

Let us now consider the case of surfaces of general type ; for such a surface S, recall the following facts :

Fact 1. S has an absolutely minimal model (unique up to isomorphism).

Fact 2. If S is minimal, then the complete linear system $|nK_S|$ is without fixed points for n sufficiently large. In fact, it can be shown that $|nK_S|$ is without fixed points for $n \geq 4$ (see [1]). Thus for any surface S of general type, ϕ_{nK} is a morphism for $n \geq 4$.

Fact 3. The canonical ring $R[S] = \bigoplus_{n\geq 0} H^0(S,nK_S)$ is a finitely generated ring over k. This follows from Fact 2, using a theorem of Zariski (see Mumford appendix to [16]).

Fact 4. Given that R[S] is finitely generated over k, we can construct a *canonical model* of S, $X = \underline{\mathrm{Proj}}_k R[S]$. We now put $R^{(m)}[S] = \bigoplus_{n\geq 0} H^0(S,nmK_S)$. Since R[S] is finitely generated, we deduce that $R^{(m)}[S] = \bigoplus_{n\geq 0} H^0(S,mK_S)^n$ for m sufficiently large, i.e. $\phi_{mK} : S \to W_m \simeq X$. It is shown in [1] that $W_m \simeq X$ for $m \geq 5$, and so in particular ϕ_{mK} is birational for $m \geq 5$.

On S, we have a formula for the plurigenera,

$$P_n(S) = \chi(\mathcal{O}_S) + 1/2(n-1)nK_{\bar{S}}^2 \quad \text{for } n > 1,$$

where $\bar{S}$ is the minimal model of S. The problem of classification up to birational equivalence then reduces down to classifying a specific type of surface of degree $25K_{\bar{S}}^2$ in $\mathbb{P}^N$, where $N = 10K_{\bar{S}}^2 + \chi(\mathcal{O}_S)-1$. This leads to the construction of a coarse moduli space for surfaces of general type (as described in [9] and [4]).

Part II : The case of threefolds, and higher dimensions

We now consider whether the facts stated for surfaces of general type in Part I of this paper can be generalized (in some form) to higher dimensions. As before, V is a d-dimensional variety of general type, where we are now interested in the case $d \geq 3$.

Question 1. In general, does there exist an absolutely minimal model of V ?

The answer to this question is No, even for threefolds of general type.

(*) Suppose that we have a diagram of birational morphisms between smooth threefolds.
$$\begin{tikzcd} & W \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ V_1 & & V_2 \end{tikzcd}$$
Suppose that on W there is a smooth surface S which has a minimal model $\mathbb{P}^1 \times \mathbb{P}^1$; we therefore have morphisms $\pi_i : S \to \mathbb{P}^1$ $(i = 1, 2)$ corresponding to the two projections from $\mathbb{P}^1 \times \mathbb{P}^1$. Finally we suppose that $f_i|_S = \pi_i$. Taking relatively minimal models of V_1 and V_2 will then in general give two non-isomorphic relatively minimal models of W.

The reader will find examples of the above type (*) constructed explicitly in [2] ; in these examples, the morphisms f_i are blow-ups of smooth rational curves on V_i , and so $S \simeq \mathbb{P}^1 \times \mathbb{P}^1$. The type of example described in (*) will cause problems in a number of contexts ; we shall for instance meet it again in Question 2. Compare also the following result of Frumkin [3].

<u>Proposition 2.1</u>. Given a birational morphism $f : X \to Y$ of smooth threefolds, there exists a threefold X' and a morphism $h : X' \to X$, where h factors into blow-ups of points or smooth *rational* curves, and $foh : X' \to Y$ factors into arbitrary blow-ups of smooth subvarieties. □

Question 2(a). Does there exist a model of V for which $|nK|$ is fixed point free for some n ?

The answer to this question is No ; there are examples of threefolds of general type where, for all smooth models V, the n-canonical system $|nK_V|$ has fixed components for all n (see [10] and [11]).

