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ON THREE DIMENSIONAL COMPACT KAEHLER MANIFOLDS OF NONNEGATIVE BISECTIONAL CURVATURE

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1. Statement of the results.

After the solution of Frankel conjecture by Mori [5] and Siu-Yau [8], it is natual to consider the classification of compact Kaehler manifolds of nonnegative bisectional curvature. In this direction there are some previous works, for example, the characterization of hyperquadrics by Siu [7], and the splitting theorem of Kaehler manifolds of nonnegative bisectional curvature by Howard, Smyth, and Wu [3],[9]. Besides these general dimensional studies there is a low dimensional result by Howard and Smyth [2], that is the complete classification of two dimensional compact Kaehler manifolds of nonnegative curvature. In this paper proceeding in this direction we consider the case of three dimension, and obtain some results which, combined together with the above results of Howard, Smyth, and Wu, [2], [3], and [9], enable us to settle the classification of three dimensional compact Kaehler manifolds of nonnegative bisectional curvature.

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Theorem 1.

Let M be a three dimensional compact Kaehler manifold of nonnegative bisectional curvature. If M has quasi-positive Ricci curvature, M is biholomorphic to one of the followings, $\mathbb{P}^3$, $\mathbb{Q}^3$, $\mathbb{P}^1 \times \mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Here quasi-positive means semi-positive definite everywhere and positive definite somewhere.

Remark.

By our proposition 3 stated below one can easily see that in the above case the assumption of quasi-positve Ricci curvature is equivalent to positivity of first Chern class.

Theorem 2.

Let M be an n-dimensional ($n \geq 3$) compact Kaehler manifold of nonnegative bisectional curvature which satisfies condition (C) at every point . If there exists some point $p \in M$ such that for all $0 \neq X \in T_pM$, $\dim_{\mathbb{C}} N(X) \leq 1$, then M is biholomorphic to either $\mathbb{P}^n$ or $\mathbb{Q}^n$.

Theorem 2 is a slight generalization of the result of Siu [7], and we used his notations , but his notation N(X) is slightly different,

TM ; the holomorphic tangent bundle,

$\mathbb{Q}^n$; n-dimensional hyperquadric,

$N(X) = \{ Y \in T_pM : R(X,\bar{X},Y,\bar{Y}) = 0,\ Y \perp X \}$,

condition (C) at p ; if $T_pM = H_1 \oplus H_2$ is a decomposition into orthogonal direct summands and if $0 \neq X_i \in H_i (i = 1,2)$, then either $H_1 \not\subset N(X_2)$ or $H_2 \not\subset N(X_1)$,

in Siu's case there is no orthogonality condition on Y.

2. Proof of theorem 2.

First remark that in this paper the sign convention of curvature is chosen so that in the case of positive bisectional curvature, $R(X,\bar{X},Y,\bar{Y})$ is to be positive for any non-zero holomorphic vectors X and Y. Since the proof of theorem 2 is almost same as that of Siu's, we will only point out the places where change must be done and write the altered argument, thus readers are asked to refer his paper [7].

He used only twice the condition corresponding to

(*) $\dim_{\mathbb{C}} N(X) \leqq 1$, for all $X \in T_p M$.

One of them is the following way. If $f : \mathbb{P}^1 \longrightarrow M$ is a non-constant holomorphic map such that $f(0) = p$ where $p \in M$ is the point that satisfies (*), one can decompose the induced holomorphic vector bundle $f^* TM$ into n holomorphic line bundles,

$$f^*TM = L_1 \oplus L_2 \oplus \cdots \oplus L_n.$$

Then we conclude first Chern class $C_1(L_i) > 0$ except at most one L_i. Under our assumption we argue as follows. Calculating the curvature of the dual L_i^* of L_i with respect to the induced metric we have $C_1(L_i) \geq 0$. Moreover if $C_1(L_i) = 0$, the bisectional curvature in the direction of $f_*T\mathbf{P}^1$ and L_i is 0, and L_i^* is trivial, there exists a non-vanishing holomorphic section u. Let X be any holomorphic tangent vector field of $\mathbf{P}^1$, then the contraction $u(f_* X)$ makes a holomorphic function on $\mathbf{P}^1$, and X vanishes somewhere, thus $u(f_* X) = 0$. Therefore, if there are two L_is with $C_1(L_i) = 0$, a contradiction arises at some point near p. (Remark that the points which satisfy (*) make an open set.) The other way Siu used (*) is very similar. Let E be the divisor of df, where f is again a holomorphic map from $\mathbf{P}^1$ to M, then $T\mathbf{P}^1 \times [E] \subset f^*TM$. In the decomposition

$$f^*TM / T\mathbf{P}^1 \times [E] = Q_1 \oplus Q_2 \oplus \cdots \oplus Q_{n-1},$$

we have $C_1(Q_i) > 0$ at most one exception. The argument is similar. Therefore we obtain theorem 2.

