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Every K3 surface is Kähler

(by Y.-T.Siu)

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§0. A K3 surface means a simply connected compact complex manifold of complex dimension two whose canonical bundle is trivial. Y.-T.Siu proved

Theorem [7,Theorem 3.3] Every K3 surface is Kähler.
The aim of this report is to introduce the reader to his proof. In §1, we shall list up 4 facts which will be used to prove the theorem. The first one is on the existence of a d-closed real 2-form whose (1,1)-part is positive definite, which we shall call a Siu form. I heard that this fact is already known to differential geometers. But Siu is the first who applied this to studying K3 surfaces. The other three facts are known to the specialists in K3 surfaces. In §2, the proof is described. In §3, we shall discuss the existence of Siu forms in a slightly generalized situation.

§1. Known facts used in the proof.

The following four facts are used to prove the theorem.

FACT 1.(The existence of Siu forms) Let M be a K3 surface and k an integer ≥ 1. Then there exists a real C^k d-closed

2-form on M whose (1,1)-part is positive definite at every point on M.

This fact will be proved in §3.

Let X be a differentiable manifold which is diffeomorphic to a K3 surface. Then, since X is simply connected, there are natural inclusions

$$H^*(X,\mathbf{Z}) \subset H^*(X,\mathbf{R}) \subset H^*(X,\mathbf{C}).$$

On $H^2(X,\mathbf{Z})$, the cup products define the inner products with signature (3,19).

FACT 2. (i) Suppose that an element $a \in H^2(X,\mathbf{C})$ satisfies $a^2 = 0$ and $a\,\bar{a} > 0$.

Put $\Delta(a) = \{\, c \in H^2(X,\mathbf{Z}) : a\cdot c = 0,\ c^2 = -2 \,\}$.

(ii) Suppose that an element $b \in H^2(X,\mathbf{R})$ satisfies $a\,b = 0$, $b^2 > 0$, and $b\cdot c \neq 0$ for all $c \in \Delta(a)$.

Then there is a kähler K3 structure N on X, a Kähler form ω_N, a non-vanishing holomorphic 2-form φ_N, and an automorphism τ of $H^2(X,\mathbf{Z})$ preserving cup products such that $\tau_{\mathbf{C}}[\varphi_N] = a$ and $\tau_{\mathbf{C}}[\omega_N] = b$, where $\tau_{\mathbf{C}} = \tau \otimes \mathbf{C}$, and [ ] indicates the cohomology class.

This fact is due to Todorov [8] and Looijenga [6]. Yau's result on the existence of Einstein-Kähler metric is used prove this fact.

Let M be a K3 structure on X, and φ_M a non-zero holomorphic 2-form. Put $\mathbf{P} = (H^2(X,C) - \{0\})/C^* \cong \mathbf{P}^{21}$. Since $C\cdot\varphi_M$ is a complex 1-dimensionmal subspace in $H^2(X,C)$, and since $\varphi_M\wedge\varphi_M = 0, \int_X \varphi_M\wedge\bar{\varphi}_M > 0$, φ_M defines a point in

$$\mathcal{D} = \{ z \in \mathbf{P}:\ z\cdot z = 0,\quad z\cdot\bar{z} > 0 \},$$

where the product is the inner product defined by the intersection form on $H^2(X,Z)$. The point $q_M \in \mathcal{D}$ defined by φ_M is called the period of the complex structure M. The mapping $M \mapsto q_M$ is called the period mapping. Every complex structure M on X defines a Hodge structure
on $H^2(X,C)$;

$$H^2(X,C) = H^{2,0}(M) \oplus H^{1,1}(M) \oplus H^{0,2}(M).$$

Put

$$H^{1,1}_{\mathbf{R}}(M) = H^2(X,R) \cap H^{1,1}(M).$$

The set

$$\{ x \in H^{1,1}_{\mathbf{R}}(M) : x^2 > 0 \}$$

is called the positive cone of M. Since the signature of the intersection form is (3,19), the positive cone has 2 connected components. If M is Kähler, every Kähler metric on M is contained in the same component of the positive cone, which is called the Kähler component.

