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Singularities of normal affine surfaces

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(I) \( k \): alg. closed field of char 0, 
\( X \): affine normal surface / \( k \) containing a cylinderlike open set \( U \cong U_0 \times \mathbb{A}^1_k \) (\( U_0 \): curve); \( X \) contains such a \( U \) if \( \exists \ G_a \)-action on \( X \).

\( d, e \): positive integers s.t. \( d > e \) and \( (d, e) = 1 \); \( G = \mathbb{Z}/d\mathbb{Z} \cong \{ d-th \ roots \ of \ 1 \} \) which acts on \( \mathbb{A}^2_k = \text{Spec}(k[x,y]) \) by \( \sigma(x,y) = (\sigma x, \sigma^e y) \) for \( \sigma \in G \); \( T = \mathbb{A}^2_k/G \) with the image point \( Q \) of \((0,0)\). \( P \in X \) has a cyclic quotient singularity of type \((d, e) \iff \mathcal{O}_{P,X} \cong \mathcal{O}_{Q,T} \cong k[[x,y]]^Q = k((x^d, y^e)) \cap k[[x,y]]. \) A cyclic quotient singular point \( P \) of \( X \) is a rational singular point, and if \( \pi: \hat{X} \to X \) is the minimal resolution of \( P \), \( \pi^{-1}(P) \) has the following dual graph:

```
  o---o---o---o
 -\alpha_1 \ -\alpha_2 \ -\alpha_3
```
where the integers $\alpha_1, \ldots, \alpha_n$ are obtained as

\[
\frac{d}{e} = \alpha_1 - \frac{1}{\alpha_2 - \frac{1}{\alpha_3 - \frac{1}{\ddots - \frac{1}{\alpha_n}}}}.
\]

**Main Theorem.** Let $X$ be a normal affine surface $\mathbb{A}^2_k$ containing a cylinder-like open set $U$. Then we have:

1. Every singularity of $X$ is a cyclic quotient singularity.

2. $X - U = \bigsqcup_i C_i$, disjoint union of irreducible components. $C_i$ is a rational curve with only one place at $\infty$ passing through at most one singular point of $X$; if $C_i \nsubseteq$ any singular point of $X$, $C_i \cong \mathbb{A}^1_k$; if $C_i$ passes through a singular point $P$, the configuration of $\pi^{-1}(C_i)$ is:

![Diagram](https://via.placeholder.com/150)
(3) Every cyclic quotient singularity of type \((d,e)\) is realized as a singular point of an affine normal surface containing a cylinderlike open set.

(II) Application to the Jacobian Problem

Recall the Jacobian Problem:

Let \(\varphi : U := \mathbb{A}^2_k \to Y := \mathbb{A}^2_k\) be an étale morphism. Is \(\varphi\) then an isomorphism?

Let \(X := \text{normalization of } Y \text{ in } k(U)\). Then

\[
\begin{array}{ccc}
U & \overset{\text{open}}{\hookrightarrow} & X \\
\varphi \downarrow & & \downarrow \tilde{\varphi} \\
\vert & & \vert \\
Y & \overset{\cong}{\longrightarrow} & Y
\end{array}
\]

where \(\tilde{\varphi}\) is a finite (ramified) covering and \(X\) is an affine normal surface containing a cylinderlike open set \(U \cong \mathbb{A}^1_k \times \mathbb{A}^1_k\). Thus, Main Theorem is applied to \(X\).

Write \(X - U = \bigcup_i C_i \cup P_1, \ldots, P_t \) : all singular points of \(X\) (if at all). If \(P_j \in \text{some } C_i\), then \(C_i \subset \text{the ramification locus of } \tilde{\varphi}\).
Conjecture 1. (i) \( X \) is nonsingular; (ii) the branch locus of \( \tilde{\xi} \) is a union of irreducible components isomorphic to \( \mathbb{A}^1_k \).

To prove Conjecture (1), it suffices to affirm:

Conjecture 2. Let \( f, g \in k[[x, y]]^G \subseteq k[[x, y]] \), where \( G = \mathbb{Z}/d\mathbb{Z} \) acts on \( k[[x, y]] \) via \( \gamma(x, y) = (\gamma x, \gamma^e y) \). Assume that \( k[[x, y]] \) is a finite \( k[[f, g]] \)-module. Then the ramification locus of \( \text{Spec}(k[[x, y]]) \to \text{Spec}(k[[f, g]]) \) has at least two irreducible components (through the point \((0, 0)\)), which are not transitive to each other by the action of \( G \).

A topological version of Conjecture 2 is:

Conjecture 3. Let \( L(3; d, e) \) be the 3-dim lens space of type \((d, e)\). Then there are no finite ramified coverings \( f : L(3; d, e) \to S^3 \), which ramifies along a single knot.

Answer is \( \text{No} \) (cf. F. Raymond, A. Fujiki).

(III) Proof of Main Theorem (Outline)

1. Embed \( X \subseteq V \) (= normal projective surface)
so that $V$ is nonsingular along $V-X$, every irreducible component of $V-X$ is nonsingular, and $V-X$ has only normal crossings as singularities.

Then, the projection $\pi: U \rightarrow U_0$ gives an irreducible pencil $\Lambda$ on $V$; we may assume $\Lambda$ has no base points because $X$ is affine. Hence, there exists a surjective morphism $\varphi: V \rightarrow C$ (= nonsingular complete curve) such that:

(i) $\varphi|U = \pi$ with $U_0 \subset C$,

(ii) general fibers $F$ of $\varphi$ are $\cong \mathbb{P}^1$,

(iii) there exists an irreducible component $S$ of $V-X$ s.t. $(S \cdot F) = 1$,

i.e., $S$ is a cross-section of $\varphi$.

