代数幾何学シンポジウム記録
1983年度 pp.12-28

On 3-dimensional terminal singularities

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Introduction. Canonical and terminal singularities are introduced by M. Reid [4], [5]. He proved that 3-dimensional terminal singularities are cyclic quotient of smooth points or cDV points [5].

Let (X,p) be a 3-dimensional terminal singularity of index m with the associated $\mathbf{Z}_m$-cover $(\tilde{X},\tilde{p}) \to (X,p)$. If $(\tilde{X},\tilde{p})$ is smooth, then it is known as Terminal Lemma (Danilov [2], D. Morrison-G. Stevens [3]) that there exist an integer a prime to m and coordinates x, y, z of $(\tilde{X},\tilde{p})$ which are $\mathbf{Z}_m$-semi-invariants such that $\sigma(x) = \zeta x$, $\sigma(y) = \zeta^{-1}y$, $\sigma(x) = \zeta^{a}z$ for the standard generator σ of $\mathbf{Z}_m$, where ζ is a primitive m-th root of 1. In this paper, we consider the case where $(\tilde{X},\tilde{p})$ is a singular point and $m > 1$. The main results are Theorems 12, 23, 25 and Remarks 12.2, 23.1, 25.1. These, together with the Terminal Lamma above, almost classifies 3-dimensional terminal singularities.

The author expresses his hearty thanks to Professor S. Iitaka who allowed him to use his word processor, and to Professor M. Ishida for his helpful comments.

§1. Criteria for terminal and canonical singularities.

Let $\mathbf{C}$ be the field of complex numbers, and $\mathbf{C}\{x\}$ denotes the ring of convergent power series in variables x.

Lemma 1. Let (X,p) be the germ of an n-dimensional terminal (resp. canonical) singularity of index m. Let (X',p') be the germ of an n-dimensional reduced Gorenstein variety and $f : (X',p') \to (X,p)$ a morphism such that f factors as

$$X' \xrightarrow{g} Y \xrightarrow{h} X,$$

where h is a blow-up of X and g is quasi-finite. Let ω be a generator of $\omega_X^{(m)}$ at p. Then $f^*\omega$, as a meromorphic section of $\omega_{X'}^{\otimes m}$, vanishes (resp. is regular) along an arbitrary irreducible divisor $D \ni p$ such that $\dim f(D) < n-1$.

Proof. Let D be a divisor as in the lemma. Let $\pi : \tilde{X}' \to X'$ be the normalization, and $X'' \subset \tilde{X}'$ the complement of the singular locus of $\tilde{X}'$. Since $\operatorname{codim}_{\tilde{X}'}(\tilde{X}'-X'') \geq 2$ and since $(\pi^*\omega_{X'})|_{X''} \supset \Omega^n_{X''}$, we may replace (X',p') by (X'',p'') for some smooth p'' such that $\pi(p'') \in D$. In other words, we may assume that X' is smooth. Hence in the factorization $X' \to Y \to X$, we may assume that Y is normal and $q = g(p)$ is a smooth point of Y (by moving p in D if necessary). Then $h^*\omega$ vanishes (resp. is regular) along $g(D)$. Since $g : (X',p') \to (Y,q)$ is a morphism of manifolds, $f^*\omega = g^*h^*\omega$ vanishes (resp. is regular) along D. q.e.d.

Let (X,p) be a 3-dimensional canonical singularity of index m such that, for the associated $\mathbf{Z}_m$-cover $\pi : (X',p') \to (X,p)$ [4,5], (X',p') is a hypersurface singularity. Then there exist $\mathbf{Z}_m$-semi-invariants $x_1, \ldots, x_4$ in the analytic local ring $\mathcal{O}_{X',p'}^h$ such that