FACT 3.(The local Torelli Theorem) All small deformations of a K3 surface are parametrized effectively and completely by the period mapping.

This is due to Andreotti-Weil, see Kodaira [3].

FACT 4.(The Torelli Theorem for Algebraic K3 surfaces) Let M and N be algebraic K3 structures on X. Suppose that there is an automorphism σ of $H^2(X,\mathbf{Z})$ preserving cup products which satisfies the following three conditions;

(iii) $\sigma_{\mathbf{C}} := \sigma \otimes \mathbf{C}$ sends $H^{p,q}(N)$ to $H^{p,q}(M)$, $p+q = 2$,

(iv) $\sigma_{\mathbf{C}}$ preserves the Kähler components,

(v) Every cohomology class represented by an effective divisor C on N with $C^2 = -2$ is sent by σ to a cohomology class represented by an effective divisor on M.

Then there is a biholomorphic mapping

$$g : M \to N$$

such that $g^* = \sigma$.

This fact is due to Burns-Rapoport [2].

§2. The proof of the theorem

Suppose that we are given a K3 structure M on X. M admits a nowhere vanishing holomorphic 2-form φ.

STEP 1. By FACT 1, there is a Siu form Θ with respect to M. Let $[\varphi]$ and $[\Theta]$ denote the corresponding cohomology classes in $H^2(X,C)$ and $H^2(X,R)$, respectively. Since the equality

$$H^{1,1}_{R}(M) = \{ c \in H^2(X,R) : [\varphi]\cdot C = 0 \}$$

holds by definition, we see that

$$(2\text{-}1) \qquad [\Theta]^{\#}_{M} := [\Theta] - \lambda [\varphi] - \bar{\lambda} [\bar{\varphi}]$$

$$\lambda = ([\Theta]\cdot[\bar{\varphi}])/([\varphi]\cdot[\bar{\varphi}]) \in C$$

is an element of $H^{1,1}_{R}(M)$. Note that $\Theta - \lambda\varphi - \bar{\lambda}\bar{\varphi}$ is also a Siu form with respect to M. We replace Θ by $\Theta - \lambda\varphi - \bar{\lambda}\bar{\varphi}$. Thus we can assume in what follows that the equality

$$(2\text{-}2) \qquad [\Theta]^{\#}_{M} = [\Theta]$$

holds.

STEP 2. We want to apply FACT 2. Put $a = [\varphi]$ and $b = [\Theta]$. We check that a and b satisfy the conditions (i) and (ii). Since $[\varphi]$ is of type (2,0), we have $a^2 = [\varphi]^2 = \int_M \varphi\wedge\varphi = 0$. Similarly $a\cdot\bar{a} = [\varphi]\cdot[\bar{\varphi}] = \int_M \varphi\wedge\bar{\varphi} > 0$. Thus (i) is satisfied. Let us check (ii). It is clear by the definition that $a\cdot b = [\varphi]\cdot[\Theta] = 0$ holds. Since

$$\Delta(a) = \Delta([\varphi]) = \{ c \in H^{1,1}_{R}(M) \cap H^2(X,Z) : c^2 = -2 \},$$

every elements of $\Delta(a)$ is the Chern class of a line bundle. By the Riemann-Roch theorem, for every line bundle ξ with $(c_1(\xi))^2 = -2$, either ξ or ξ^{-1} is defined by an effective divisor on M. Thus that $b^2 > 0$ and $bc \neq 0$ for $^{\forall}c \in \Delta(a)$ follow from (1) and (2) of the following proposition.

Proposition 1 ([7,Prop.3.1])

(1) $[\Theta]\cdot C > 0$, if C is represented by an effective divisor.

(2) $([\Theta])^2 > 0$.

(3) If M is Kähler, then $[\Theta]$ is on the Kähler component of the positive cone of M.