Corollary.

Let M be an n-dimensional ($n \geq 3$) compact Kaehler manifold of nonnegative bisectional curvature. If (*) holds at evry point on M, M is biholomorphic to either $\mathbf{P}^n$ or $\mathbf{Q}^n$.

We will use this corollary in the case n = 3 to prove theorem 1. Our methold is that assuming the manifold is biholomorphic to neither $\mathbf{P}^3$ nor $\mathbf{Q}^3$, we deduce that there exists at least one $\mathbf{P}^1$ factor and apply the classification of two dimensional case. Beforehand we have to prepare some propositions on Hamitlon's equation.

3. Hamilton's equation.

In his paper [1] Hamilton introduced a evolution equation on metric tensors, and proved it has local solvability. He considered real three dimensional case and proved that the equation preserves nonnegativity of Ricci curvature, but we will use it in complex three dimensional case and prove that it preseves nonnegativity of bisectional curvature.

Proposition 1.

Let M be a three dimensional compact Kaehler manifold and consider the following equation on Kaehler metrics.

$$d/dt\, g_{i\bar{j}} = - R_{i\bar{j}}, \quad g_{i\bar{j}}\big|_{t=0} = \text{a given metric}.$$

Then it preserves nonnegativity of bisectional curvature, and if the initial metric has positive bisectional curvature, it remains so.

The proof is almost same as Hamilton's case. But his proof had a slight gap and these kinds of propositions are very important in our argument and we will employ them many times, thus we write a proof of proposition 1 for completeness. But we will state other facts, which can be similarly proved, as propositions without proof.

Proof of proposition 1.

Calculating directly we get

$$d/dt\, R_{i\bar{j}k\bar{l}} = \Box R_{i\bar{j}k\bar{l}} + R_{i\bar{\alpha}\beta\bar{l}} R_{\alpha\bar{j}k\bar{\beta}} - R_{i\bar{\alpha}k\bar{\beta}} R_{\alpha\bar{j}\beta\bar{l}}$$

$$+ R_{i\bar{j}\alpha\bar{\beta}}R_{\beta\bar{\alpha}k\bar{l}} - 1/2\ [R_{i\bar{\alpha}}R_{\alpha\bar{j}k\bar{l}} + R_{\alpha\bar{j}}R_{i\bar{\alpha}k\bar{l}}$$

$$+ R_{k\bar{\alpha}}R_{i\bar{j}\alpha\bar{l}} + R_{\alpha\bar{l}}R_{i\bar{j}k\bar{\alpha}}]\ ,$$

where

$$\Box = -1/2\ \nabla^{*}\nabla,$$

that is a real operator. Set,

$$R^{c}{}_{i\bar{j}k\bar{l}} = 1/2[g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}}]\ ,$$

that is a parallel tensor. Then,

$$d/dt\ R^{c}{}_{i\bar{j}k\bar{l}} = -1/2[R_{i\bar{\alpha}}R^{c}{}_{\alpha\bar{j}k\bar{l}} + R_{\alpha\bar{j}}R^{c}{}_{i\bar{\alpha}k\bar{l}}$$

$$+ R_{k\bar{\alpha}}R^{c}{}_{i\bar{j}\alpha\bar{l}} + R_{\alpha\bar{l}}R^{c}{}_{i\bar{j}k\bar{\alpha}}].$$

For short we write the above equations in the following way.

$$d/dt\ R = \Box R + F(R) - 1/2\ Ric * R\ .$$

$$d/dt\ R^{c} = \Box R^{c} - 1/2\ Ric * R^{c}\ .$$

We postpone the proof of the property (#) of F(R) below.

(#) if $R(X,\bar{X},Y,\bar{Y}) \geqq 0$ for all X and $Y \in T_pM$ and there exist two unit vectors X_o and Y_o such that $R(X_o,\bar{X}_o,Y_o,\bar{Y}_o) = 0$, then $F(R)(X_o,\bar{X}_o,Y_o,\bar{Y}_o) \geqq 0$.

