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Examples of false ruled surfaces

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I describe a construction of false ruled surfaces, which were discovered by Peter Russell. We show that this surfaces are of general typ by calculating the self-intersection of the canonical class, and we show that these surfaces have at least one global vector-field, i.e. that the schema of automorphisms of these surfaces is a finite, non-reduced group schema. We also give a formula for the Euler-characteristic $\chi(\mathcal{O}_W)$ of these surfaces.

I would like to thank Mc Gill university and in particular my friend Peter Russell for their hospitality during my visite in September and October 1981, where this work was done.

1. Some generalities

In the following we consider smooth complete algebraic varieties over an algebraic closed ground field k. If W is a variety of general typ, there exist an algebraic group schema A, which is finite over k, and an action $A \times W \to W$, which represents the cofunctor on the category of k-schemas

$$S \longmapsto \mathrm{Aut}_S(V \times S)$$

In particulare, for $S = \mathrm{Spec}\, k[t]/(t^2) \supset \mathrm{Spec}(k)$ we obtain the Lie-algebra of A by

$$\mathrm{Lie}(A) = \mathrm{Ker}\,(A(S) \to A(\mathrm{Spec}(k))) \simeq H^0(W; \Theta_W)$$

where Θ_W denotes the sheaf of vector fields on W. Over fields of characteristic 0 any algebraic group schema is smooth, hence $H^0(W; \Theta_W) = 0$ in this case. For curves of general typ this is true in any characteristic, since Θ_W is then a line bundle of negative degree. According to my knowledge this was an open question for surfaces of general typ.

An other classical result for algebraic surfaces W over fields of characteristic zero is the following one: If the Euler-number $e(W)$ is negative then W is a ruled surface over a curve of genus $g \geq 2$.

Raynaud's counterexample to Kodaira's vanishing theorem shows that this is no longer true in positive characteristic. Therefore it is interesting to characterize surfaces with negative Euler number in characteristic p, this was our starting point which we discussed with P. Russell.

The following construction is described in Seminaire Chevalley, Variétés de Picard, Exposees of Seshadri: If V is an algebraic variety over a field of characteristic p and $\mathcal{F} \subset \Theta_V$ a coherent subsheaf, we get a new algebraic variety with the same underlying space and with the structure sheaf $\mathcal{O}_V^{\mathcal{F}}$ = annihilator of $\mathcal{F}$ in $\mathcal{O}_V$. Let us denote this variety by $V^{\mathcal{F}}$, since

$\mathcal{O}_V{}^p \subseteq \mathcal{O}_V{}^{\mathfrak{f}} \subseteq \mathcal{O}_V$ we have a factorization

$$V \xrightarrow{\pi} V^{\mathfrak{f}} \xrightarrow{\pi'} V'$$

(where V' denotes the variety with the structure sheaf $\mathcal{O}_V{}^p$). The following conditions ensure that $V^{\mathfrak{f}}$ is again smooth:

(i) $\mathfrak{f}$ is a subsheaf of p-closed sub-Lie-algebras i.e. if $\theta_1, \theta_2 \in \mathfrak{f}$, then $[\theta_1, \theta_2] \in \mathfrak{f}$ and $\theta_1^p \in \mathfrak{f}$

(ii) $\Theta_V/\mathfrak{f}$ is locally free

In this case case, π is a purely inseparable finite flat morphisme of degree p^r, $r =$ rank$(\mathfrak{f})$, and $\mathfrak{f} = \Theta_{V/W}$ (where we denote $V^{\mathfrak{f}}$ by W). The following sequences are then exact:

$$0 \to \Theta_{V/W} \to \Theta_V \to \pi^* \Theta_{W/V'} \to 0$$

$$0 \to \Theta_{W/V'} \to \Theta_W \to \pi'^* \Theta_{V'/W'} \to 0$$

Therefore

$$\det(\Theta_V) \simeq \det(\Theta_{V/W}) \otimes \pi^* \det(\Theta_{W/V'})$$

$$\det(\Theta_{W/V'}) \simeq \det(\Theta_W) \otimes \pi'^* \det(\Theta_{V'/W'})^{-1}$$