(1.1) $\rho(x_i) = \zeta^{c_i} x_i \quad (i = 1, \ldots, 4)$

(1.2) $\mathcal{O}^h_{X',p'} \cong \mathbf{C}\{x_1, \ldots, x_4\}/(\varphi)$,

where ρ (resp. ζ) is a generator of $\mathbf{Z}_m$ (resp. μ_m), $c = (c_1, \ldots, c_4) \in \mathbf{Z}^4$ be such that $\gcd(c,m) = \text{g.c.d.}\{c_1, \ldots, c_4, m\} = 1$, and φ is a semi-invariant. Since π is unramified outside p', one has

$\{q \in X' \mid x_i(q) = 0 \text{ if } c_i \not\equiv 0 \pmod{d}\} = \{p'\}$

for any divisor $d > 1$ of m. Hence, for any divisor $d > 1$ of m, one has

(1.3) $\gcd(c,m) = 1, \quad \#\{i \in [1,4] \mid c_i \equiv 0 \pmod{d}\} \leq 1.$

Now we reverse the process:

Notation (2.0). Let $\mathbf{Z}_m$ act on $\mathbf{C}\{x_1, \ldots, x_4\}$ by (1.1) with ρ (resp. ζ) a generator of $\mathbf{Z}_m$ (resp. μ_m), where $c \in \mathbf{Z}^4$ satisfies (1.3) for an arbitrary divisor $d > 1$ of m. Let φ be a semi-invariant of $\mathbf{C}\{x_1, \ldots, x_4\}$ such that $\mathbf{C}\{x_1, \ldots, x_4\}/(\varphi)$ is normal, and let (X',p') be the germ of a hypersurface at 0 defined by (1.2), and $(X,p) = (X',p')/\mu_m$.

Then we have

Theorem 2. Under Notation (2.0), let σ be an arbitrary element of $\mathbf{Z}_m$, and $a = (a_1, \ldots, a_4)$ a 4-ple of arbitrary integers ≥ 0 such that $\sigma(x_i) = \zeta^{a_i} x_i$ $(i = 1, \ldots, 4)$ and that at least three of $a_1, \ldots, a_4$ are positive. Let $e(a) = \max\{j \mid \varphi(x_1 t^{a_1}, \ldots, x_4 t^{a_4}) \equiv 0 \pmod{t^j}\}$ and $|a| = a_1 + \cdots + a_4$. Then if (X,p) is terminal (resp. canonical), then $|a| - m - e(a) > 0$ (resp. ≥ 0).

Proof. (2.1) Let X_0' be $\mathbf{C}^4$ with global coordinates $x_1, \ldots, x_4$, and let $\mathbf{Z}_m$ act on X_0' by (1.1). Then $T' = (\mathbf{C}^*)^4 \subset \mathbf{C}^4 = X_0'$ is an affine torus embedding, and to the affine torus embedding $T' \subset X_0'$ correspond the group $\Gamma(T') \cong \mathbf{Z}^4$ of 1 parameter subgroups of T' and a cone $C(X_0')$ of $\Gamma(T') \otimes_{\mathbf{Z}} \mathbf{Q}$. Let $\Gamma(T') = \mathbf{Z}^4$ and $C(X_0') = \mathbf{Q}_+^4$ in the standard way, where $\mathbf{Q}_+ = \{q \in \mathbf{Q} \mid q \geq 0\}$. Then, to $T = T'/\mathbf{Z}_m \to X_0 = X_0'/\mathbf{Z}_m$, correspond $\Gamma(T) = \mathbf{Z}^4 + \mathbf{Z}\, c/m$ and $C(X_0) = \mathbf{Q}_+^4$. By the definition of a, there exist integers β, γ such that $\beta c_i \equiv \gamma a_i$ mod. m, where γ is prime to m. Hence $a/m \in \Gamma(T)$.