Assuming this property we first complete the proof. Since to prove the proposition it is sufficient to do for a short time, we work in some short time interval whithout specification. First let us define some notations. For two tensors u and v which have the same symmetric properties as curvature tensor, we define $u_p \geq v_p$ ($u_p > v_p$) if $u_p(X,\bar{X},Y,\bar{Y}) \geq v_p(X,\bar{X},Y,\bar{Y})$ ($u_p(X,\bar{X},Y,\bar{Y}) > v_p(X,\bar{X},Y,\bar{Y})$ respetively) for all non-zero X and $Y \in T_pM$, and $u \geq v$ ($u > v$) if $u_p \geq v_p$ ($u_p > v_p$ respectively) for all $p \in M$. Since F(R) is smooth in R and $R^o > 0$, there exists a positve constant C such that

$$F(R) \geq F(R + uR^c) - C|u|R^c, \text{ for } u \in \mathbb{R}, \ |u| < 1.$$

Let f be a real valued function and $e > 0$ a small real number.

$$\begin{aligned} d/dt\,(R + efR^c) &= \square(R + efR^c) + F(R) \\ &\quad - 1/2\ Ric * (R + efR^o) \\ &\quad + e(d/dt\ f - \square f)R^c \\ &\geq \square(R + efR^c) + F(R + efR^c) \\ &\quad - 1/2\ Ric * (R + efR^c) \\ &\quad + e(d/dt\ f - \square f - C|f|)R^c. \end{aligned}$$

We choose f to be the solution of the following equation.

$$d/dt\ f = \square f + Cf + 1,$$

$$f|_{t=0} = 1.$$

Then $f > 0$ and we get

$$d/dt\,(R + efR^c) > \square(R + efR^c) + F(R + efR^c)$$

$$- 1/2 \, Ric * (R + efR^c) \, .$$

Here we can prove $R + efR^c > 0$. If it is not true, there is the first time $t_c > 0$ so that it fails to hold, because $R|_{t=0} \geq 0$, $f|_{t=0} > 0$. By the definition it follows that at t_c

$$(R + efR^c)(X,\bar{X},Y,\bar{Y}) \geq 0 \quad \text{for all } X \text{ and } Y \in T_p M,$$

and there exist a point $p \in M$ and unit vectors X and $Y \in T_p M$ such that

$$(R + efR^c)(X_c,\bar{X}_c,Y_c,\bar{Y}_c) = 0.$$

thus we get at t_c,

$$F(R + efR^c)(X_c,\bar{X}_c,Y_c,\bar{Y}_c) \geq 0,$$

and if one of the vectors in the expression $(R + efR^c)(X_c,\bar{X}_c,Y_c,\bar{Y}_c)$ is replaced by an arbitrary vector, it gives zero, in particular,

$$Ric * (R + efR^c)(X_c,\bar{X}_c,Y_c,\bar{Y}_c) = 0.$$

We extend X_c, Y_c to be vector fields such that ∇X_c, $\nabla Y_c = 0$ at (t_c, p). Then we get at (t_c, p)

$$0 \geq d/dt \, [(R + efR^c)(X_c,\bar{X}_c,Y_c,\bar{Y}_c)]$$

$$= [d/dt \, (R + efR^c)](X_c,\bar{X}_c,Y_c,\bar{Y}_c)$$

$$> [\Box \, (R + efR^c)](X_c,\bar{X}_c,Y_c,\bar{Y}_c)$$

$$= \Box [(R + efR^c)(X_c,\bar{X}_c,Y_c,\bar{Y}_c)] \geq 0$$

This is a contradiction. Thus we get $R + efR^{0} > 0$ for all sufficiently small $e > 0$, that means $R \geqq 0$.

The proof of the last statement of the proposition is similar.

$$d/dt\,(R - fR^{0}) \geqq \Box(R - fR^{0}) + F(R - fR^{0})$$

$$- 1/2\ \mathrm{Ric} * (R - fR^{0})$$

$$+ (-d/dt\, f + \Box f - C|f|)R^{0}.$$

We choose the function f so that

$$R - fR^{0}\big|_{t=0} \geqq 0, \quad f\big|_{t=0} \geqq 0, \quad f\big|_{t=0} \not\equiv 0,$$

$$d/dt\, f = \Box f - Cf.$$

Then we get by the above aurgument $R - fR^{0} \geqq 0$, and it is well-known that under the conditions f is positive for $t > 0$, (c.f. [6]).