The last statement (3) is used in STEP 4 (see Prop.2 (6)). The proof is an easy calculation of forms under type considerations. Since the similar calculations also appear in STEP 4, we omit the details here.

Now we can apply FACT 2 to our situation and obtain

Lemma 1. There is a Kähler K3 structure N on X, a Kähler form ω on N, a non-vanishing holomorphic 2-form ψ on N, and an automorphism τ of $H^2(X,Z)$ preserving cup products such that

(2-3) $\tau_C[\psi] = [\varphi]$ and $\tau_C[\omega] = [\Theta]$.

STEP 3. Let W and W′ be small open neighborhoods of $0 := q_M$ and $0' := q_N$ in $\mathcal{D}$, respectively. Let $\mathcal{M} = \{M_s\}_{s \in W}$, $M = M_0$, be the universal family of small deformations of M.

Let $\mathfrak{N} = \{N_t\}_{t \in W'}$, $N = N_{0'}$, be the univeral family of small deformations of N of Lemma 1. By FACT 3, we can find such families $\mathfrak{M}$ and $\mathfrak{N}$. Note that τ_C of Lemma 1 induces a holomorphic automorphism $\bar{\tau}$ of $\mathcal{D} \subset \mathbf{P} \cong \mathbf{P}^{21}$, and that $\bar{\tau}(0') = 0$. Therefore we can replace the parametrization t by s so that the equality

$$(2\text{-}4) \qquad \bar{\tau}(q_{N_s}) = q_{M_s}$$

holds for all $s \in W$.

STEP 4. Let $\{\varphi_s\}_{s \in W}$ be a family of non-zero 2-forms such that φ_s is holomorphic with respect to M_s, depends continuously in s, and satisfies $[\varphi_s]\cdot[\bar{\varphi}_s] = 1$. For $s \in W$, we define $[\Theta]^{\#}_{M_s} \in H^{1,1}_{\mathbf{R}}(M_s)$ by

$$(2\text{-}5) \qquad [\Theta]^{\#}_{M_s} = [\Theta] - \mu_s[\varphi_s] - \bar{\mu}_s[\bar{\varphi}_s]$$

$$\mu_s = ([\Theta]\cdot[\bar{\varphi}_s]) \in \mathbf{C}.$$

Since W is sufficiently small, by continuity argument on s, we can assume that both the (1,1)-part of Θ with respect to M_s and that of ω with respect to N_s are positive definite.

Proposition 2.

(4) $[\Theta]^{\#}_{M_s}\cdot C > 0$, if C is represented by an effective divisor.

(5) $([\Theta]^{\#}_{M_s})^2 > 0$.

(6) If M_s is Kähler, $[\Theta]^{\#}_{M_s}$ is on the Kähler component of the positive cone of M_s.

Proof. (4) Let D be an effective divisor which represents

C. Let $\Theta = \alpha_s + \eta_s + \bar{\alpha}_s$ be the decomposition of Θ into types, where α_s is a (2,0)-form, and η_s is a real positive (1,1)-form. Since $[\varphi_s]\cdot D = [\bar{\varphi}_s]\cdot D = 0$, it follows from (2-5) that

$$[\Theta]^{\#}_{M_s}\cdot D = [\Theta]\cdot D = \int_D \eta_s > 0.$$

(5) Since

$$([\Theta])^2 = (\lambda_s[\varphi_s] + [\Theta]^{\#}_{M_s} + \bar{\lambda}[\bar{\varphi}_s])^2$$
$$= ([\Theta]^{\#}_{M_s})^2 + 2|\lambda_s|^2,$$

we have

$$(2\text{-}6)\quad ([\Theta]^{\#}_{M_s})^2 = ([\Theta])^2 - 2|\lambda_s|^2$$
$$= \int_X \Theta \wedge \Theta - 2|\lambda_s|^2$$
$$= \int_X \eta_s\wedge\eta_s + 2\int_X \alpha_s\wedge\bar{\alpha}_s - 2|\lambda_s|^2.$$

It is clear that $\int_X \eta_s\wedge\eta_s > 0$. Note that

$$\lambda_s = [\Theta]\cdot[\bar{\varphi}_s] = \int_X \alpha_s\wedge\bar{\varphi}_s \quad\text{and}\quad \int_X \varphi_s\wedge\bar{\varphi}_s = 1$$

hold. Therefore, by Schwarz lemma, we have

$$\int_X \alpha_s\wedge\bar{\alpha}_s \geq |\int_X \alpha_s\wedge\bar{\varphi}_s|^2 = |\lambda_s|^2.$$

Hence $([\Theta]^{\#}_{M_s})^2 > 0$ follows from (2-6).

