Every K3 surface is Kähler
(by Y.-T. Siu)

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§0. A K3 surface means a simply connected compact complex
manifold of complex dimension two whose canonical bundle is trivial.
Y.-T. Siu proved

Theorem [7, Theorem 3.3] Every K3 surface is Kähler.
The aim of this report is to introduce the reader to his proof.
In §1, we shall list up 4 facts which will be used to prove the
theorem. The first one is on the existence of a d-closed real
2-form whose (1,1)-part is positive definite, which we shall
call a Siu form. I heard that this fact is already known to
differential geometers. But Siu is the first who applied this
to studying K3 surfaces. The other three facts are known to
the specialists in K3 surfaces. In §2, the proof is described.
In §3, we shall discuss the existence of Siu forms in a slightly
generalized situation.

§1. Known facts used in the proof.
The following four facts are used to prove the theorem.

FACT 1. (The existence of Siu forms) Let \( M \) be a K3 surface
and \( k \) an integer \( \geq 1 \). Then there exists a real \( C^k \) d-closed
2-form on $M$ whose $(1,1)$-part is positive definite at every point on $M$.

This fact will be proved in §3.

Let $X$ be a differentiable manifold which is diffeomorphic to a K3 surface. Then, since $X$ is simply connected, there are natural inclusions

$$H^*(X;\mathbb{Z}) \subset H^*(X;\mathbb{R}) \subset H^*(X;\mathbb{C}).$$

On $H^2(X;\mathbb{Z})$, the cup products define the inner products with signature $(3,19)$.

**FACT 2.** (i) Suppose that an element $a \in H^2(X;\mathbb{C})$ satisfies $a^2 = 0$ and $a \bar{a} > 0$.

Put $\Delta(a) = \{ c \in H^2(X;\mathbb{Z}) : a \cdot c = 0, c^2 = -2 \}$.

(ii) Suppose that an element $b \in H^2(X;\mathbb{R})$ satisfies $b = 0$, $b^2 > 0$, and $b \cdot c \neq 0$ for all $c \in \Delta(a)$.

Then there is a kähler K3 structure $N$ on $X$, a Kähler form $\omega_N$, a non-vanishing holomorphic 2-form $\eta_N$, and an automorphism $\tau$ of $H^2(X;\mathbb{Z})$ preserving cup products such that $\tau_C[\eta_N] = a$ and $\tau_C[\omega_N] = b$, where $\tau_C = \tau \circ C$, and $[\ ]$ indicates the cohomology class.

This fact is due to Todorov [8] and Looijenga [6]. Yau's result
on the existence of Einstein-Kähler metric is used prove this
fact.

Let $M$ be a K3 structure on $X$, and $\varphi_M$ a non-zero holomorphic
2-form. Put $P = (H^2(X,\mathbb{C}) - \{0\})/\mathbb{C}^* \cong \mathbb{P}^{21}$. Since $C \cdot \varphi_M$ is a
complex 1-dimensionnal subspace in $H^2(X,\mathbb{C})$, and since $\varphi_M^2 = 0$, $\varphi_M$ defines a point in
$$O = \{ z \in P : z \cdot z = 0, \quad z \cdot \bar{z} > 0 \},$$
where the product is the inner product defined by the intersection
form on $H^2(X,\mathbb{Z})$. The point $q_M \in O$ defined by $\varphi_M$ is called
the period of the complex structure $M$. The mapping $M \mapsto q_M$
is called the period mapping. Every complex structure $M$ on
$X$ defines a Hodge structure
on $H^2(X,\mathbb{C})$;
$$H^2(X,\mathbb{C}) = H^{2,0}(M) \oplus H^{1,1}(M) \oplus H^{0,2}(M).$$
Put
$$H^{1,1}_R(M) = H^2(X,\mathbb{R}) \cap H^{1,1}(M).$$
The set
$$\{ x \in H^{1,1}_R(M) : x^2 > 0 \}$$
is called the positive cone of $M$. Since the signature of the
intersection form is $(3,19)$, the positive cone has 2 connected
components. If $M$ is Kähler, every Kähler metric on $M$ is
contained in the same component of the positive cone, which is
called the Kähler component.
FACT 3. (The local Torelli Theorem) All small deformations of a K3 surface are parametrized effectively and completely by the period mapping.