Question 2(b). For a d-fold V of general type, does there exist a smooth model for which the n-canonical system is base point free (i.e. defines a morphism) for all n sufficiently large ?

This question is very closely related to the fundamental Question 3, concerning the canonical ring $R[V] = \bigoplus_{n \geq 0} H^0(V, nK_V)$. A weaker form of Question 3, would be to ask whether R[V] is finitely generated under the assumption that the answer to Question 2(b) is Yes ; even this however seems to be a very difficult question.

The converse to this weaker form of Question 3 is easy. If R[V] is finitely generated, then the answer to Question 2(b) is Yes. To see this, simply take m such that $R^{(m)}[V]$ is generated by its degree 1 terms (i.e. by $H^0(V, mK_V)$), and resolve the base locus of $|nK_V|$ for $m \leq n < 2m$. It is then easy to verify that the resulting model has the required property.

Question 2(c). In the case of surfaces, $|nK_V|$ was base point free for n sufficiently large and for *any* model V. Does this generalize at all to higher dimensions ?

If V is a threefold of general type with R[V] finitely generated, then we do have a suitable generalization.

Proposition 2.2. If V is a threefold of general type with R[V] finitely generated, then $|nK_V|$ is base point free for n sufficiently large, apart from threefolds arising from examples of type (*). In particular, if B is a base curve of $|nK_V|$ for arbitrarily large n, then B must be rational.

In the second example constructed in [2], the mobile part of $|nK_{V_2}|$ on V_2 meets the rational curve $\pi_2(S)$ in a finite number of points for n sufficiently large. It then follows easily that on V_1, the rational curve $\pi_1(S)$ must be a base curve of $|nK_{V_1}|$ for all n sufficiently large. Thus the counterexamples in (2.2) do actually occur. The proof of (2.2) is postponed until later.

Question 3. If V is a d-dimensional variety of general type, is the canonical ring R[V] finitely generated ?

As things stand, there is not yet enough evidence to indicate whether we should try to prove that R[V] is finitely generated in general, or to find an example where it is not. I would like to quote here some results which will indicate some of the problems involved (see [15] for proofs and further details).

Proposition 2.3. Let D be a divisor on an algebraic variety W of dimension d, where $\kappa(W,D) = d$. Let $R[W,D] = \bigoplus_{n\geq 0} H^0(W,nD)$; then R[W,D] is finitely generated if and only if there exists $m\geq 1$ and a blow-up $f : \tilde{W} \to W$ with $\tilde{D} = f^*D$, such that the map $\phi_{m\tilde{D}}$ on $\tilde{W}$ is a birational morphism which contracts every fixed component of $|m\tilde{D}|$. □

If R[W,D] is finitely generated, it is in fact sufficient to take m such that $R^{(m)}[W,D] = \bigoplus_{n\geq 0} H^0(W,nmD)$ is generated by its degree 1 terms, and then resolve the base locus of $|mD|$ (see [15], and cf. [10]).

Corollary 2.4. The canonical ring R[V] of an algebraic variety of general type is finitely generated if and only if there exists $m\geq 1$ and a birationally equivalent model $\tilde{V}$, such that the map $\phi_{mK_{\tilde{V}}}$ on $\tilde{V}$ is a birational morphism which contracts every fixed component of $|mK_{\tilde{V}}|$. □

The above result leads us to ask about the irreducible divisors on V which are fixed components of $|nK_V|$ for all $n \in \mathbb{N}(V,K_V)$. It is easy to see that if E is such a component, then $p_g(E) = 0$. In the case when V is a threefold, we can say rather more (see [15]), although it is not yet known whether or not $\kappa(E) = -\infty$.

Let us now concentrate on the case of threefolds of general type ; if in general the answer to Question 3 is then No, any examples with R[V] not finitely generated will necessarily be far more complicated than the known examples where R[V,D] is not finitely generated for some divisor D (see [16] and [7]).