(2) We may assume that $U$ is nonsingular.

$P_1, \ldots, P_t$: all singular points of $X$; $\forall P_i \notin U$.

$\pi: W \rightarrow V$: minimal good resolution of $P_1, \ldots, P_t$,

i.e.,

a) $W - \pi^{-1}\{P_1, \ldots, P_t\} \sim V - \{P_1, \ldots, P_t\},$

b) every irreducible component of $\pi^{-1}(P_i)$ is nonsingular,

c) $\pi^{-1}(P_i)$ has only normal crossings as singularities,

d) there exist no exceptional curves (of the first kind)
meeting at most two other components.

Let $L := \pi' \Lambda$ (the proper transform) and let
$
\psi := \varphi \cdot \pi : W \to C; \text{ general fibers of } \psi \text{ are } \cong \mathbb{P}^1_k.
$
Let $P$ be one of $P_1, \ldots, P_k$, let $F_o := \varphi^{-1}(\varphi(P))$
and let $G_o := \psi^{-1}(\varphi(P))$. Then $P \in F_o$ and $\pi^{-1}(P) \subset G_o$.
Hence, every irreducible component of $\pi^{-1}(P)$ is $\cong \mathbb{P}^1_k$, the
dual graph of $\pi^{-1}(P)$ is a tree, and $\not\exists$ no exceptional
curves of the first kind meeting $3$ or more other
components. Hence, $\pi^{-1}(P)$ $\not\exists$ exceptional curves of
the first kind. Let $P := \varphi |_X : X \to C$ and let
$Q := \rho(P)$. Then $\psi^{-1}(Q)$ is either $\mathbb{P}^1_k$ or a degenerate
curve of $\mathbb{P}^1_k$. If $\psi^{-1}(Q)$ is a degenerate curve of $\mathbb{P}^1_k$, one irreducible component of $\psi^{-1}(Q)$ is an exceptional curve of the first kind; after its contraction, $\psi^{-1}(Q)$ becomes $\mathbb{P}^1_k$ or a degenerate curve of $\mathbb{P}^1_k$; if it is still a
degenerate curve, we can find an exceptional curve of the first kind among irreducible components. After
a succession of finitely many contractions, we get $\mathbb{P}^1_k$. This implies that the dual graph of a degenerate
curve of $\mathbb{P}^1_k$ contains no circular chains.
(3) **Claim:** (i) Every irreducible component of $p^{-1}(Q)$ is a connected component of $p^{-1}(Q)$.

(ii) Every irreducible component of $p^{-1}(Q)$ is a rational curve with only one place at $\infty$.

\[ \pi^{-1}(S) \cong S, \quad \pi^{-1}(S) \subset W - \pi^{-1}(x) \cong V - X \quad \text{and} \quad (\psi^{-1}(Q) \cdot \pi^{-1}(S)) = 1. \]

Then $\psi^{-1}(Q) - \psi^{-1}(Q) \cap \pi^{-1}(x) (\equiv \phi^{-1}(Q) - \phi^{-1}(Q))$ is connected.

Suppose an irreducible component of $p^{-1}(Q)$ is not a connected component. Then we have one of the configurations as shown below:

We have contradictions in both cases.
(4) **Claim:** $C_i$ passes through at most one singular point.

Let $Y$ be an irreducible component of $\varphi^*(Q)$ through $P$. Then, there exists an ample divisor $D > 0$ such that $\text{Supp}(D) = V - X$ (cf. Goodman [ ]). Hence, there exists an ample divisor $D' > 0$ such that $\text{Supp}(D') = (V - X) \cup \varphi^*(Q) - Y$. Replacing $V$ by $V - \text{Supp}(D')$, we may assume that $\varphi^*(Q) = Y$. We may assume that $\varphi^*(Q) - Y$ contains no exceptional curves of the first kind. Suppose $P_1, \ldots, P_s \in Y$. Let $\bar{Y}$ be the closure of $\pi'(Y)$ in $W$. Then, the configuration of $\varphi^*(Q)$ is one of the following:

![Diagram](attachment:image.png)

Note that $\bar{Y}$ is the unique exceptional curve of the first kind in $\varphi^*(Q)$. By contradicting $\bar{Y}$, we get contradictions in both cases.
(5) **Claim:** The dual graph of \( \pi^{-1}(P) \) is a linear chain.

\[
\begin{array}{ccc}
\bar{Y} & Z_1 & Z_2 \\
& & \\
& & \\
& & \vdots
\end{array}
\]

\[
\begin{array}{ccc}
& & T_1 \\
& & \\
& & \\
& & T_2 \\
& & \\
& & Z_{e-2}
\end{array}
\]

No other components of \( \pi^{-1}(P) \) meet any one of these curves.

\( \bar{Y} \) is the unique exceptional curve of the first kind in \( \psi^{*}(Q) \). Before contracting \( T_1 \) or \( T_2 \), \( Z_e \) must become contractable after a succession \( \tau \) of contractions. If \( \tau(Z_e) \cap \tau(S) \neq \emptyset \) then \( \sigma \tau(T_i) \cap \sigma \tau(S) \neq \emptyset \) for \( i = 1, 2 \), where \( \sigma \) is the contraction of \( \tau(Z_e) \). If \( \tau(Z_e) \cap \tau(S) = \emptyset \), \( \exists \) irreducible component \( C \) of \( \tau(\psi^{*}(Q)) \) s.t. \( C \neq \tau(Z_e) \), \( \tau(T_i) \) (\( i = 1, 2 \)) and \( C \cap \tau(Z_e) \neq \emptyset \). Both cases lead to contradictions.

(6) Finally, note that if \( \pi^{-1}(P) \) has the linear dual
graph, \( P \) is a cyclic quotient singular point.

Q.E.D.

References