This is due to Andreotti-Weil, see Kodaira [3].

FACT 4. (The Torelli Theorem for Algebraic K3 surfaces)
Let $M$ and $N$ be algebraic K3 structures on $X$. Suppose that there is an automorphism $\sigma$ of $H^2(X,\mathbb{Z})$ preserving cup products which satisfies the following three conditions:

(iii) $\sigma_C := \sigma \circ C$ sends $H^p,q(N)$ to $H^p,q(M)$, $p+q = 2$,

(iv) $\sigma_C$ preserves the Kähler components,

(v) Every cohomology class represented by an effective divisor $C$ on $N$ with $C^2 = -2$ is sent by $\sigma$ to a cohomology class represented by an effective divisor on $M$.

Then there is a biholomorphic mapping $g : M \to N$ such that $g^* = \sigma$.

This fact is due to Burns-Rapoport [2].

§2. The proof of the theorem
Suppose that we are given a K3 structure $M$ on $X$. $M$ admits a nowhere vanishing holomorphic 2-form $\Omega$. 
STEP 1. By FACT 1, there is a Siu form \( \Theta \) with respect to \( M \). Let \([\mathcal{F}]\) and \([\Theta]\) denote the corresponding cohomology classes in \( H^2(X,\mathbb{C}) \) and \( H^2(X,\mathbb{R}) \), respectively. Since the equality
\[
H_{\mathbb{R}}^{1,1}(M) = \{ c \in H^2(X,\mathbb{R}) : [\mathcal{F}] \cdot c = 0 \}
\]
holds by definition, we see that
\[
(2-1) \quad [\Theta]_M^\# := [\Theta] - \lambda [\mathcal{F}] - \tau [\overline{\mathcal{F}}]
\]
\[
\lambda = ([\Theta] \cdot [\mathcal{F}]/([\mathcal{F}] \cdot [\overline{\mathcal{F}}]) \in \mathbb{C}
\]
is an element of \( H_{\mathbb{R}}^{1,1}(M) \). Note that \( \Theta - \lambda \mathcal{F} \) is also a Siu form with respect to \( M \). We replace \( \Theta \) by \( \Theta - \lambda \mathcal{F} \). Thus we can assume in what follows that the equality
\[
(2-2) \quad [\Theta]_M^\# = [\Theta]
\]
holds.

STEP 2. We want to apply FACT 2. Put \( a = [\mathcal{F}] \) and \( b = [\Theta] \). We check that \( a \) and \( b \) satisfy the conditions (i) and (ii). Since \([\mathcal{F}]\) is of type \((2,0)\), we have \( a^2 = [\mathcal{F}]^2 = \int_M \mathcal{F} \cdot \mathcal{F} = 0 \). Similarly \( a \cdot \overline{a} = [\mathcal{F}] \cdot [\overline{\mathcal{F}}] = \int_M \mathcal{F} \cdot \overline{\mathcal{F}} > 0 \). Thus (i) is satisfied. Let us check (ii). It is clear by the definition that \( a \cdot b = [\mathcal{F}] \cdot [\Theta] = 0 \) holds. Since
\[
\Delta(a) = \Delta([\mathcal{F}]) = \{ c \in H_{\mathbb{R}}^{1,1}(M) \cap H^2(X,\mathbb{Z}) : c^2 = -2 \}
\]
every elements of \( \Delta(a) \) is the Chern class of a line bundle. By the Riemann–Roch theorem, for every line bundle \( \xi \) with \( (c_1(\xi))^2 = -2 \), either \( \xi \) or \( \xi^{-1} \) is defined by an effective
divisor on $M$. Thus that $b^2 > 0$ and $bc \neq 0$ for $\nu_c \in \Delta(a)$ follow from (1) and (2) of the following proposition.