and since

$$\pi^* \pi'^* \det(\Theta_{V'/W'}) = (\pi' \circ \pi)^* \det(\Theta_{V'/W'})$$
$$\simeq \det(\Theta_{V/W})^{\otimes p}$$

and $\Theta_{V/W} \simeq \mathcal{F}$ we get the following formula (Rudakov – Shafarevich) for the canonical classes

$$\omega_V \simeq \pi^* \omega_W \otimes \det(\mathcal{F})^{\otimes p-1}$$

2. The construction of Peter Russell

The surfaces W will be of the type $W = V^{\mathcal{F}}$, where V is a ruled surface over a curve B of genus $g \geq 2$ such that for some integer $n > 0$ holds $p(np-1) \mid 2g-2$ and $np > n+2$. The surface V will be a ruled surface of the type $V = \mathbb{P}(\mathcal{O}_B \oplus L^{\otimes p})$ where L is a line bundle such that $L^{\otimes p(np-1)} \simeq \Theta_B$ (observe that $(np-1)p \mid \deg(\Theta_B) = 2-2g$). The problem is to find a suitable p-closed subsheaf $\mathcal{F}$ such that $\Theta_V / \mathcal{F}$ is locally free of rank 1. Any locally free subsheaf of rank 1 of Θ_V is of the form $\mathcal{F} = \mathcal{O}_V(\Delta)\theta$, where θ is a rational vector field on V and $\Delta = \operatorname{div}(\theta)$

(if $\mathcal{O}_V/\mathcal{G}$ is torsionfree). If t is an affine coordinate on the generic fibre of the ruling $V \xrightarrow{\beta} B$ the function field of V is $k(V) = k(B)(t)$ and we can extend any rational vector field δ on B to a rational vector field, also denoted by δ, on V by assuming $\delta(t) = 0$. Therefore any rational vector field on V is parallel to a vector field of the form $\theta = \delta + h\frac{\partial}{\partial t}$, $h \in k(V)$.

The divisor $\Delta = \mathrm{div}(\theta)$ of a rational vector field is defined as follows: On open sets $U \subset V$ where there exist regulare functions x, y such that $dx \wedge dy$ has no zeros we can write

$$\theta = f\left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right)$$

where (for sufficient small U) a, b are regulare functions on U which have at most isolated common zeros on U. Then $\mathcal{O}_V(\Delta)|U = \mathcal{O}_V\frac{1}{f}|U$. Then $\mathcal{O}_V(\Delta)\theta \subseteq \Theta_V$ and $\Theta_V/\mathcal{O}_V(\Delta)\theta$ is locally free if and only if a, b have no common zero, i.e. if θ has only divisorial singularities.

If we choose θ in the form $\theta = \delta + h^p\frac{\partial}{\partial t}$

then $\theta^p = \delta^p$ and therefore the condition of p-closedness is satisfied, if $\delta^p = 0$.

<u>Lemma</u> Assume δ is a rational vector field on the curve B such that $\delta^p = 0$ and the divisor $\operatorname{div}(\delta)$ is of the form $-pmE$, $m>0$. Then there exist a covering $B = U \cup U^*$ and regular functions x on U, x^* on U^* such that $\operatorname{supp}(E) \subset U^* - U$, $\theta = \frac{d}{dx}$ and

$$dx = \mu^{pm} dx^*, \quad \omega_B | U = \mathcal{O}_B dx | U, \quad \omega_B | U^* = \mathcal{O}_B dx^* | U^*$$

where μ is a local equation of E on U^* i.e. $\mathcal{O}_B(E)|U^* = \frac{1}{\mu} \mathcal{O}_B | U^*$.

Prove: We use the following formula (Hochschild formula) $(f\delta)^p = f^p \delta^p + f\, \delta^{p-1}(f^{p-1})\theta$.
If $\delta = f\frac{d}{dy}$, y a rational function on B such that $dy \neq 0$, then $0 = \delta^p = f\left(\frac{d}{dy}\right)^{p-1}(f^{p-1})\frac{d}{dy}$, hence $f^{p-1} = \sum_{\nu=0}^{p-2} a_\nu^p y^\nu = \frac{dg}{dy}$, where a_ν are rational functions on B and $g = \sum_{\nu=0}^{p-2} (\nu+1)^{-1} a_\nu y^{\nu+1}$.
If $x = \frac{g}{f^p}$, then $\frac{dx}{dy} = \frac{1}{f}$, hence $f\frac{d}{dy} = \frac{d}{dx}$.