Finally we prove (#). First we get the following inequality considering the second variation.

$$(\#\#)\ \ [\,|R(X_0,\overline{X},Y_0,\overline{Y})| + |R(X_0,\overline{X},Y,\overline{Y}_0)|\,]^2$$

$$\leqq R(X_0,\overline{X}_0,Y,\overline{Y})R(X,\overline{X},Y_0,\overline{Y}_0)$$

$$\text{for all } X,\ Y \in T_pM.$$

$$F(R)(X_0,\overline{X}_0,Y_0,\overline{Y}_0)$$

$$= \sum_{i,j} [\,|R(X_0,\overline{E}_i,E_j,\overline{Y}_0)|^2 - |R(X_0,\overline{E}_i,Y_0,\overline{E}_j)|^2$$

$$+R(X_0,\overline{X}_0,E_i,\overline{E}_j)R(E_j,\overline{E}_i,Y_0,\overline{Y}_0)]$$

$$\geqq \sum_{i,j}[-|R(X_0,\overline{E}_i,Y_0,\overline{E}_j)|^2 + R(X_0,\overline{X}_0,E_i,\overline{E}_j)R(E_j,\overline{E}_i,Y_0,\overline{Y}_0)],$$

where $\{E_i\}$ is an orthogonal basis of T_pM.

We divide into two cases.

First case ; $X_0 /\!/ Y_0$.

In this case we can assume $X_0 = Y_0$. Then,

$$F(R)(X_0,\overline{X}_0,Y_0,\overline{Y}_0)$$

$$\geqq \sum_{i,j}[-|R(X_0,\overline{E}_i,X_0,\overline{E}_j)|^2 + |R(X_0,\overline{X}_0,E_i,\overline{E}_j)|^2] .$$

By the symmetry of curvature tensor we can choose E_i so that

$$R(X_0,\overline{E}_i,X_0,\overline{E}_j) = 0 \quad \text{if } i \neq j .$$

Then,

$$F(R)(X_0,\overline{X}_0,Y_0,\overline{Y}_0)$$

$$\geqq \sum_i[-|R(X_0,\overline{E}_i,X_0,\overline{E}_i)|^2 + |R(X_0,\overline{X}_0,E_i,\overline{E}_i)|^2]$$

$$\geqq 0 \quad \text{by (\#\#)}.$$

Second case ; $X_0 \not\parallel Y_0$.

Because T_pM is of three dimension, there is a unique unit vector E (of cause unique up to constant multiplication) which is orthogonal to X_0 and Y_0. Then,

$$F(R)(X_0,\overline{X}_0,Y_0,\overline{Y}_0)$$

$$\geq -|R(X_0,\overline{E},Y_0,\overline{E})|^2 + \sum_{i,j} R(X_0,\overline{X}_0,E_i,\overline{E}_j)R(E_j,\overline{E}_i,Y_0,\overline{Y}_0) ,$$

where we used the fact that if one of the vectors in the expression $R(X_0,\overline{X}_0,Y_0,\overline{Y}_0)$ is replaced by an arbitrary vector, it gives zero. We have two cases.

Case 1 ; $R(X_0,\overline{X}_0,X_0,\overline{X}_0) = 0$ or $R(Y_0,\overline{Y}_0,Y_0,\overline{Y}_0) = 0$.

Case 2 ; $R(X_0,\overline{X}_0,X_0,\overline{X}_0) \neq 0$ and $R(Y_0,\overline{Y}_0,Y_0,\overline{Y}_0) \neq 0$.