Proposition 1 ([7, Prop.3.1])
(1) $[\Theta] \cdot C > 0$, if $C$ is represented by an effective divisor.
(2) $([\Theta])^2 > 0$.
(3) If $M$ is Kähler, then $[\Theta]$ is on the Kähler component of the positive cone of $M$.

The last statement (3) is used in STEP 4 (see Prop.2 (6)). The proof is an easy calculation of forms under type considerations. Since the similar calculations also appear in STEP 4, we omit the details here.

Now we can apply FACT 2 to our situation and obtain

Lemma 1. There is a Kähler K3 structure $N$ on $X$, a Kähler form $\omega$ on $N$, a non-vanishing holomorphic 2-form $\psi$ on $N$, and an automorphism $\tau$ of $H^2(X,\mathbb{Z})$ preserving cup products such that

$$\cap_C[\psi] = [\Psi] \text{ and } \cap_C[\omega] = [\Theta].$$

STEP 3. Let $W$ and $W'$ be small open neighborhoods of $0 := q_M$ and $0' := q_N$ in $D$, respectively. Let $M = (M_s)_{s \in W}$, $M = M_0$, be the universal family of small deformations of $M$. 
Let $\mathcal{N} = \{ N_t \}_{t \in W'}$, $N = N_0'$, be the universal family of small deformations of $N$ of Lemma 1. By FACT 3, we can find such families $M$ and $N$. Note that $\tau_C$ of Lemma 1 induces a holomorphic automorphism $\tau_C$ of $\mathcal{O} \subset \mathbb{P}^2$, and that $\tau_C(0') = 0$. Therefore we can replace the parametrization $t$ by $s$ so that the equality

$$\tau(q_{N_s}) = q_{M_s}$$

holds for all $s \in W$.

STEP 4. Let $(\varphi_s)_s \in W$ be a family of non-zero 2-forms such that $\varphi_s$ is holomorphic with respect to $M_s$, depends continuously in $s$, and satisfies $[\varphi_s] \cdot [\bar{\varphi}_s] = 1$. For $s \in W$, we define

$$[\Theta]_{M_s}^\# \in H_{\mathbb{R}}^{1,1}(M_s)$$

by

$$[\Theta]_{M_s}^\# = [\Theta] - \nu_s[\varphi_s] - \bar{\nu}_s[\bar{\varphi}_s]$$

$$\nu_s = ([\Theta] \cdot [\varphi_s]) \in \mathbb{C}.$$

Since $W$ is sufficiently small, by continuity argument on $s$, we can assume that both the $(1,1)$-part of $\Theta$ with respect to $M_s$ and that of $\omega$ with respect to $N_s$ are positive definite.

Proposition 2.

(4) $[\Theta]_{M_s}^\# \cdot C > 0$, if $C$ is represented by an effective divisor.

(5) $([\Theta]_{M_s}^\#) = 0$.

(6) If $M_s$ is Kähler, $[\Theta]_{M_s}^\#$ is on the Kähler component of the positive cone of $M_s$.

Proof. (4) Let $D$ be an effective divisor which represents
C. Let $\Theta = \alpha_s + \eta_s + \bar{\alpha}_s$ be the decomposition of $\Theta$ into types, where $\alpha_s$ is a $(2,0)$-form, and $\eta_s$ is a real positive $(1,1)$-form. Since $[\varphi_s]_D = [\bar{\varphi}_s]_D = 0$, it follows from (2-5) that

$$[\Theta]^\#_{M_s} \cdot D = [\Theta]_{M_s} \cdot D = \int_D \eta_s > 0.$$ 

(5) Since

$$([\Theta]^\#_{M_s})^2 = (\lambda_s[\varphi_s] + [\Theta]^\#_{M_s} + \chi[\bar{\varphi}_s])^2$$

$$= ([\Theta]^\#_{M_s})^2 + 2|\lambda_s|^2,$$

we have

$$(2-6) \quad ([\Theta]^\#_{M_s})^2 = ([\Theta]^\#_{M_s})^2 - 2|\lambda_s|^2$$

$$= \int_X \Theta \wedge \Theta - 2|\lambda_s|^2$$

$$= \int_X \eta_s \wedge \eta_s + 2\int_X \alpha_s \wedge \bar{\alpha}_s - 2|\lambda_s|^2.$$