Let $\mathcal{U}$ be an open set such that x is regular on $\mathcal{U}$ and $\operatorname{supp}(E) \cap \mathcal{U} = \phi$. Then $\theta = \frac{d}{dx}$ and $\omega_B|\mathcal{U} = \mathcal{O}_B\, dx\, |\mathcal{U}$.

Let y be an arbitrary function on B which has simple zeros in the finite many points of $B \smallsetminus \mathcal{U}$, then in a small neighbourhood $\mathcal{U}^*$ of $B \smallsetminus \mathcal{U}$ we have $\omega_B|\mathcal{U}^* = \mathcal{O}_B\, dy\, |\mathcal{U}^* = \frac{1}{\mu^{pm}} \mathcal{O}_B\, dx\, |\mathcal{U}^*$, where μ is a local equation of E on $\mathcal{U}^*$.

Therefore $\frac{dx}{dy} = \varepsilon \mu^{pm}$, $\varepsilon \in \mathcal{O}_B(\mathcal{U}^*)^*$ and $0 = \frac{d^p x}{dy^p} = \frac{d^{p-1}\varepsilon}{dy^{p-1}} \mu^{pm}$, hence

$$\varepsilon = a_0^p + a_1^p y + \cdots + a_{p-2}^p y^{p-2} \quad , a_\nu \in \mathcal{O}_B(\mathcal{U}^*).$$

The function $x^* = y\,(a_0^p + \frac{1}{2} a_1^p y + \cdots + \frac{1}{p-1} a_{p-2}^p y^{p-2})$ has therefore simple zeros in the points of $B \smallsetminus \mathcal{U}$ and $dx^* = \varepsilon\, dy$, hence $dx = \mu^{pm} dx^*$ q.e.d.

We assume now that B is a curve with a rational vector field δ such that $\delta^p = 0$ and $\operatorname{div}(\delta) = -p(np-1)E$ and we choose $\mathcal{U}, \mathcal{U}^*, x$, x^* and μ as in the lemma. The conditions on B seems to be rather special but here are some examples of such curves:

(i) $p=2$, $n=3$

$B \subset \mathbb{P}^2$ defined by $y^4 + x^5 + y = 0$ (in inhomogeneous coordinates), $\delta = \frac{d}{dx}$

(ii) In general, given $p \geq 2$ and n, let $f(x)$ be a polynomial of degree $p(np-1)+3$ with only simple zeros such that $(\frac{d}{dx})^{p-1}(f^{\frac{p-1}{2}}) = 0$, and let B be the hyperelliptic curve defined by

$$y^2 = f(x)$$

and $\delta = y\frac{d}{dx}$, E the point at infinity P_∞.

Clearly $\operatorname{div}(\delta) = -(2g-2)P_\infty$. On the other hand $\delta^p = y(\frac{d}{dx})^{p-1}(y^{p-1})\frac{d}{dx} = 0$ if and only if $(\frac{d}{dx})^{p-1}(y^{p-1}) = (\frac{d}{dx})^{p-1}(f^{\frac{p-1}{2}}) = 0$.

Examples are: $p=3$, $n=5$, $y^2 = x^{45} + x$

$p=5$, $n=3$, $y^2 = x^{73} + x^{37} + 2x$

etc.

Let L be the line bundle $\mathcal{O}_B(-E)$ and choose sections $a_i \in H^0(B, \mathcal{O}_B(ipE))$, $i=1,\dots,n$