(6) Suppose that we are given a Kähler form ξ_s with respect to M_s. It is enough to show that, for any $t \in [0,1]$, the inequality

$$(t[\xi_s] + (1-t)[\Theta]^{\#}_{M_s})^2 > 0$$

holds, where $[\xi_s]$ is the corresponding cohomology class of ξ_s in $H^2(X,R)$. Note that

$$t[\xi_s] + (1-t)[\Theta]^{\#}_{M_s} = [t\xi_s + (1-t)\Theta]^{\#}_{M_s}$$

and that

$t\xi_s + (1-t)\Theta$ is a Siu form for all $t \in [0,1]$ with respect to M_s. Hence, by (5), $(t[\xi_s] + (1-t)[\Theta]^{\#}_{M_s})^2 > 0$ holds for all $t \in [0,1]$. Thus $[\xi_s]$ and $[\Theta]^{\#}_{M_s}$ are in the same component of the positive cone.∎

Lemma 2. $\tau_{\mathbf{C}}[\omega]^{\#}_{N_s} = [\Theta]^{\#}_{M_s}$

Proof. Note that

$[\omega]^{\#}_{N_s} = [\omega] - \nu_s[\psi_s] - \bar{\nu}_s[\bar{\psi}_s]$,

$\nu_s = [\omega]\cdot[\bar{\psi}_s]$,

$[\Theta]^{\#}_{M_s} = [\Theta] - \mu_s[\varphi_s] - \bar{\mu}_s[\bar{\varphi}_s]$, and

$\mu_s = [\Theta]\cdot[\bar{\varphi}_s]$.

By the equations (2-3) and (2-4), we have $\tau_{\mathbf{C}}[\psi_s] = \kappa_s[\varphi_s]$ for some $\kappa_s \in \mathbf{C}$, $|\kappa_s| = 1$. Since $\tau_{\mathbf{C}}$ preserves cup products, $\nu_s = \kappa_s\mu_s$ holds. Then the lemma follows immediately from (2-3).∎

Lemma 3. There is a dense subset $A \subset W$ such that for every $s \in A$, both N_s and M_s are algebraic.

Proof. A K3 structure Y is algebraic if and only if there is an element $\xi \in H^{1,1}_{\mathbf{R}}(Y) \cap H^2(Y,\mathbf{Z})$ such that $\xi^2 > 0$. Since this is a condition only on the periods and the intersection forms, we infer that N_s is algebraic if and only if so is M_s.∎

Now we want to check the conditions in FACT 4. Set M =

M_s, $N = N_s$, and $\sigma = \tau$ in FACT 4. Then σ_C sends $H^{p,q}(N_s)$ to $H^{p,q}(M_s)$ by (2-4). Hence (iii) holds. For any point s of A in Lemma 3, (iv) holds by Lemma 2 and Proposition 2 (6), since ω is a Siu form on N_s. To prove that (v) holds, take any cohomology class represented by an effective divisor C with $C^2 = -2$ with respect to N_s. Since τ preserves cup products, $\tau(C)^2 = -2$. Therefore, by the Riemann-Roch theorem, either $\tau(C)$ or $-\tau(C)$ is represented by an effective divisor with respect to N_s. Suppose that $-\tau(C)$ is represented by an effective divisor. Then, by Proposition 2 (4), we have

$$[\Theta]^{\#}_{M_s} \cdot [-\tau(C)] > 0.$$

On the other hand, we have, by Lemma 2,

$$[\Theta]^{\#}_{M_s} \cdot [-\tau(C)] = -\tau[\omega]^{\#}_{N_s} \cdot \tau[C] = -[\omega]^{\#}_{N_s} \cdot [C] < 0,$$

because ω is a Siu form on N_s. This is a contradiction. Thus $\tau(C)$ is represented by an effective divisor. This proves (v).