It is clear that $\int_X \eta_s \wedge \eta_s > 0$. Note that

$$\lambda_s = \Theta \cdot [\varphi_s] = \int_X \alpha_s \wedge \bar{\varphi}_s$$

and $\int_X \varphi_s \wedge \bar{\varphi}_s = 1$

hold. Therefore, by Schwarz lemma, we have

$$\int_X \alpha_s \wedge \bar{\alpha}_s \geq |\int_X \alpha_s \wedge \bar{\phi}_s|^2 = |\lambda_s|^2.$$

Hence $([\Theta]^\#_{M_s})^2 > 0$ follows from (2-6).

(6) Suppose that we are given a Kähler form $\xi_s$ with respect to $M_s$. It is enough to show that, for any $t \in [0,1]$, the inequality

$$(t[\xi_s] + (1-t)[\Theta]^\#_{M_s})^2 > 0$$

holds, where $[\xi_s]$ is the corresponding cohomology class of $\xi_s$ in $H^2(X,\mathbb{R})$. Note that

$$t[\xi_s] + (1-t)[\Theta]^\#_{M_s} = t\xi_s + (1-t)[\Theta]^\#_{M_s}$$

and that

$$t\xi_s + (1-t)\Theta$$

is a Siu form for all $t \in [0,1]$ with respect
to $M_s$. Hence, by (5), $(t[\xi_s] + (1-t)[\Theta]_{M_s})^2 > 0$ holds for all $t \in [0,1]$. Thus $[\xi_s]$ and $[\Theta]_{M_s}$ are in the same component of the positive cone.

Lemma 2. $\tau_C[\omega]_{N_s}^\# = [\Theta]_{M_s}^\#$

Proof. Note that

$[\omega]_{N_s}^\# = [\omega] - \nu_s[\psi_s] - \bar{\nu}_s[\bar{\psi}_s],$

$\nu_s = [\omega] \cdot [\bar{\psi}_s],$

$[\Theta]_{M_s}^\# = [\Theta] - \mu_s[\phi_s] - \bar{\mu}_s[\bar{\phi}_s],$ and

$\mu_s = [\Theta] \cdot [\phi_s].$

By the equations (2-3) and (2-4), we have $\tau_C[\psi_s] = \chi_s[\phi_s]$ for some $\chi_s \in \mathbb{C}, |\chi_s| = 1$. Since $\tau_C$ preserves cup products, $\nu_s = \chi_s \mu_s$ holds. Then the lemma follows immediately from (2-3).

Lemma 3. There is a dense subset $A \subset W$ such that for every $s \in A$, both $N_s$ and $M_s$ are algebraic.

Proof. A K3 structure $Y$ is algebraic if and only if there is an element $\xi \in H^{1,1}_R(Y) \cap H^2(Y, \mathbb{Z})$ such that $\xi^2 > 0$. Since this is a condition only on the periods and the intersection forms, we infer that $N_s$ is algebraic if and only if so is $M_s$.

Now we want to check the conditions in FACT 4. Set $M =$
Ms, N = Ns, and σ = τ in FACT 4. Then \( a^*_C \) sends \( H^{p,q}(N_s) \) to \( H^{p,q}(M_s) \) by (2-4). Hence (iii) holds. For any point \( s \) of \( A \) in Lemma 3, (iv) holds by Lemma 2 and Proposition 2 (6), since \( w \) is a Siu form on \( N_s \). To prove that (v) holds, take any cohomology class represented by an effective divisor \( C \) with \( C^2 = -2 \) with respect to \( N_s \). Since \( τ \) preserves cup products, \( τ(C)^2 = -2 \). Therefore, by the Riemann-Roch theorem, either \( τ(C) \) or \( -τ(C) \) is represented by an effective divisor with respect to \( N_s \). Suppose that \( -τ(C) \) is represented by an effective divisor. Then, by Proposition 2 (4), we have
\[
[θ]^{\#}_{M_s} \cdot [-τ(C)] > 0.
\]
On the other hand, we have, by Lemma 2,
\[
[θ]^{\#}_{M_s} \cdot [-τ(C)] = -τ[ω]^{\#}_{N_s} \cdot [C] = -[ω]^{\#}_{N_s} \cdot [C] < 0,
\]
because \( w \) is a Siu form on \( N_s \). This is a contradiction. Thus \( τ(C) \) is represented by an effective divisor. This proves (v).