Let V be the surface $\mathbb{P}(\mathcal{O}_B \oplus L^{\otimes p})$ and S_0 and S $\subset V$ the sections corresponding to

the projections $\mathcal{O}_B \oplus L^{\otimes p} \to \mathcal{O}_B$ resp. $\mathcal{O}_B \oplus L^{\otimes p} \to L^{\otimes p}$.
Then $(S \cdot S_0) = 0$, $(S^2) = -(S_0^2) = \deg(L^{\otimes p}) = -p \deg E$.
We can choose trivializations

$$V|U \simeq U \times \mathbb{P}^1$$
$$V|U^* \simeq U^* \times \mathbb{P}^1$$

such that $t_0^* = t_0$, $t_1^* = \mu^p t_1$ for the corresponding homogenous coordinates, S is given by $t_0 = t_0^* = 0$, S_0 by $t_1 = t_1^* = 0$.
We denote by t, t^* the affine coordinates $t = \frac{t_1}{t_0}$, $t^* = \frac{t_1^*}{t_0^*} = \mu^p t$. At infinity, i.e. in a neighbourhood of the section S we have to use the coordinates $s = \frac{1}{t}$ and $s^* = \frac{1}{t^*}$
Let θ be the vector field

$$\theta = \frac{\partial}{\partial x} + h^p \frac{\partial}{\partial t}, \quad h = t^n + a_1 t^{n-1} + \dots + a_n$$

It has no singularities on $\beta^{-1}U - S$.
If $h_0 = 1 + a_1 s + \dots + a_n s^n$

$$\theta_0 = \mathcal{D}^{np-2} \frac{\partial}{\partial x} - h_0(s) \frac{\partial}{\partial s}$$

then θ_0 has no singularity along $S \cap \beta^{-1}U$ and

$$\theta = s^{-(np-2)} \theta_0$$

If $\quad h^*(t^*) = t^{*n} + \mu^p a_1 t^{*n-1} + \cdots + \mu^{np} a_n$ and

$$\theta^* = \frac{\partial}{\partial x^*} + h^*(t^*)\frac{\partial}{\partial t^*}$$

then θ^* has no singularity on $\beta^{-1}\mathcal{U}^* - S$ and

$$\theta = \mu^{-p(np-1)}\,\theta^*$$

In the same way, using $h_0^*(s^*) = 1 + \mu^p a_1 s^* + \cdots +$ $+ \mu^{pn} a_n s^{*n}$ and

$$\theta_0^* = \frac{\partial}{\partial x^*} - h_0^*(s^*)\frac{\partial}{\partial s^*}$$

we get

$$\theta^* = s^{*\,-(np-2)}\,\theta_0^*$$

and θ_0^* has no singularity on $S \cap \beta^{-1}\mathcal{U}^*$.
Therefore, θ has only divisorial singularities and

$$\Delta = \mathrm{div}(\theta) = -(np-2)S - p(np-1)\beta^* E$$

and the surface $W = V^{\delta}$, $\delta = \mathcal{O}_V(\Delta)\theta$, is a smooth algebraic surface. We get a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\pi} & W \\ \beta\downarrow & & \downarrow\gamma \\ B & \xrightarrow{F} & B' \end{array}$$

where π is a homeomorphisme, which is birational on the geometric fibres.

3. Numerical invariants

We have for the Euler numbers

$$e(W) = e(V) = 4-4g = -2p(np-1)\deg E$$

We can compute (ω_W^2) by using the formula

$$\pi^*\omega_W \cong \omega_V(-(p-1)\Delta)$$

(since $f \simeq \mathcal{O}_V(\Delta)$), hence

$$p(\omega_W^2) = (\omega_V^2) - 2(p-1)(\omega_V\cdot\Delta) + (p-1)^2(\Delta^2)$$

For ruled surfaces V we have

$$(\omega_V^2) = 8(1-g) = -4p(np-1)\deg E$$

$$-(\omega_V\cdot\Delta) = (np-2)(\omega_V\cdot S) + p(np-1)\deg E\,(\omega_V\cdot F)$$

(F a fibre)

$$(\omega_V\cdot S) = 2g-2-(S^2) = p(np-1)\deg E + p\deg E$$
$$= np^2\deg(E)$$

$$(\omega_V\cdot F) = -2$$

hence $-(\omega_V\cdot\Delta) = p(n^2p^2 - 4np + 2)$

$$(\Delta^2) = -(np-2)^2 p\deg E + 2p(np-1)(np-2)\deg E$$
$$= \qquad p^2 n(np-2)\deg E$$

$$(\omega_W^2) = [p^2(p^2-1)n^2 - 2p(p^2+2p-1)n + 4p]\deg E$$

If $n > \frac{2}{p(p-1)}$ this integer is positive, and since W has no exceptional curves of the first kind

and the Albanesemap is not trivial, it follows that W is a surface of general typ.
Applying Noethers formula $\chi(\mathcal{O}_W) = \frac{1}{12}((\omega_W^2) + e(W))$ we get

$$\chi(\mathcal{O}_W) = \left[\frac{p^2(p^2-1)}{12} n^2 - \frac{p(p^2+3p-1)}{6} n + \frac{p}{2}\right] \deg(E)$$