(2.2) Let φ' be a convergent power series defined by

$$\varphi'(w, y_1, \ldots, y_4) = w^{-e(a)} \cdot \varphi(y_1 w^{a_1}, \ldots, y_4 w^{a_4}).$$

Then $\varphi'(0,y)$ is a non-zero weighted homogeneous polynomial of weight $e(a)$ in $y_1, \ldots, y_4$ for weight $y_i = a_i$ (cf. (*) below). Since $y_1 y_2$ and $y_3 y_4$ are coprime, $\varphi'(0,y)$ has a prime factor which is prime to $y_1 y_2$ or $y_3 y_4$. By symmetry, we may assume that $\varphi'(0,y)$ has a prime factor prime to $y_1 y_2$ and $a_1 > 0$. Since

(*) $\varphi'(0, y_1 t^{a_1}, \ldots, y_4 t^{a_4}) = t^{e(a)} \varphi'(0, y_1, \ldots, y_4)$,

one can find r_2, r_3, $r_4 \in \mathbf{C}$ such that

$$\varphi'(0, 1, r_2, r_3, r_4) = 0.$$

(2.3). Let $e_2 = (0, 1, 0, 0)$, $e_3 = (0, 0, 1, 0)$, $e_4 = (0, 0, 0, 1) \in \mathbf{Z}^4$, and let C be the cone spanned by a, e_2, e_3, e_4 in $\mathbf{Q}^4$ and $M = \mathbf{Z}\, e_2 \oplus \mathbf{Z}\, e_3 \oplus \mathbf{Z}\, e_4 \subset \mathbf{Q}^4$. Then the commutative diagram

$$\begin{array}{ccc} \mathbf{Z}\, a \oplus M & \longrightarrow & \mathbf{Z}\, \frac{a}{m} \oplus M \\ \downarrow & & \downarrow \\ \mathbf{Z}^4 & \longrightarrow & \mathbf{Z}^4 + \mathbf{Z}\, \frac{c}{m} \end{array}$$

gives a commutative diagram of tori:

$$\begin{array}{ccc} R' & \longrightarrow & R \\ \downarrow & & \downarrow \\ T' & \longrightarrow & T \end{array}$$

and $C \subset \Gamma(R') \otimes \mathbf{Q}$ gives a torus embedding

$$R' \subset Z_0' = \mathrm{Spec}\, \mathbf{C}[w, \bar{x}_2, \bar{x}_3, \bar{x}_4],$$

where a, e_2, e_3, e_4 of $\Gamma(R')$ correspond to w, $\bar{x}_2$, $\bar{x}_3$, $\bar{x}_4$. Then

$$R \subset Z_0 = \mathrm{Spec}\, \mathbf{C}[\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4],$$

with $\bar{x}_1 = w^m$, is the torus embedding corresponding to

$C \subset \Gamma(R) \otimes \mathbf{Q}$, and commutative diagrams

$$\begin{array}{ccccc} \Gamma(R') \otimes \mathbf{Q} & \longrightarrow & \Gamma(R) \otimes \mathbf{Q} & \longrightarrow & \Gamma(T) \otimes \mathbf{Q} \\ \cup & & \cup & & \cup \\ C & \longrightarrow & C & \longrightarrow & \mathbf{Q}_+^4 \end{array}$$

and

$$\begin{array}{ccccc} \Gamma(R') \otimes \mathbf{Q} & \longrightarrow & \Gamma(T') \otimes \mathbf{Q} & \longrightarrow & \Gamma(T) \otimes \mathbf{Q} \\ \cup & & \cup & & \cup \\ C & \longrightarrow & \mathbf{Q}_+^4 & \longrightarrow & \mathbf{Q}_+^4 \end{array}$$

give a commutative diagram

$$\begin{array}{ccc} Z_0' & \longrightarrow & Z_0 \\ f' \downarrow & & f \downarrow \\ X_0' & \longrightarrow & X_0, \end{array}$$

where f' is given by

$$x_1 = w^{a_1},\ x_2 = \bar{x}_2 w^{a_2},\ x_3 = \bar{x}_3 w^{a_3},\ x_4 = \bar{x}_4 w^{a_4},$$

If $T \subset Y_0$ is the torus embedding corresponding to $C \subset \Gamma(T) \otimes \mathbf{Q}$, then $Z_0 \to Y_0$ is finite and $Y_0 \to X_0$ is a blow-up.