Now we can apply FACT 4. For any \( s \in A \), there is a diffeomorphism \( g_s : X \to X \) with \( g_s^* = τ \) such that \( g_s \) is a biholomorphic map of \( M_s \) onto \( N_s \). Then the graph \( Γ_s \) of \( g_s \) is a complex analytic subvariety with respect to \( N_s \times M_s \).

Lemma 4. The volume of the graph \( Γ_s \) is uniformly bounded as \( s \) approaches 0.

Proof. Since \( N = N_0 \) is a Kähler structure, by Kodaira-
Spencer [5], there is a differentiable family of 2-forms \( \omega_s \in \mathcal{W} \) such that every \( \omega_s \) is a Kähler form on \( N_s \). Then, for every \( s \in \mathcal{W} \), \( p_1^*\omega_s + p_2^*\Theta \) is a Siu form on \( X \times X \) with respect to \( N_s \times M_s \), where \( p_v : X \times X \to X \), \( v = 1, 2 \), are the projections to the \( v \)-th component. Then the volume of \( \Gamma_s \) can be measured by the \( (1,1) \)-part of \( p_1^*\omega_s + p_2^*\Theta \) with respect to \( N_s \times M_s \).

Thus

\[
\text{Vol}(\Gamma_s) = \int_{\Gamma_s} (p_1^*\omega_s + \Theta_s)^2,
\]

where \( \Theta_s \) is the \( (1,1) \)-part of \( p_2^*\Theta \) with respect to \( M_s \). Letting \( p_2^*\Theta = \Theta_s + \eta_s + \bar{\eta}_s \) with a \((2,0)\)-form \( \eta_s \) on \( N_s \times M_s \), we have

\[
\text{Vol}(\Gamma_s) = \int_{\Gamma_s} (p_1^*\omega_s + p_2^*\Theta - \eta_s - \bar{\eta}_s)^2
= \int_{\Gamma_s} (p_1^*\omega_s + p_2^*\Theta)^2 - 2\int_{\Gamma_s} (p_1^*\omega_s + p_2^*\Theta)\Lambda(\eta_s + \bar{\eta}_s)
+ \int_{\Gamma_s} (\eta_s + \bar{\eta}_s)^2
\]

Since \( p_1^*\omega_s \) and \( \Theta_s \) are \((1,1)\)-forms on \( N_s \times M_s \), we have

\[
\int_{\Gamma_s} (p_1^*\omega_s + p_2^*\Theta)\Lambda(\eta_s + \bar{\eta}_s)
= \int_{\Gamma_s} p_2^*\Theta\Lambda(\eta_s + \bar{\eta}_s)
= \int_{\Gamma_s} (\eta_s + \bar{\eta}_s)^2
= 2\int_{\Gamma_s} \eta_s\Lambda\bar{\eta}_s
\]

Hence

\[
\text{Vol}(\Gamma_s) = \int_{\Gamma_s} (p_1^*\omega_s + p_2^*\Theta)^2 - 2\int_{\Gamma_s} \eta_s\Lambda\bar{\eta}_s
\leq \int_{\Gamma_s} (p_1^*\omega_s + p_2^*\Theta)^2
= \int_X (\omega_s + (g_s^{-1}\Theta)) =: I(s)
\]

Since \((g_s^{-1})^*\Theta\) is a d-closed real 2-form which represents \( \tau_C^{-1}(\Theta) \), the value \( I(s) \) depends only on \( \omega_s \). Since \( \omega_s \) varies continuously
on $s$ as $s \to 0$, we conclude that $\text{Vol}(\Gamma_s)$ is uniformly bounded.