Proposition 2.5. Let V be a threefold of general type. For all n sufficiently large, suppose that $|nK_V|$ has no base locus, and the fixed components are smooth and disjoint. Then the canonical ring is finitely generated. □

The above result shows that in the case of the canonical ring (unlike the known examples where R[V,D] is not finitely generated), any counterexamples will depend crucially on the fixed components interacting with one another. Note that we may consider a base curve as a fixed component merely by blowing it up.

Question 4. Does there exist N (independent of V) such that ϕ_{NK_V} is birational ?

Recall, by (1.4), this is if and only if there is an n (independent of V) such that ϕ_{nK_V} is generically finite. For $d\leq 2$, the answer to Question 4 is of course Yes. Let us therefore consider the case when dim(V)=3. Furthermore, let us assume that R[V] is finitely generated.

Recall first the following result of Matsusaka [6].

Proposition 2.6. If X is a normal projective variety of dimension d, D an ample divisor on X with $h^0(nD) > d-1 + n^{d-1}D^d$, then ϕ_{nD} is generically finite. □

Remark 2.7. If we apply (2.6) and (1.4) to the case when V is a canonically polarized threefold, we can deduce immediately that ϕ_{25K} is birational (see [13]). Note that a threefold of general type is canonically polarized if and only if $K_V.C > 0$ for all curves C on V (see [14]).

Returning however to the case when V is an arbitrary threefold of general type with R[V] finitely generated, we shall see that the answer to Question 4 is 'almost certainly' No ; this is due to a specific reason shown by Reid in [10]. We shall now quickly summarize the relevant results from [10].

Given that R[V] is finitely generated, we have a canonical model $X = \underline{\mathrm{Proj}}_k R[V]$, which is easily seen to be normal. Choosing m such that $R^{(m)}[V]$ is generated by its degree 1 terms, we have $\phi_{mK} : V \dashrightarrow W_m \simeq X$. By resolving the base locus of $|mK_V|$, we may assume $\phi = \phi_{mK}$ is a morphism on V. It then follows without difficulty that ϕ_{mK} contracts down the fixed components of $|mK_V|$ (see(2.3)).

Now let ω_X denote the sheaf of differentials (i.e. d-forms) on X regular in codimension 1, and $\omega_X^{[n]}$ denote the sheaf of n-differentials regular in codimension 1. The above reasoning shows that there exists m such $\omega_X^{[m]}$ is invertible, and corresponds to a hyperplane section of $X \simeq \mathrm{Im}(\phi_{mK})$.

Let r be the minimum m such that $\omega_X^{[m]}$ is invertible ; it is an easy lemma that if $\omega_X^{[n]}$ is invertible for some n, then r divides n. We call r the *index* of V (a birational invariant).

Let rK_X be the divisor on X corresponding to $\omega_X^{[r]}$; we define $K_X^3 = (rK_X)^3/r^3 \in \mathbb{Q}$. By computing $P_{mr+1}(V) = h^0(V, K_V+m\phi^*(rK_X))$ by means of Riemann-Roch, and using the vanishing theorem of Grauert-Riemenschneider, we see that $P_n(V) \sim (1/6)n^3K_X^3 + O(n^2)$ for large n.

If therefore we can find V with K_X^3 arbitrarily small, then we see that the answer to Question 4 is No. It is shown in [10] that there exist V of arbitrarily high index, and that one can write down threefolds with for instance $K_X^3 = 1/10$. Thus, it seems almost certain that there exist threefolds with arbitrarily small K_X^3, in which case the answer to Question 4 would be No.

Let us digress here to give the promised proof of (2.2).

Proof of (2.2). Using Theorem 6.2 of [16], we need only consider the case of base *curves*. Suppose that there is a curve B on V such that we can find n arbitrarily large with B a base curve of $|nK_V|$. Then I claim that V arises from an example of type (*).

We choose m so that $R^{(m)}[V]$ is generated by its degree 1 terms *and* B is a base curve of $|mK_V|$. By blowing V up in points, we may assume that B is smooth.