Now we can apply FACT 4. For any $s \in A$, there is a diffeomorphism $g_s : X \to X$ with $g_s^* = \tau$ such that g_s is a biholomorphic map of M_s onto N_s. Then the graph Γ_s of g_s is a complex analytic subvariety with respect to $N_s \times M_s$.

Lemma 4. The volume of the graph Γ_s is uniformly bounded as s approches 0.

Proof. Since $N = N_0$ is a Kähler structure, by Kodaira-

Spencer [5], there is a differentiable family of 2-forms $(\omega_s)_{s \in W}$ such that every ω_s is a Kähler form on N_s. Then, for every $s \in W$, $p_1^*\omega_s + p_2^*\Theta$ is a Siu form on $X \times X$ with respect to $N_s \times M_s$, where $p_\nu : X \times X \to X$, $\nu = 1,2$, are the projections to the ν-th component. Then the volume of Γ_s can be measured by the (1,1)-part of $p_1^*\omega_s + p_2^*\Theta$ with respect to $N_s \times M_s$. Thus

$$\mathrm{Vol}(\Gamma_s) = \int_{\Gamma_s} (p_1^*\omega_s + \Theta_s)^2,$$

where Θ_s is the (1,1)-part of $p_2^*\Theta$ with respect to M_s. Letting $p_2^*\Theta = \Theta_s + \eta_s + \bar{\eta}_s$ with a (2,0)-form η_s on $N_s \times M_s$, we have

$$\begin{aligned}\mathrm{Vol}(\Gamma_s) &= \int_{\Gamma_s} (p_1^*\omega_s + p_2^*\Theta - \eta_s - \bar{\eta}_s)^2 \\ &= \int_{\Gamma_s} (p_1^*\omega_s + p_2^*\Theta)^2 - 2\int_{\Gamma_s} (p_1^*\omega_s + p_2^*\Theta)\wedge(\eta_s + \bar{\eta}_s) \\ &\quad + \int_{\Gamma_s} (\eta_s + \bar{\eta}_s)^2\end{aligned}$$

Since $p_1^*\omega_s$ and Θ_s are (1,1)-forms on $N_s \times M_s$, we have

$$\begin{aligned}&\int_{\Gamma_s} (p_1^*\omega_s + p_2^*\Theta)\wedge(\eta_s + \bar{\eta}_s) \\ &\quad = \int_{\Gamma_s} p_2^*\Theta\wedge(\eta_s + \bar{\eta}_s) \\ &\quad = \int_{\Gamma_s} (\eta_s + \bar{\eta}_s)^2 \\ &\quad = 2\int_{\Gamma_s} \eta_s\wedge\bar{\eta}_s\end{aligned}$$

Hence

$$\begin{aligned}\mathrm{Vol}(\Gamma_s) &= \int_{\Gamma_s} (p_1^*\omega_s + p_2^*\Theta)^2 - 2\int_{\Gamma_s} \eta_s\wedge\bar{\eta}_s \\ &\le \int_{\Gamma_s} (p_1^*\omega_s + p_2^*\Theta)^2 \\ &= \int_X (\omega_s + (g_s^{-1*}\Theta))^2 =: I(s)\end{aligned}$$

Since $(g_s^{-1})^*\Theta$ is a d-closed real 2-from which represents $\tau_{\mathbf{C}}^{-1}([\Theta])$, the value $I(s)$ depends only on ω_s. Since ω_s varies continuously on s as $s \to 0$, we conclude that $\mathrm{Vol}(\Gamma_s)$ is uniformly bounded.∎

By a theorem Bishop [1], it follows from Lemma 4 that the limit set Γ_0 of Γ_s defines an analytic subvariety with respect to $N_0 \times M_0$. Choosing a suitable irreducible component of Γ_0, we can prove the following

Proposition 3. There is a biholomorphic mapping of N_0 onto M_0.

