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Dual Graphs of Curves on Surfaces of Class VII_0

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§0 Introduction. A compact complex surface is in class VII_0 if the first Betti number of it is equal to one and if it is minimal. We know many examples of surfaces of class VII_0 (in short VII_0 surfaces) with curves — minimal surfaces with global spherical shells [2],[6],[7]. Some of them are characterized by the existence of certain curves [1],[4],[5]. Here we recall a table of classification of VII_0 surfaces from [4],[5].

(0.1) Table.

curves	structure of surfaces
1) (more than) 3 elliptic curves	elliptic VII_0 surfaces
2) two elliptic curves	Hopf surfaces
3) an elliptic curve, no cycles	Hopf surfaces
4) an elliptic curve and a cycle	parabolic Inoue surfaces
5) two cycles	hyperbolic Inoue surfaces
6) a cycle C with $C^2=0$	$S_{n,\beta,t}$ (see [1])
7) a cycle C with $C^2<0$	
7-1) $b_2(S)=b_2(C)$	half Inoue surfaces
7-2) $b_2(S)>b_2(C)$	examples exist
8) no elliptic curves,no cycles	?

It is necessary to study the class 7) in the table in more detail in order to complete the classification of VII_0 surfaces. The purpose of this note is to complete the LHS of 7) in (0.1) under the assumption that the VII_0 surfaces has at least (in fact, then exactly) b_2 rational curves. Here we make a brief review on curves on VII_0 surfaces. An irreducible curve on a VII_0 surface is either a nonsingular rational curve, or an elliptic curve, or a rational curve with a node. If a VII_0 surface has a cycle of rational curves, then it has positive b_2 and no meromorphic functions except constants. The number of elliptic curves and cycles of rational curves on a VII_0 surface is at most two in total if the surface has no meromorphic functions except constants. If a VII_0 surface has a cycle C of rational curves with $C^2<0$, then there are no elliptic curves on the surface. If moreover it has another cycle of rational curves, then (0.1) 5) asserts that it is a hyperbolic Inoue surface. Therefore in (0.1) 7), any VII_0 surface has a unique cycle C with $C^2<0$, but no elliptic curves.

Let S be a VII_0 surface in (0.1) 7). Assume moreover that S has b_2 rational curves. If the unique cycle C on S has no branches (see (2.1)), then S is a half Inoue surface (see (2.4)). If C has branches, then our main theorem asserts that the dual graph of all the curves on S is the same as one of those of known surfaces — minimal surfaces with global sphericall shells. We also give a uniform characterization of Inoue surfaces with $b_2>0$ by the Dloussky invariant (§3).

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Notations.

O_S	the sheaf of germs over S of holomorphic functions
Ω_S^1	the sheaf of germs over S of holomorphic one forms
K_S	the canonical line bundle of S
$c_1(L)$	the first Chern class of a line bundle L
$b_2(E)$	#(irreducible components of a reduced divisor E)
$b_i(S)$	the i-th Betti number of S
$H^q(S,\mathcal{F})$	q-th cohomology group of S with $\mathcal{F}$ coefficients
$h^q(S,\mathcal{F})$	dim $H^q(S,\mathcal{F})$
$\chi(S,\mathcal{F})$	$\sum_{q=0}^{\infty} (-1)^q h^q(S,\mathcal{F})$

§1 Smoothability of a cycle by deformations

(1.1) Theorem. <u>Let</u> S <u>be a</u> VII_0 <u>surface with</u> $b_2>0$. <u>Then</u> $H^2(S,\Theta_S) = 0$.

<u>Proof</u>. Assume the contrary to derive a contradiction. By Serre duality, $H^0(S,\Omega^1_S(K_S)) \neq 0$, K_S being the canonical line bundle of S. Let D be the maximum effective divisor of S such that $H^0(S,\Omega^1_S(K_S-D))\neq 0$ and let ω be a nonzero element of $H^0(S, \Omega^1_S(K_S-D))$. By definition, zero($\omega$) is isolated. Then we have,

(1.1.1) Lemma. <u>The following is exact;</u>

$$0 \to O_S(K_S-D) \underset{f}{\to} \Omega^1_S \underset{g}{\to} O_S(2K_S-D)$$

where $f(a) = a\omega$, $g(b)= b\wedge\omega$.