By a theorem Bishop [1], it follows from Lemma 4 that the limit set $\Gamma_0$ of $\Gamma_s$ defines an analytic subvariety with respect to $N_0 \times M_0$. Choosing a suitable irreducible component of $\Gamma_0$, we can prove the following

Proposition 3. There is a biholomorphic mapping of $N_0$ onto $M_0$.

The proof of the proposition is the same as Burns-Rapoport [2, pp.248-250]. The details are omitted here. Since $N_0$ is Kähler, the theorem follows from Proposition 3.

§3. The outline of the proof of FACT 1

Let $S$ be a compact complex surface. For $k \in \mathbb{Z}$, we let $A_k$ denote the real Hilbert space of all real 2-currents on $S$ whose coefficients are in the Sobolev $k$-space. Let $\lVert \cdot \rVert_k$ be the norm on $A_k$ which is defined by using partition of unity and Fourier transformations on a torus. The pairing

$$(\xi, \eta) = \int_S \xi \wedge \eta$$

of 2-forms extends uniquely to a complete pairing on $A_k \times A_{-k}$. By means of this pairing, we can identify $(A_{-k}, \lVert \cdot \rVert_{-k})$ with the dual space of $(A_k, \lVert \cdot \rVert_k)$ with the weak topology.
Let \( \{ U_y \}_y^1 = 1 \) and \( \{ U^*_y \}_y^1 = 1 \) be open coverings of \( S \) such that \( U_y \) is a relatively compact subset in \( U^*_y \) for all \( y \). Put

\[
S = \{ \lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2 : |\lambda_1|^2 + |\lambda_2|^2 = 1 \}.
\]

For \( 1 \leq y \leq 1, \ p \in U_y \), and \( \lambda \in S \), we denote by \( T_{y,p,\lambda} \) the \((1,1)\)-current on \( S \) such that

\[
T_{y,p,\lambda}(\varphi) = \sum_{i,j=1}^{2} \partial_\bar{\partial} \varphi_{i,j} \lambda_i \lambda_j
\]

for all real smooth \((1,1)\)-forms

\[ \varphi = \sqrt{-1} \sum_{i,j=1}^{2} \partial_\bar{\partial} \varphi_{i,j} dz^i \wedge d\bar{z}^j \]

on \( S \). Since

\[
\sum_{\xi \in \mathbb{Z}^4} (1 + \|\xi\|^2)^{-(2+\varepsilon)} < +\infty
\]

for all \( \varepsilon > 0 \), we see easily from the definition that \( T_{y,p,\lambda} \) is an element of \( A_{-3} \). Hence \( T_{y,p,\lambda} \in A_{-k} \) for all \( k \geq 3 \).

Suppose that \( k \geq 4 \) in what follows. From the resonance theorem, it follows that

\[
\lambda_1 := \sup_{1 \leq y \leq 1, \ p \in U_y, \ \lambda \in S} |T_{y,p,\lambda}|^{-k}
\]

is finite. We fix a positive definite hermitian form \( \omega \) on \( S \). Put

\[
\lambda_2 := \inf_{1 \leq y \leq 1, \ p \in U_y, \ \lambda \in S} T_{y,p,\lambda}(\omega)
\]

Obviously, \( \lambda_2 \) is a finite positive number. Let \( P \) denote the set of all positive semi-definite \((1,1)\)-forms. Put

\[
E = \left\{ \eta \in A_{-k} : (i) \ \eta \text{ is of type } (1,1), \right. \\
\left. (ii) |\eta|_{-k} \leq \lambda_1, \right. \\
\left. (iii) \ \eta(\omega) \geq \lambda_2. \right. 
\]

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Lemma 5. If the first Betti number $b_1$ of $S$ is even, then $E \cap F = \emptyset$.

Proof. Suppose that $u \in E \cap F$. Since $u \in F$, there is a 1-current $\zeta$ on $S$ such that $u = d\zeta$. Put $\zeta = \beta + \beta$, where $\beta$ is a $(1,0)$-current. Obviously $u = \beta \beta + \beta \beta$, $\beta \beta = \beta \beta = 0$.