Now blow V up in B, obtaining $f : V' \longrightarrow V$ and a surface E' on V' which is ruled over B. Consider the base locus of $|mK_{V'}|$. If there is a base curve B' such that $f(B')=B$, then we repeat the proceedure. Eventually, we obtain a smooth threefold W, containing a smooth curve C which is a base curve of $|mK_W|$; moreover, if we blow W up in C, say $g : W' \to W$, there does not exist a base curve of $|mK_{W'}|$ whose image under g is C. Let S' denote the surface on W' corresponding to C on W ; S' is therefore ruled over C.

Now resolve the base locus of $|mK_{W'}|$, say $h : \widetilde{W} \to W'$. On $\widetilde{W}$, the linear system $|mK_{\widetilde{W}}|$ has no base locus, and the morphism ϕ_{mK} contracts down all the fixed components (see (2.3) again).

Let $\widetilde{S}$ be the surface on $\widetilde{W}$ corresponding to S' on W' ; thus $\widetilde{S}$

is birationally ruled over C. Clearly $\tilde{S}$ is a fixed component of $|mK_{\tilde{W}}|$, and so it is contracted by $\phi_{mK_{\tilde{W}}}$. From our assumption on the base curves of $|mK_{W'}|$ and since C is a base curve of $|mK_W|$, we deduce that the general fibre F of $\tilde{S}$ over C is *not* contracted by ϕ_{mK}. Thus the morphism $\phi_{mK} : \tilde{W} \to X$ contracts $\tilde{S}$ down to a curve Δ on X, but does not contract the general fibre F of $\tilde{S} \to C$. In particular therefore, the curve Δ is rational.

From Corollary 1.14 of [10], apart from at a finite number of points, every point on Δ has an analytic neighbourhood which is either non-singular or isomorphic to (rational double point)$\times \mathbb{A}^1$. Thus $\tilde{S}$ is birationally ruled over Δ, and the mobile part of $|mK_{\tilde{W}}|$ meets $\tilde{S}$ in fibres of this ruling. As rational singularities do not affect adjunction, and $\tilde{S}$ is a fixed component of $|mK_{\tilde{W}}|$, it follows that we can (generically) contract $\tilde{S}$ down to a curve C' of smooth points, in such a way that the general fibre F of the ruling $\tilde{S} \to C$ is mapped surjectively onto C'.

We may assume that C' is a smooth curve, and we can resolve the indeterminacies of the map involved, obtaining a diagram of the form

$$\begin{tikzcd} & W^* \arrow[dl] \arrow[d] \arrow[dr] & \\ W_1 \arrow[d] & \tilde{W} \arrow[dl] \arrow[dr] & X_1 \arrow[d] \\ W & & X \end{tikzcd}$$

where

$$\begin{tikzcd} & W^* \arrow[dl] \arrow[dr] & \\ W_1 & & X_1 \end{tikzcd}$$

is now of type (*).

Moreover, a general fibre F' of $\tilde{S} \to C'$ maps surjectively onto C, and thus C is rational. This in turn implies that our original base curve B is rational. $\square$

Finally, we return to Question 4. Given a threefold V of general type with R[V] finitely generated, what can we say about N such that ϕ_{NK} is birational ?

Theorem 2.8. There exists a polynomial Q(x) with rational coefficients, such that for n sufficiently large,

$$P_n(V) = Q(n) + \text{term of period } r .$$

Furthermore, there exists N = N(Q,r) (depending only on the index r and the polynomial Q) such that ϕ_{NK_V} is birational.

Proof. The first part comes from a more careful version of the above calculation of $P_{mr+1}(V) = h^0(V, K_V+m\phi^*(rK_X))$ (see [10]). The second part then follows easily using (1.4) and (2.6). For more details, see [14]. □

Conjecture 2.9. The value of N such that ϕ_{NK} is birational depends only on the index r.

The above conjecture is the natural generalization of the results for surfaces of general type and for canonically polarized threefolds. In both these cases the index is 1, and so it is no surprise that there exists an N which is independent of the variety chosen.

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