Case 1 is easy, for example in the case $R(X_0,\overline{X}_0,X_0,\overline{X}_0) = 0$, we choose $\{E_i\}$ to be $\{X', Y_0, E\}$, then,

$$\sum_{i,j} R(X_0,\overline{X}_0,E_i,\overline{E}_j)R(E_j,\overline{E}_i,Y_0,\overline{Y}_0)$$

$$= R(X_0,\overline{X}_0,E,\overline{E})R(E,\overline{E},Y_0,\overline{Y}_0) ,$$

$$F(R)(X_0,\overline{X}_0,Y_0,\overline{Y}_0)$$

$$\geq -|R(X_0,\overline{E},Y_0,\overline{E})|^2 + R(X_0,\overline{X}_0,E,\overline{E})R(E,\overline{E},Y_0,\overline{Y}_0)$$

$$\geq 0 \quad \text{by (\#\#)}.$$

In case 2 we argue as follows. By (##) we get that for any complex numbers $s, t \in \mathbb{C}$,

$$|R(X_0,\overline{E},Y_0,\overline{E})|^2 = |R(X_0,\overline{E - tY_0},Y_0,\overline{E - sX_0})|^2$$

$$\leq R(X_0,\overline{X}_0,E - sX_0,\overline{E - sX_0})R(E - tY_0,\overline{E - tY_0},Y_0,\overline{Y}_0) .$$

Choosing s, t suitably, we get that

$$|R(X_0,\overline{E},Y_0,\overline{E})|^2$$

$$\leq [R(X_0,\bar{X}_0,E,\bar{E}) - |R(X_0,\bar{X}_0,E,\bar{X}_0)|^2/R(X_0,\bar{X}_0,X_0,\bar{X}_0)]$$

$$\times [R(E,\bar{E},Y_0,\bar{Y}_0) - |R(Y_0,\bar{E},Y_0,\bar{Y}_0)|^2/R(Y_0,\bar{Y}_0,Y_0,\bar{Y}_0)] .$$

We choose $\{E_i\}$ to be $\{X', Y_0, E\}$, $X' = aX_0 + bY_0$.

$$\sum_{i,j} R(X_0,\bar{X}_0,E_i,\bar{E}_j)R(E_j,\bar{E}_i,Y_0,\bar{Y}_0)$$

$$= R(X_0,\bar{X}_0,E,\bar{E})R(E,\bar{E},Y_0,\bar{Y}_0) + R(X_0,\bar{X}_0,X',\bar{X}')R(X',\bar{X}',Y_0,\bar{Y}_0)$$

$$+ R(X_0,\bar{X}_0,E,\bar{X}')R(X',\bar{E},Y_0,\bar{Y}_0) + R(X_0,\bar{X}_0,X',\bar{E})R(E,\bar{X}',Y_0,\bar{Y}_0)$$

$$= R(X_0,\bar{X}_0,E,\bar{E})R(E,\bar{E},Y_0,\bar{Y}_0)$$

$$+ |ab|^2 R(X_0,\bar{X}_0,X_0,\bar{X}_0)R(Y_0,\bar{Y}_0,Y_0,\bar{Y}_0)$$

$$+ 2\mathrm{Re}[\bar{a}bR(X_0,\bar{X}_0,E,\bar{X}_0)R(Y_0,\bar{E},Y_0,\bar{Y}_0)] .$$

$$F(R)(X_0,\bar{X}_0,Y_0,\bar{Y}_0)$$

$$\geq -[R(X_0,\bar{X}_0,E,\bar{E}) - |R(X_0,\bar{X}_0,E,\bar{X}_0)|^2/R(X_0,\bar{X}_0,X_0,\bar{X}_0)]$$

$$\times [R(E,\bar{E},Y_0,\bar{Y}_0) - |R(E,\bar{Y}_0,Y_0,\bar{Y}_0)|^2/R(Y_0,\bar{Y}_0,Y_0,\bar{Y}_0)]$$

$$+ R(X_0,\bar{X}_0,E,\bar{E})R(E,\bar{E},Y_0,\bar{Y}_0)$$

$$- |R(X_0,\bar{X}_0,E,\bar{X}_0)|^2/R(X_0,\bar{X}_0,X_0,\bar{X}_0)$$

$$\times |R(E,\bar{Y}_0,Y_0,\bar{Y}_0)|^2/R(Y_0,\bar{Y}_0,Y_0,\bar{Y}_0)$$

$$= R(X_0,\bar{X}_0,E,\bar{E})\,|R(E,\bar{Y}_0,Y_0,\bar{Y}_0)|^2/R(Y_0,\bar{Y}_0,Y_0,\bar{Y}_0)$$

$$+ |R(X_0,\bar{X}_0,E,\bar{X}_0)|^2/R(X_0,\bar{X}_0,X_0,\bar{X}_0)\ R(E,\bar{E},Y_0,\bar{Y}_0)$$

$$- 2|R(X_0,\bar{X}_0,E,\bar{X}_0)|^2/R(X_0,\bar{X}_0,X_0,\bar{X}_0)$$

$$\times |R(E,\bar{Y}_c,Y_c,\bar{Y}_c)|^2/R(Y_c,\bar{Y}_c,Y_c,\bar{Y}_c)$$

$$\geq 0 \ .$$

We state several properties on the solutions of the Hamilton's equation which can be proved similarly. We always assume that the considered manifold M is of three dimension and the initial metric has nonnegative bisectional cuvature, so the solution metric remains so.