Since $np > 2+n$ we always have

$$\chi(\mathcal{O}_W) > 0.$$

4. Vector fields on W

On $\gamma^{-1}(U) - \pi(S)$ we have the functions

$$y = h(t)^p x - t \ , \ \xi = x^p \text{ and } \tau = t^p$$

They satisfy the relation

$$y^p = g(\tau)^p \xi - \tau$$

where $g(\tau) = \tau^n + a_1^p \tau^{n-1} + \cdots + a_n^p$, and the subschema of $U' \times A^2$ defined by this equation (y, τ affine coordinates of A^2) is smooth over U', therefore equal to $\gamma^{-1}(U) - \pi(S)$.
In a neighbourhood of $\pi(S) \cap \gamma^{-1}(U)$ we can use the functions

$$z = h_0(s)^p x - s^{np-1} \quad \text{and} \quad \sigma = s^p$$

to define an embedding into $U' \times \mathbb{A}^2$, given by the relation

$$z^p = g_0(\sigma)^p \xi - \sigma^{np-1}$$

(where $g_0(\sigma) = 1 + a_1^p \sigma + \cdots + a_n^p \sigma^n$). Each fibre of γ has therefore precisely on singularity, namely the point $\pi(S \cap F)$, which is isomorphic to the cusp $u^p + v^{np-1} = 0$.

The coordinates z and σ are related to y and τ by

$$\sigma = \frac{1}{\tau} \quad , \quad z = \frac{y}{\tau^n}$$

On $\gamma^{-1}(U^*)$ we use the functions

$$y^* = h^*(t^*)^p x^* - t^* \quad , \quad \xi^* = x^{*p} \, , \quad \tau^* = t^{*p}$$

and

$$z^* = h_0^*(s^*)^p x^* - s^{*\,np-1} \quad , \quad \sigma^* = s^{*p}$$

Since $dx = \mu^{p(np-1)} dx^*$, we have

$$x = \mu^{p(np-1)}(x^* + a^p) \quad , \quad a \in k(B).$$

Then $\varkappa = \mu^p$ and $b = a^p$ are functions on B' and

$$\xi = \varkappa^{p(np-1)}(\xi^* + b^p)$$

$$y = \varkappa^{-1}(y^* + b\, g^*(\tau^*))$$

$$\tau = \frac{\tau^*}{\varkappa^p}$$

$$\sigma = x^p \sigma^*$$
$$z = x^{np-1}\,(z^* + b\, g_0^*(\sigma^*))$$

Therefore the vector field

$$\frac{\partial}{\partial y} = x\frac{\partial}{\partial y^*} = \sigma^n \frac{\partial}{\partial z} = x\,\sigma^{*n}\frac{\partial}{\partial z^*}$$

is regular on W.

The vector field $\frac{\partial}{\partial y}$ is a section of the subbundle $\Theta_{W|V'}$ of Θ_W and

$$\Theta_{W|V'} = \mathcal{O}_W(nT + \gamma^* E')\frac{\partial}{\partial y}$$

where $T = \pi_*(S)$, $E' = F_*(E)$.

If we consider the factorization

$$V \xrightarrow{\pi} W \xrightarrow{\pi'} V'$$

we get the exact sequence

$$0 \to \Theta_{W|V'} \to \Theta_W \to \pi'^*\Theta_{V'|W'} \to 0$$

and $\pi^*\pi'^*\Theta_{V'|W'} \cong \mathcal{f}^{\otimes p}$. Therefore $H^0(W, \pi'^*\Theta_{V'|W'}) = 0$ and

$$H^0(W, \Theta_W) = H^0(W, \Theta_{W|V'}) \cong H^0(W, \mathcal{O}_W(nT + \gamma^* E'))$$
$$= H^0(B', \gamma_* \mathcal{O}_W(nT) \otimes \mathcal{O}_{B'}(E'))$$

Since $d\xi \wedge d\eta = -\frac{\sigma^{np-2-n}}{g_0(\sigma)^p}\, d\sigma \wedge dz = x^{p(np-1)-1}\, d\xi^* \wedge dy^*$, the divisor

$$K = (np-2-n)T + [p(np-1)-1]\,\gamma^* E'$$

is a canonical divisor on W.