(2.4). Let $s' = V(w, \bar{x}_2 - r_2, \bar{x}_3 - r_3, \bar{x}_4 - r_4) \in Z_0'$ (resp. s = the image of s' in Z_0), and let

$$\psi(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4) = w^{-e(a)} \cdot \varphi(w^{a_1}, \bar{x}_2 w^{a_2}, \bar{x}_3 w^{a_3}, \bar{x}_4 w^{a_4}),$$

(note that the right hand side is a holomorphic function in

$\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4$ defined near s.) Let $(Z,s) \subset (Z_0,s)$ (resp. $(Z',s') \subset (Z_0',s')$) be defined by $\psi = 0$. Then f (resp. f') induces $f : (Z,s) \to (X,p)$ (resp. $f' : (Z',s') \to (X',p')$), which satisfies the conditions of Lemma 1 by (2.3). We have the following commutative diagram of natural morphisms:

$$\begin{array}{ccc} Z' & \xrightarrow{\tau} & Z \\ f' \downarrow & & f \downarrow \\ X' & \xrightarrow{\pi} & X. \end{array}$$

By Poincaré residue formula,

$$\omega_{X'} = \mathrm{Res}\ \frac{dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4}{\varphi} = -\frac{dx_1 \wedge dx_2 \wedge dx_3}{\varphi_4},$$

$$f'^*\omega_{X'} = -\frac{d(w^{a_1}) \wedge d(\bar{x}_2 w^{a_2}) \wedge d(\bar{x}_3 w^{a_3})}{f'^*\varphi_4},$$

where $\varphi_4 = \partial\varphi/\partial x_4$. By calculation, one sees

$$d(w^{a_1}) \wedge d(\bar{x}_2 w^{a_2}) \wedge d(\bar{x}_3 w^{a_3})$$

$$= a_1 \cdot w^{a_1+a_2+a_3-1} \cdot dw \wedge d\bar{x}_2 \wedge d\bar{x}_3.$$

Since $f'^*\varphi = w^{e(a)}\psi$, it follows from the chain rule that

$$(f'^*\varphi_4)\cdot w^{a_4} = w^{e(a)}\ \psi_4,$$

where $\psi_4 = \partial\psi/\partial\bar{x}_4$. Thus

$$f'^*\omega_{X'} = -a_1 \cdot w^{|a|-e(a)-1} \cdot \frac{dw \wedge d\bar{x}_2 \wedge d\bar{x}_3}{\psi_4}.$$

One also has

$$\omega_{Z'} = \mathrm{Res}\ \frac{dw \wedge d\bar{x}_2 \wedge d\bar{x}_3 \wedge d\bar{x}_4}{\psi} = -\ \frac{dw \wedge d\bar{x}_2 \wedge d\bar{x}_3}{\psi_4}.$$

One has

$$f'^{*}\omega_{X'} = a_1 \cdot w^{|a|-e(a)-1}\ \omega_{Z'}.$$

By construction, we have

$$\tau^{*}\omega_{Z} = m \cdot w^{m-1}\ \omega_{Z'}.$$

Hence we have

$$f'^{*}\omega_{X'} = (a_1/m) \cdot w^{|a|-e(a)-m}\ \tau^{*}\omega_{Z}.$$

Since $\pi : X' \to X$ is unramified in codimension 1 by (1.3), one has $\pi^{*}\ \omega_X^{(m)} = (\mathrm{unit}) \cdot \omega_{X'}^{\otimes m}$ near p' (by abuse of language, $\omega_X^{(m)}$ denotes one of its generators at p), and

$$f^{*}\ \omega_X^{(m)} = (\mathrm{unit}) \cdot \bar{x}_1^{\ |a|-e(a)-m}\ \omega_Z^{\otimes m}\quad \text{near}\ \ s.$$

Since $\{\bar{x}_1 = 0\}$ is a divisor through s collapsed by f, one has $|a| - e - m > 0$ (resp. ≥ 0). q.e.d.