The proof of the proposition is the same as Burns-Rapoport [2,pp.248-250]. The details are omitted here. Since N_0 is Kähler, the theorem follows from Proposition 3.

§3. The outline of the proof of FACT 1

Let S be a compact complex surface. For $k \in \mathbf{Z}$, we let A_k denote the real Hilbert space of all real 2-currents on S whose coefficients are in the Sobolev k-space. Let $|\ |_k$ be the norm on A_k which is defined by using partition of unity and Fourier transformations on a torus. The pairing

$$(\xi, \eta) = \int_S \xi \wedge \eta$$

of 2-forms extends uniquely to a complete pairing on $A_k \times A_{-k}$. By means of this pairing, we can identify $(A_{-k}, |\ |_{-k})$ with the dual space of $(A_k, |\ |_k)$ with the weak topology.

Let $\{U_\nu\}_{\nu=1}^{l}$ and $\{U_\nu^*\}_{\nu=1}^{l}$ be opne coverings of S such that U_ν is a relatively compact subset in U_ν^* for all ν. Put

$$S = \{ \lambda = (\lambda_1, \lambda_2) \in \mathbf{C}^2 : |\lambda_1|^2 + |\lambda_2|^2 = 1 \}.$$

For $1 \leq \nu \leq l$, $p \in U_\nu$, and $\lambda \in S$, we denote by $T_{\nu,p,\lambda}$ the (1,1)-current on S such that

$$T_{\nu,p,\lambda}(\varphi) = \Sigma_{i,j=1}^{2} \varphi_{i,j}(p)\lambda_i \bar{\lambda}_j$$

for all real smooth (1,1)-forms

$$\varphi = \sqrt{-1}\ \Sigma_{i,j=1}^{2} \varphi_{i,j} dz_i^\nu \wedge d\bar{z}_j^\nu$$

on S. Since

$$\Sigma_{\xi \in \mathbf{Z}^4} (1+\|\xi\|^2)^{-(2+\varepsilon)} < +\infty$$

for all $\varepsilon > 0$, we see easily from the definition that $T_{\nu,p,\lambda}$ is an element of A_{-3}. Hence $T_{\nu,p,\lambda} \in A_{-k}$ for all $k \geq 3$. Suppose that $k \geq 4$ in what follows. From the resonance theorem, it follows that

$$\lambda_1 := \sup_{1 \leq \nu \leq l,\ p \in U_\nu,\ \lambda \in S} |T_{\nu,p,\lambda}|_{-k}$$

is finite. We fix a positive definite hermitian form ω on S. Put

$$\lambda_2 := \inf_{1 \leq \nu \leq l,\ p \in U_\nu,\ \lambda \in S} T_{\nu,p,\lambda}(\omega)$$

Obviously, λ_2 is a finite positive number. Let P denote the set of all positive semi-definite (1,1)-forms. Put

$$E = \left\{ \eta \in A_{-k} : \begin{array}{l} \text{(i) } \eta \text{ is of type } (1,1), \\ \text{(ii) } |\eta|_{-k} \leq \lambda_1, \\ \text{(iii) } \eta(\omega) \geq \lambda_2, \end{array} \right.$$

(iv) $\eta(\xi) \geq 0$ for $\forall \xi \in P$,

and

$F = \{ \eta \in A_{-k} : \eta = d\zeta$, where ζ is a 1-current $\}$.

Then both E and F are convex and closed. Moreover E is compact. Obviously, $T_{\nu,p,\lambda} \in E$ and $0 \notin E$.

Lemma 5. If the first Betti number b_1 of S is even, then $E \cap F = \emptyset$.