5. Some special examples

Example 1 : $k = \bar{\mathbb{F}}_2$

B the completion of the curve

$$u^5 + v^4 + v = 0$$

in $\mathbb{P}^2$, $\delta = \frac{d}{du}$, $\vartheta = \frac{\partial}{\partial u} + t^6 \frac{\partial}{\partial t}$.

This curve has one point E at infinity, in a neighbourhood of this point E is defined by $\mu = 0$, $\mu = \frac{u}{v}$.

If we choose $x = u$, $x^* = \frac{v^7}{u^9}$, then x^* has a simple zero at E and $dx = \mu^{10} dx^*$.

Furthermore

$$x = (v\mu^3)^2 + \mu^{10} x^*$$

Let W be the corresponding surface, then

$$e(W) = -20$$

$$(\omega_W^2) = 32$$

$$\chi(\mathcal{O}_W) = 1$$

Using the notation of § 4 we have

$$d\xi \wedge dy = \sigma \, d\sigma \wedge dz = x^9 \, d\xi^* \wedge dy^*$$

Using the notation $w = v^p$, the coordinate ring of U' is

$$A = k[\xi] + k[\xi]w + k[\xi]w^2 + k[\xi]w^3$$

and $\gamma^{-1}U' \setminus \pi(S)$ has the coordinate ring

$$A[\tau] + A[\tau]y .$$

If $f, g \in A[\tau]$, the 2-form $\eta = (f+yg)d\xi \wedge d\eta$ is regular on $\gamma^{-1}U'$ if and only if the function $(f+yg)\sigma$ is regular in a neighbourhood of $\gamma^{-1}U' \cap T$, i.e. $f = a + b\tau$, $g = 0$. Therefore the 2-form η is regular on W if and only if

$$\mathrm{ord}_{E'}(ax^9) \geq 0 \quad , \mathrm{ord}_{E'}(bx^7) \geq 0 ,$$

i.e. $\quad a \in H^0(B', \mathcal{O}_{B'}(+9E'))$

$$b \in H^0(B', \mathcal{O}_{B'}(+7E')) .$$

Since $\mathrm{ord}_{E'}(x) = 1$, $\mathrm{ord}_{E'}(\xi) = -4$, $\mathrm{ord}_{E'}(w) = -5$ we get

$$H^0(B', \mathcal{O}_{B'}(+9E')) = k + k\xi + k\xi^2 + kw + k\xi w$$

$$H^0(B', \mathcal{O}_{B'}(+7\ E')) = k + k\xi$$

Therefore $\quad p_g(W) = q(W) = 8 \quad$ and

$$\dim(\mathrm{Alb}(W)) = \dim(\mathrm{Jac}(B')) = 6$$

Example 2 : Let k be an algebraically closed field of characteristic 3, $n = 2m+1 \geq 3$ an odd integer and $f(u) \in k[u]$ a monic polynomial of degree $3n$. We consider the hyperelliptic curve

$$B \quad : \quad v^2 = f(u)^3 + u$$

and $\delta = \frac{d}{dv}$, $\theta = \frac{\partial}{\partial v} + t^{3n}\frac{\partial}{\partial t}$.

By E we denote the point at infinity. For the corresponding surface W we get

$$e(W) = 6 - 18n \quad , \quad (\omega_W^2) = 12(6n^2 - 7n + 1)$$

$$\chi(\mathcal{O}_W) = 6n^2 - \frac{17n-3}{2}$$

$$\dim Alb(W) = \frac{9n-1}{2} = 4n + m$$

We want to compute the irregularity $q = h^1(\mathcal{O}_W)$ of W.