Under Notation (2.0), let $\chi_1, \ldots, \chi_r \in \mathbf{C}\{x_1, \ldots, x_4\}$ $(r \geq 2)$ be $\mathbf{Z}_m$-semi-invariants with the same character such that $\chi_i \mathbf{C}\{x_1, \ldots, x_4\} = u_i \mathbf{C}\{x_1, \ldots, x_4\}$ for some monomial u_i in $x_1, \ldots, x_4$ $(i = 1, \ldots, r)$ and that $u_1, \ldots, u_r$ are linearly independent over $\mathbf{C}$ and the locus defined by $\chi_1 = \cdots = \chi_r = 0$ is of dimension ≤ 1. Let Φ be the linear system generated by $\chi_1, \ldots, \chi_r$, and assume that our φ is written as

$$\varphi = \sum_{i=1}^{r} \lambda_i \chi_i$$

for some $\lambda = (\lambda_1, \ldots, \lambda_r) \in (\mathbf{C}^*)^r$. By Bertini's theorem, $\varphi = 0$ defines a normal variety for general λ since the base locus of Φ is of dimension ≤ 1. Let σ, a, ζ be as in Theorem 2, then the value of $e(a)$ given in Theorem 2 does not depend on the choice of $\lambda \in (\mathbf{C}^*)^r$. Then under the notation of Theorem 2, the following is the corollary to the proof of Theorem 2.

Corollary 2.1. If $|a| - m - e(a) > 0$ (resp. ≥ 0) for arbitrary σ, a, ζ as in Theorem 2, then (X,p) is terminal (resp. canonical) for general λ. If (X,p) has an isolated singular point at p (resp. canonical singularity outside p) for general λ, then one can add the extra conditions $a_1, \ldots, a_4 > 0$ on a in the statement above.

Proof. Let $X_0 = \mathbf{C}^4/\mathbf{Z}_m$ with respect to the action given above. Then Φ induces a linear system Φ_0 (of Weil divisors) in a neighborhood of 0 in X_0 which is free from fixed components. By the conditions on χ_i's, there exists a toric resolution

$h : U_0 \to X_0$ such that the proper transform Ψ_0 of Φ_0 to U_0 is free from base points (principalizer of the coherent sheaf associated to Φ_0). Then a general member X of Φ_0 is normal at 0 and its proper transform $U = h^{-1}[X]$ is smooth *in a neighborhood of* $h^{-1}(0)$. Thus it is enough to show that $h^*\omega_X^{(m)}$, as a meromorphic section of $\omega_U^{\otimes m}$, vanishes (resp. is regular) along $D_0 \cap U$, where $D_0 \subset U_0$ is an arbitrary exceptional divisor of h. We now use the notation of the *proof of Theorem 2*. Let L_+ be the 1-simplex $\subset \mathbf{Q}_+^4 \subset \Gamma(T) \otimes \mathbf{Q}$ corresponding to D_0. Let $a = (a_1, \ldots, a_4)$ be a 4-ple of integers ≥ 0 such that $\mathbf{Z}a/m = \mathbf{Q}L_+ \cap \Gamma(T)$. By (1.3), the singular locus of X_0 is of dimension ≤ 1, and the base locus of Φ_0 *is of dimension* ≤ 1. Thus *one may assume* that the image of D_0 to X_0 is of dimension ≤ 1, whence at least 3 of $a_1, \ldots, a_4$ are positive. Then the notation of (2.3) can be used, and the torus embedding $T \subset V_0 = T \cup D_0$ corresponds to $L_+ \subset \Gamma(T) \otimes \mathbf{Q}$. $L_+ \subset \Gamma(R) \otimes \mathbf{Q}$ *corresponds to the open* subset W_0 of Z_0 defined by $\bar{x}_3\bar{x}_4 \neq 0$. Since $\mathbf{Z}a/m = \mathbf{Q}a/m \cap \Gamma(T)$, $\{\bar{x}_1 = 0\}$ is not in the branch locus of $W_0 \to V_0$. Let V, W be the proper transforms of X to V_0 and W_0, respectively. Then, since Ψ_0 is free from base points, $g : W \to V$ is unramified over general points of arbitrary irreducible components of $V \cap D_0$. Thus by

$$g^*h^*\omega_X^{(m)} = (\text{unit})\cdot \bar{x}_1^{\,|a|-e(a)-m}\,\omega_W^{\otimes m} \quad \text{along} \quad W \cap D_0,$$

(2.4), we have corollary 2.1. q.e.d.