By the assumption $b_1 = 0 \mod 2$ and by Kashiwara's lemma (see [4, pp. 124-126]), there is a distribution $\eta$ on $S$ such that $\beta \eta = \beta \eta$. Hence $u = \sqrt{-1} \beta \eta$, where $\tau = \sqrt{-1} (\eta - \eta)$. Since $u$ is positive, $\tau$ is a plurisubharmonic function. Since $S$ is compact, $\tau$ reduces to a constant. Thus $u = 0$. This contradicts $0 \notin E$. \qed

Lemma 6. If $E \cap F = \emptyset$, then there is a Siu form of class $\mathcal{C}^{k-3}$.

Proof. Since $E$ and $F$ are closed convex with $E \cap F = \emptyset$, and since $E$ is compact, there is a continuous linear functional $f : A_{-k} \to \mathbb{R}$ such that
\[ \sup_{F} f \leq c_1 < c_2 \leq \inf_{E} f \]

for some real constants \( c_1 \) and \( c_2 \) by the separation theorem. If \( F \) being a subspace, we have easily \( f|F = 0 \). Since \((A_k, \| \cdot \|_k)\) is the dual space of \((A_{-k}, \| \cdot \|_{-k})\) with the weak topology, there is an element \( \xi \in A_k \) such that \( f(\eta) = (\xi, \eta) \) for all \( \eta \in A_{-k} \). Since \( k \geq 4 \), we can assume that \( \xi \) is a form of class \( C^{k-3} \) by the Sobolev lemma. Obviously \( f|F = 0 \) implies \( d\xi = 0 \). Let

\[ \xi^{1,1} = \sum_{j,k} \xi_{j,k} dz_j \wedge dz_k \]

be the \((1,1)\)-part of \( \xi \). From \( T_{v,p,\lambda} \in E \) it follows that

\[ 0 \leq c_1 < c_2 \leq \inf_{E} f \leq f(T_{v,p,\lambda}) = \sum_{j,k} \xi_{j,k}(p) \lambda^j \lambda^k \]

Thus \( \xi \) is a Siu form of class \( C^{k-3} \).

FACT 1 follows from Lemmas 5 and 6. As an appendix, we consider the converse question. We shall show

Lemma 7. If a compact complex surface \( S \) admits a Siu form of class \( C^1 \), then the first Betti number \( b_1 \) of \( S \) is even.

Proof. Suppose that \( b_1 \) is odd. Let \( \Theta \) denote the Siu form. Write \( \Theta \) as \( \Theta = \alpha + \eta + \bar{\alpha} \), where \( \alpha \) is a \((2,0)\)-form and \( \eta \) is a real positive definite \((1,1)\)-form. Suppose that \( C \) is a curve on \( S \). Then

\[ [\Theta] \cdot [C] = \int_{C} \eta > 0 \]
Hence every effective divisor on $S$ is not homologous to zero. This implies that $S$ is not of Class VI. Hence $S$ is of Class VII. Then the intersection form is negative definite. But we have

$$[\Theta] \cdot [\Theta] = \int_S (\alpha + \eta + \tilde{\alpha}) \Lambda (\alpha + \eta + \tilde{\alpha})$$
$$= 2\int_S \alpha \Lambda \tilde{\alpha} + \int_S \eta \Lambda \eta > 0$$

This is a contradiction.

Thus we have the following

**Proposition 4.** Let $S$ be a compact complex surface. Then the followings are equivalent:

(i) the first Betti number of $S$ is even,

(ii) $E \cap F = \emptyset$,

(iii) there is a Siu form of class $C^1$,

(iv) there is a Siu form of class $C^k$, $k \geq 1$,

(v) $S$ is Kähler.

As a corollary to this proposition, we have

**Proposition 5.** Let $S$ be a compact complex surface. Then the followings are equivalent:

(i) the first Betti number of $S$ is odd,

(ii) $E \cap F \neq \emptyset$, 

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(iii) there is a non-trivial $d$-exact positive $(1,1)$-current.
(iv) $S$ is non-Kähler.

I understand that similar fact as Proposition 5 is known to I.Enoki earlier.

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