We have $\quad \mathrm{ord}_E(v) = -(2g+1) = -9n$

$$\mathrm{ord}_E(u) = -2$$

hence $\quad \mu = \frac{v}{u^{9m+5}}$ has a simple zero at E and if

$$x^* = \frac{u^{9m+6}}{f(u)^3 \mu^{18m+5}}$$

we get $\quad \mathrm{ord}_E(x^*) = 1$

$$dx = \mu^{18m+6} dx^* = \mu^{3(3n-1)} dx^*$$

$$x = \mu^{18m+6} x^* + \frac{v^3}{f(u)^3}$$

If we use the notation

$$v^3 = \eta \ , \ u^3 = \xi \ , \ f(u)^3 = g(\xi)$$

the curve B' is defined by the equation

$$\eta^2 = g(\xi)^3 + \xi$$

and $$y = \frac{y^*}{\varkappa} + \frac{\eta}{\varkappa^{3n} g(\xi)} \tau^{*n}$$

(using the notation of § 4). The divisor

$$K = (2n-2)T + (9n-4)\gamma^* E'$$

is canonical on W, hence

$$p_g = h^0(\gamma_* \mathcal{O}_W((2n-2)T) \otimes \mathcal{O}_{B'}((9n-4)E'))$$

If A is the coordinate ring of the affine curve $B' - \{E'\} = U'$ we have

$$\gamma_* \mathcal{O}_W((2n-2)T) = \sum_{\nu=0}^{2n-2} A\tau^\nu + \sum_{\nu=0}^{n-2} A\tau^\nu y$$

A function $f = \sum_{\nu=0}^{2n-2} a_\nu \tau^\nu + \sum_{\mu=0}^{n-2} b_\mu \tau^\mu y$ is a section of $\mathcal{O}_W(K)$ if and only if the coefficients of

$$\begin{aligned} f\varkappa^{9n-4} &= \sum_{\nu=0}^{n-1} a_\nu \varkappa^{9n-4-3\nu} \tau^{*\nu} \\ &+ \sum_{\mu=0}^{n-2} \left(a_{n+\mu} + \eta \frac{b_\mu}{g}\right) \varkappa^{6n-4-3\mu} \tau^{*n+\mu} \\ &+ \sum_{\mu=0}^{n-2} b_\mu \varkappa^{9n-5-3\mu} \tau^{*\mu} y^* \end{aligned}$$

are regular at E'.

Since $\mathrm{ord}_{E'}(\eta) = -9n$, $\mathrm{ord}_{E'}(\xi) = -2$ this implies that a_ν, $\nu < n$ and b_μ $\mu < n-2$ are polynomials of $k[\xi]$

$$\deg(a_\nu) \le \frac{9n-4-3\nu}{2} \qquad (1)$$

$$\deg(b_\mu) \le \frac{9n-5-3\mu}{2}$$

If $a_{n+\mu} = r_\mu + \eta q_\mu$, $r_\mu, q_\mu \in k[\xi]$, then
$a_{n+\mu} + \eta \frac{b_\mu}{g} = r_\mu + \frac{\eta}{g}(g q_\mu + b_\mu)$. With the notation

$$p_\mu = g q_\mu + b_\mu$$

we get the conditions

$$\deg(r_\mu) \le \frac{6n-4-3\mu}{2} \qquad (2)$$

$$\deg(q_\mu) \le \frac{3n-5-3\mu}{2} \qquad (3)$$

$$\deg(p_\mu) \le \frac{3n-4-3\mu}{2} \qquad (4)$$

Because b_μ is determined by $b_\mu = p_\mu - g q_\mu$ we get by a straightforward calculation from (1)-(4)

$$p_g(W) = 8n^2 - 11n + 21 + \frac{m(m-1)}{2}$$

According to § 3 we have

$$\lambda(Q_W) = 6n^2 - 8n + 1 - m$$

and because of $q(W) = p_g(W) + 1 - \chi(\mathcal{O}_W)$ we get

$$q(W) = 2n^2 - 3n + 21 + \frac{m(m+1)}{2}$$

Therefore the Picard variety of W is not reduced and

$$q(W) - \dim \mathrm{Alb}(W) = 2n^2 - 7n + 21 + \frac{m(m-1)}{2}$$

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