Proposition 2.

For the solution metric at $t > 0$, if $R(X,\bar{X},Y,\bar{Y}) = 0$ with some $0 \neq X$, $Y \in T_pM$, then $R(X,\bar{\cdot},\cdot,\bar{Y}) = 0$.

Proof.

Extend X,Y to be vector fields as in the proof of proposition 1. Then, from the proof

$$0 = d/dt\,[R(X,\bar{X},Y,\bar{Y})] = \square\,[R(X,\bar{X},Y,\bar{Y})] + F(R)(X,\bar{X},Y,\bar{Y})$$

$$\geq \sum_{i,j} |R(X,\bar{E}_i,E_j,\bar{Y})|^2 \ .$$

Proposition 3.

If the initial metric has quasi-positive Ricci curvature then for $t > 0$ the solution metric has positive Ricci curvature.

Proposition 4.

If the initial metric has the property (*) at some point then for $t > 0$ the solution metric has the property (*)

everywhere. In particular M must be biholomorphic to either $\mathbb{P}^3$ or $\mathbb{Q}^3$.

Corollary.

M is biholomorphic to neither $\mathbb{P}^3$ nor $\mathbb{Q}^3$, then for every point $p \in M$ there exists non-zero vector $X \in T_p M$ such that

$$(**)\quad R(X,\overline{X},Y,\overline{Y}) = 0$$

$$\text{for all } Y \perp X,\ Y \in T_p M.$$

Proposition 5.

If the initial metric has a point $p \in M$ such that

$$R(X,\overline{X},Y,\overline{Y}) + R(Y,\overline{Y},Z,\overline{Z}) + R(Z,\overline{Z},X,\overline{X}) > 0$$

$$\text{for any orthonormal basis } \{X,Y,Z\} \text{ of } T_p M,$$

then the solution metric at $t > 0$ has the above property everywhere.

Remark.

In the proof of proposition 4,5 we must be careful about the orthogonality condition on vectors. It must be compensate by the term $-1/2$ Ric $*$ R.

4. Proof of theorem 1.

From now on we assume, besides the assumption of the theorem, that M is biholomorphic to neither $\mathbb{P}^3$ nor $\mathbb{Q}^3$. We understand that when we mention metric, curvature and so on we always mean those of a solution metric of Hamilton's equation. Especially we have positive Ricci curvature and (**) is satisfied by some non-zero vector $X \in T_p M$ for every p M.

We have two cases at each point $p \in M$.

Case 1.

The vetor $X \in T_p M$ satisfying (**) is unique up to constant multiplication.

Case 2.

The vector $X \in T_p M$ satisfying (**) is not unique up to constant multiplication.

First we look at case 2. By the proposition 2 for such $0 \neq X \in T_p M$ we have

$$R(X, \bar{\cdot}, \cdot, \bar{Y}) = 0 \quad \text{for all } Y \perp X,\ Y \in T_p M.$$

Let X' be another vector satisfying (**), which is linearly independent of X, satisfying (**), then since $\dim_{\mathbb{C}} T M = 3$, there exists a unit vector Y such that

$$R(X, \bar{\cdot}, \cdot, \bar{Y}) = R(X', \bar{\cdot}, \cdot, \bar{Y}) = 0\ ,\ Y \perp X,\ X'.$$

This means that Y also satisfies (**). Replacing X' by Y in the above argument we obtain an orthonormal basis $\{X, Y, Z\}$ which cosists of vectors satisfying (**). It is easy to see that the orthonormal basis of such properties is unique up to constant multiplication and that every vector satisfying (**) is a constant multiplication of some element of the orthonormal basis. We remark that the positivity of Ricci curvature is used here.