Corollary 2.2. Under the assumptions of Theorem 2, assume that m is odd, and

$$\varphi = x_1^2 + f(x_2, x_3, x_4) \quad (f \in \mathbf{C}\{x_2, x_3, x_4\}).$$

Let $n = \max\{j \mid \varphi(x_2 t^{a_2}, x_3 t^{a_3}, x_4 t^{a_4}) \equiv 0 \ (t^j)\}$. Then

$$a_2+a_3+a_4 > (\text{resp.} \geq) \begin{cases} m + n/2 & \text{if } n \text{ is even} \\ m/2 + n/2 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. One has $2 \cdot \mathrm{wt}\, x_1 \equiv n \pmod{m}$. If n is even, then we choose $a_1 = n/2$, keeping a_2, a_3, a_4 the same. Then $n/2 + a_2 + a_3 + a_4 > (\text{resp.} \geq)\ m + n$. If n is odd, then we choose $a_1 = (m+n)/2$, keeping a_2, a_3, a_4 the same. Then $(n + m)/2 + a_2 + a_3 + a_4 > (\text{resp.} \geq)\ m + n$. q.e.d.

For the approximation of φ, we need the following which easily follows by the argument of [1]:

Theorem 3. Let $\varphi \in \mathbf{C}\{x,y,z,u\}$. Assume that φ has an isolated singular point at (O), and that $\mathbf{Z}_m$ acts on $\mathbf{C}\{x,y,z,u\}$ in such a way that x, y, z, u, and φ are semi-invariants. Then for an arbitrary integer $b > 0$, there exists an integer $n > 0$ such that, for an arbitrary semi-invariant $\psi \in \mathbf{C}\{x,y,z,u\}$ with the property $\psi \equiv \varphi \quad (x,y,z,u)^n$, there exists an analytic $\mathbf{C}$-automorphism σ of $\mathbf{C}\{x,y,z,u\}$ commuting with $\mathbf{Z}_m$-action (will be called a $\mathbf{Z}_m$-automorphism, for short) such that $\sigma(\varphi)\mathbf{C}\{x,y,z,u\} = \psi\mathbf{C}\{x,y,z,u\}$, $\sigma \equiv \mathrm{id.}$ modulo $(x,y,z,u)^b$.

Corollary 4. Under the notation and the assumptions of Theorem 3, if $\varphi \in (x,y,z,u)^2$ and if xy appears in φ, then there exist a $\mathbf{Z}_m$-automorphism σ of $\mathbf{C}\{x,y,z,u\}$ such that

$$\sigma(x) - x \in (y,z,u) + (x,y,z,u)^2,$$
$$\sigma(y) - y \in (x,z,u) + (x,y,z,u)^2,$$
$$\sigma(z) - z,\ \sigma(u) - u \in (x,y,z,u)^2,$$
$$(\sigma(\varphi)) = (xy + f(z,u)) \text{ as ideals for some } f \in \mathbf{C}\{z,u\}.$$

§2. Notation and classification.

Let φ be an element of $(x,y,z,u)^2\mathbf{C}\{x,y,z,y\}$ which has a $\mathbf{Z}_m$-action $(m > 1)$ such that x,y,z,u,φ are semi-invariants. Assume that φ has an isolated cDV singularity at the origin (0), that the quotient of $\{\varphi = 0\}$ by $\mathbf{Z}_m$ has a terminal singularity at (0), and that the action of $\mathbf{Z}_m$ is free on $U - (0)$, where $U \ni (0)$ is an open set of $\{\varphi = 0\}$. By a $\mathbf{Z}_m$-automorphism, we mean an analytic $\mathbf{C}$-automorphism of $\mathbf{C}\{x,y,z,u\}$ commuting with $\mathbf{Z}_m$-action unless otherwise mentioned. We will keep these assumptions and notation, unless otherwise mentioned.