Proof. Suppose that $u \in E \cap F$. Since $u \in F$, there is a 1-current ζ on S such that $u = d\zeta$. Put $\zeta = \beta + \bar{\beta}$, where β is a (1,0)-current. Obviously $u = \bar{\partial}\beta + \partial\bar{\beta}$, $\partial\beta = \bar{\partial}\bar{\beta} = 0$. By the assumption $b_1 \equiv 0 \bmod 2$ and by Kashiwara's lemma (see [4,pp.124-126]), there is a distribution η on S such that $\partial\bar{\beta} = \partial\bar{\partial}\eta$. Hence $u = \sqrt{-1}\,\partial\bar{\partial}\tau$, where $\tau = \sqrt{-1}(\bar{\eta} - \eta)$. Since u is positive, τ is a plurisubharmonic function. Since S is compact, τ reduces to a constant. Thus $u = 0$. This contradicts $0 \notin E$. ∎

Lemma 6. If $E \cap F = \emptyset$, then there is a Siu form of class C^{k-3}.

Proof. Since E and F are closed convex with $E \cap F = \emptyset$, and since E is compact, there is a continuous linear functional

$$f : A_{-k} \to R$$

such that

$$\sup_F f \leq c_1 < c_2 \leq \inf_E f$$

for some real constants c_1 and c_2 by the separation theorem. F being a subspace, we have easily $f|F = 0$. Since $(A_k, |\ |_k)$ is the dual space of $(A_{-k}, |\ |_{-k})$ with the weak topology, there is an element $\xi \in A_k$ such that $f(\eta) = (\xi, \eta)$ for all $\eta \in A_{-k}$. Since $k \geq 4$, we can assume that ξ is a form of class C^{k-3} by the Sobolev lemma. Obviously $f|F = 0$ implies $d\xi = 0$. Let

$$\xi^{1,1} = \Sigma_{j,k}\xi_{j,k}dz_j^\nu \wedge d\bar{z}_k^\nu$$

be the (1,1)-part of ξ. From $T_{\nu,p,\lambda} \in E$ it follows that

$$0 \leq c_1 < c_2 \leq \inf_E f \leq f(T_{\nu,p,\lambda}) = \Sigma_{j,k}\xi_{j,k}(p)\lambda^j\bar{\lambda}^k$$

Thus ξ is a Siu form of class C^{k-3}.∎

FACT 1 follows from Lemmas 5 and 6. As an appendix, we consider the converse question. We shall show

Lemma 7. If a compact complex surface S admits a Siu form of class C^1, then the first Betti number b_1 of S is even.

Proof. Suppose that b_1 is odd. Let Θ denote the Siu form. Write Θ as $\Theta = \alpha + \eta + \bar{\alpha}$, where α is a (2,0)-form and η is a real positive definite (1,1)-form. Suppose that C is a curve on S. Then

$$[\Theta]\cdot[C] = \int_C \eta > 0$$

Hence every effective divisor on S is not homologous to zero. This imples that S is not of Class VI. Hence S is of Class VII. Then the intersection form is negative definite. But we have

$$[\Theta]\cdot[\Theta] = \int_S(\alpha + \eta + \bar{\alpha})\wedge(\alpha + \eta + \bar{\alpha})$$
$$= 2\int_S\alpha\wedge\bar{\alpha} + \int_S\eta\wedge\eta > 0$$

This is a contradiction.∎

Thus we have the following

Proposition 4. Let S be a compact complex surface. Then the followings are equivallent;

(i) the first Betti number of S is even,

(ii) $E \cap F = \emptyset$,

(iii) there is a Siu form of class C^1,

(iv) there is a Siu form of class C^k, $k \geq 1$,

(v) S is Kähler.

As a corollary to this proposition, we have

Proposition 5. Let S be a compact complex surface. Then the followings are equivallent;

(i) the first Betti number of S is odd,

(ii) $E \cap F \neq \emptyset$,

(iii) there is a non-trivial d-exact positive (1,1)-current.

(iv) S is non-Kähler.

I understand that similar fact as Proposition 5 is known to I.Enoki earlier.

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