Thus by proposition 5 we know that either case 1 holds everywhere or case 2 holds everywhere, and in both cases vectors with the property (**) are unique in appropriate sense. Here, again using the positivity of Ricci curvature we can prove in both cases such vectors make differentiable distributions. We want to prove that these distributions are parallel. Since we know that M is simply connected because of positive Ricci curvature (c.f. [9]), we can apply the de Rham decomposition theorem and get that M is biholomorphic to $\mathbf{P}^1 \times N^2$ in case 1, or biholomorphic to $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ in case 2. Here we applied the classification of two dimension case (c.f. [2]), and get N is biholomorphic to either $\mathbf{P}^2$ or $\mathbf{P}^1 \times \mathbf{P}^1$. This is the desired result.

The proof of parallelism of the distributions is actually the same in both cases, we will consider only case 1. Since parallelism is a local property, we work locally from now on. Let $0 \neq X \in T_pM$ be a vector with the property (**) and $Y \in T_pM$ be an arbitrary vector orthogonal to X. We extend them to be vector fields. Then we have at p that

$$0 = d/dt\,[R(X,\bar{X},Y,\bar{Y})] = [d/dt\,R](X,\bar{X},Y,\bar{Y})$$

$$= [\Box R](X,\bar{X},Y,\bar{Y}) + F(R)(X,\bar{X},Y,\bar{Y})$$

$$- 1/2\,[Ric * R](X,\bar{X},Y,\bar{Y})$$

$$\geq [\Box R](X,\bar{X},Y,\bar{Y})$$

$$0 \leq \Box\,[R(X,\bar{X},Y,\bar{Y})]$$

$$= [\Box R](X,\bar{X},Y,\bar{Y}) + [\nabla R](\nabla X,\bar{X},Y,\bar{Y}) + [\nabla R](X,\nabla\bar{X},Y,\bar{Y})$$

$$+ [\nabla R](X,\bar{X},\nabla Y,\bar{Y}) + [\nabla R](X,\bar{X},Y,\nabla\bar{Y})$$

$$+ R(\Box X,\bar{X},Y,\bar{Y}) + R(X,\Box\bar{X},Y,\bar{Y}) + R(X,\bar{X},\Box Y,\bar{Y})$$

$$+ R(X,\bar{X},Y,\Box\bar{Y}) + R(\nabla X,\nabla\bar{X},Y,\bar{Y}) + R(\nabla X,\bar{X},\nabla Y,\bar{Y})$$

$$+ R(\nabla X,\bar{X},Y,\nabla\bar{Y}) + R(X,\nabla\bar{X},\nabla Y,\bar{Y})$$

$$+ R(X,\nabla\bar{X},Y,\nabla\bar{Y}) + R(X,\bar{X},\nabla Y,\nabla\bar{Y})$$

$$\leqq [\nabla R](\nabla X,\bar{X},Y,\bar{Y}) + [\nabla R](X,\nabla\bar{X},Y,\bar{Y}) + [\nabla R](X,\bar{X},\nabla Y,\bar{Y})$$

$$+[\nabla R](X,\bar{X},Y,\nabla\bar{Y}) + R(\nabla X,\nabla\bar{X},Y,\bar{Y}) + R(X,\bar{X},\nabla Y,\nabla\bar{Y}),$$

since $R(X,\bar{\cdot},Y,\bar{\cdot}) = 0$ by (##). The above inequality holds for any extension X,Y, thus if one of the vectors in the expression $[\nabla R](X,\bar{X},Y,\bar{Y})$ is replaced by an arbitrary vector, we get 0.

Next we choose X as a vector field satisfying (**) and Y as any vector field orthogonal to X. Then we get for any vetor field Z

$$0 = \nabla[R(Z,\bar{X},Y,\bar{Y})]$$

$$= [\nabla R](Z,\bar{X},Y,\bar{Y}) + R(\nabla Z,\bar{X},Y,\bar{Y}) + R(Z,\nabla\bar{X},Y,\bar{Y})$$

$$+ R(Z,\bar{X},\nabla Y,\bar{Y}) + R(Z,\bar{X},Y,\nabla\bar{Y}) .$$

We know that except $R(Z,\nabla\bar{X},Y,\bar{Y})$ all other terms vanish, thus

$$R(Z, \nabla \bar{X}, Y, \bar{Y}) = 0$$ for any vector Z, and any vetor $Y \perp X$.

This implies $\nabla X \parallel X$. It means that X gives rise to a parallel distribution.

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