Fixing a primitive m-th root ζ of 1, and given the $\mathbf{Z}_m$-action above, we associate to each $\sigma \in \mathbf{Z}_m$ a weight modulo m (denoted by σ-wt mod. m); $\sigma\text{-wt}(x) \equiv a(\sigma)$, ..., $\sigma\text{-wt}(u) \equiv d(\sigma)$ (mod. m) are determined by $\sigma(x) = \zeta^{a(\sigma)}\cdot x$, ..., $\sigma(u) = \zeta^{d(\sigma)}\cdot u$. If σ is a generator of $\mathbf{Z}_m$, we may simply call σ-wt a wt., if there is no danger of confusion. By Theorem 2 and Corollary 2.2, one can almost classify such φ as above. The results are as follows.

Theorem 5. If xy appears in φ, then one of the following holds (after exchanging z,u if necessary):

(1) $\text{wt } x + \text{wt } y \equiv \text{wt } z \equiv 0$ mod. m, and wt u, wt x, wt y are prime to m.

(2) $m = 4$ and there exists a generator σ of $\mathbf{Z}_4$ such that σ-wts of x,y,z,u are 1,1,2,3 mod. 4.

By Theorem 5, one can easily prove

Lemma 6. If x^2 and y^2 appear in φ and $m = 2$, then

(1) if $\mathrm{wt}\ x \equiv \mathrm{wt}\ y$ mod. 2, then after exchanging z, u (if necessary), one has wts of x, y, z, u equal to 1, 1, 0, 1 mod. 2, and modulo $\mathbf{Z}_2$-automorphism, $(\varphi) = (x^2 + y^2 + f(z, u^2))$ for some $f \in \mathbf{C}\{z,u\}$,

(2) if $\mathrm{wt}\ x \not\equiv \mathrm{wt}\ y$ mod. 2, then after exchanging x, y (if necessary), one has wts of x, y, z, u equal to 1, 0, 1, 1 mod. 2, and modulo $\mathbf{Z}_2$-automorphism, $(\varphi) = (x^2 + y^2 + f(z, u))$ for some $f \in \mathbf{C}\{z,u\}$.

Thus one gets

Theorem 7. If the quadratic part φ_2 of φ has rank ≥ 2 as a quadratic form, then after permutation of x, y, z, u (if necessary), one of the following holds.

(1) $\mathrm{wt}\ x + \mathrm{wt}\ y \equiv 0$; $\mathrm{wt}\ z \equiv \mathrm{wt}\ \varphi \equiv 0$; wt x, wt y, wt u are prime to m, and modulo $\mathbf{Z}_m$-automorphism, $(\varphi) = (xy + f(z, u^m))$ for some $f \in \mathbf{C}\{z,u^m\}$.

(2) $m = 4$: wts of x, y, z, u, φ are 1, 3, 2, 1, 2 mod. 4 (after choosing a generator of $\mathbf{Z}_4$), and modulo $\mathbf{Z}_4$-automorphism, $(\varphi) = (x^2 + y^2 + f(z, u^2))$ for some $f \in \mathbf{C}\{z,u^2\}$.

(3) $m = 2$: wts of x, y, z, u, φ are 1, 0, 1, 1, 0 mod. 2, and modulo $\mathbf{Z}_2$-automorphism, $(\varphi) = (x^2 + y^2 + f(z, u))$ for some $f \in (z,u)^4\mathbf{C}\{z,u\}$.

Remark 7.1. In case (1) (resp. (2), (3)) of Theorem

7, if f is a general linear combination of a finite number of monomials in $(z^2, u^m)\mathbf{C}\{z, u^m\}$ (resp. $(z^3, u^2)\mathbf{C}\{z^2, zu^2, u^4\}$, $(z^4, z^3u, z^2u^2, zu^3, u^4)\mathbf{C}\{z^2, zu, u^2\}$) and if f has an isolated singular point at 0, then (X,p) is a terminal singularity. This follows from Corollary 2.1.

Lemma 8. If rk $\varphi_2 \leq 1$ and u^2 appears in φ_2, then one has

$$(\varphi) = (u^2 + f(x,y,z)) \text{ modulo } \mathbf{Z}_m\text{-automorphism},$$

where $f \in (x,y,z)^3\mathbf{C}\{x,y,z\}$ has non-zero cubic term f_3.

This follows from the definition of cDV points.

Theorem 9. Under the situation of Lemma 8, if f_3 is not a cube of a linear factor, then after permutation of x, y, z and choice of generator σ of $\mathbf{Z}_m$, one of the following holds.

(1) $m = 2$, wts of x, y, z, u, φ are 1, 1, 0, 1, 0 mod.2, and xyz or y^2z appears in f_3.

(2) $m = 3$, σ-wts of x, y, z, u, φ are 1, 2, 2, 0, 0 mod. 3, and $f_3 = x^3 + y^3 + z^3$, $x^3 + yz^2$, or $x^3 + y^3$ modulo $\mathbf{Z}_3$-automorphism of $\mathbf{C}\{x,y,z\}$.

Remark 9.1. In cases (1), (2) of Theorem 9, it is easy to see that modulo $\mathbf{Z}_m$-automorphism and units of $\mathbf{C}\{x, y, z, u\}$, one may put φ in one of the following forms by Theorem 3:

Case 1. $m = 2$, wts of x, y, z, u are 1, 1, 0, 1 mod.

2,

(9.1.1) $\varphi = u^2 + xyz + x^{2a} + y^{2b} + z^c$,

(9.1.2) $\varphi = u^2 + y^2z + \lambda yx^{2a-1} + g$,

$(a, b \geq 2, c \geq 3,\ \lambda \in k,\ g \in (x^4, x^2z^2, z^3)\mathbf{C}\{x^2, z\})$,

Case 2. $m = 3$, wts of x, y, z, u are 1, 2, 2, 0 mod. 3,

(9.2.1) $\varphi = u^2 + x^3 + y^3 + z^3$,

(9.2.2) $\varphi = u^2 + x^3 + yz^2 + \lambda y^{3r+1}x + \mu y^{3s}$,

(9.2.3) $\varphi = u^2 + x^3 + y^3 + xyz^3\cdot\alpha + xz^4\cdot\beta + yz^5\cdot\gamma + z^6\cdot\delta$,

$(r \geq 1,\ s \geq 2,\ \lambda, \mu \in k,\ \alpha, \beta, \gamma, \delta \in \mathbf{C}\{z^3\})$.

For (9.1.1) and (9.2.1), (X, p) is terminal. If φ is a general linear combination of a finite number of monomials as in (9.1.2), (9.2.2), or (9.2.3), and if φ has an isolated singularity at 0, then (X, p) is terminal.

Lemma 10. Under the situation of Lemma 8, if $f_3 = x^3$, then modulo $\mathbf{Z}_m$-automorphism,

$$(\varphi) = (u^2 + x^3 + g(y,z)x + h(y,z)),$$

where $g, h \in \mathbf{C}\{y, z\}$, $g \in (y,z)^3$, $h \in (y, z)^4$.

Theorem 11. Under the situation of Lemma 10, one has $m = 2$ and wts of x, y, z, u, φ are 0, 1, 1, 1, 0 mod. 2, and $h \notin (y,z)^5$.

Remark 11.1. If $m = 2$ and wts of x, y, z, u are 0, 1, 1, 1 mod. 2 and if an even polynomial g (resp. h) in y, z is a general linear combination of a finite number of monomials $\in (y, z)^4$ and $h \notin (y,z)^5$, and if

$$\varphi = u^2 + x^3 + g(y, z)x + h(y, z)$$

has an isolated singularity at O, then (X,p) is terminal.

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