<u>Proof</u>. Clearly f is injective. Take $b\in \mathrm{Ker}\ g$. Then $b\wedge\omega= 0$. Hence $b = h\omega$ locally for a germ h of a meromorphic function. Then pole(h)$\subset$ zero(ω)=isolated, whence h is holomorphic. Hence h is contained in Im f. □

Let H be Coker g. Then the following is exact;

$$0 \to O_S(K_S-D) \to \Omega^1_S \to O_S(2K_S-D) \to H \to 0$$

We remark that supp(H) is isolated points, so that $H^1(S,H) = 0$. We also see $h^0(S,\Omega^1_S)=h^2(S,\Omega^1_S)=0, h^1(S,\Omega^1_S)=b_2$ by [3],or [5,p.405]. Therefore by taking Euler-Poincare characterictics, we see

$$\begin{aligned} b_2 = -\chi(S,\Omega^1_S) &= -\chi(S,-K_S+D)-\chi(S,2K_S-D)+h^0(S,H) \\ &=-2\chi(S,-K_S+D)+h^0(S,H) \\ &= -2K_S^2+3K_SD-D^2+h^0(S,H) \\ &= 2b_2+3K_SD-D^2+h^0(S,H) \qquad (\text{by } -K_S^2=b_2), \end{aligned}$$

and therefore,

$$b_2+3K_SD-D^2+h^0(S,\mathcal{H}) = 0.$$

(1.1.2) Lemma. $K_SD \geqq 0$, $D^2 \leqq 0$.

See [3] or [5,p. 399,(2.1.2) and (2.2)].

Consequently, $b_2 = 0$, $K_SD = D^2 = h^0(S,\mathcal{H}) = 0$. This contradicts the assumption $b_2 > 0$. q.e.d.

(1.2) Theorem. Let S be a VII_0 surface with C a cycle of rational curves. Let E be a reduced divisor containing C. Then $H^2(S,\Theta_S(-\log E)) = 0$.

Proof. Assume the contrary. Then as in (1.1), we see

$$4 \geqq b_2(E)+(K_SE+E^2)/2.$$

Let $E = C+H$, $H = \sum H_j$, H_j irreducible. Then

$$\begin{aligned} b_2(E)+(K_SE+E^2)/2 &= b_2(C)+CH+b_2(H)+(K_SH+H^2)/2 \\ &= b_2(C)+CH+\sum_{i<j} H_iH_j \end{aligned}$$

whence $4 \geqq b_2(C)$. Take a fivefold unramified covering $\pi: S^* \to S$. Then since $H^2(S^*,\Theta_{S^*}(-\log E^*)) \neq 0$ for $E^*=\pi^*E$, we see $4 \geqq b_2(C^*)$, C^* being π^*C. However $b_2(C^*)=5b_2(C)$ since π is five-sheeted. This is a contradiction. q.e.d.

A geometric consequence of this theorem is

(1.3) Theorem. Let S be a VII_0 surface with C a cycle of rational curves, H a reduced divisor with no common components with C, E = C+H. Then there is a smooth proper family $\pi : \mathcal{S} \to \Delta$ with π-flat divisors $\mathcal{C}$ and $\mathcal{H}$ of $\mathcal{S}$ such that

(1.3.1) $(\mathcal{S}_0,\mathcal{C}_0,\mathcal{H}_0) \cong (S,C,H)$,

(1.3.2) $\mathcal{H}_t = H$ (together with embeddings) for any t Δ

(1.3.3) $\omega := \pi|_{\mathcal{C}} : \mathcal{C} \to \Delta$ is a versal deformation of C with natural isomorphism $\mathrm{Ext}^1(\Omega^1_C,\mathcal{O}_C) \cong T_0(\Delta)$.

(1.3.4) S_t <u>is either a blown-up</u> primary <u>Hopf surface or a blown-up parabolic Inoue surface if</u> C_t <u>is an elliptic curve</u>.

We note that (1.3.4) is a consequence of (0.1). See [5,(12.1) and (12.2)]. Since any parabolic Inoue surface is deformed into a blown-up primary Hopf surface, we may assume that S_t in (1.3.4) is a blown-up <u>primary</u> Hopf surface for generic t.

(1.4) Corollary. <u>Let</u> S,C <u>and</u> H_j <u>be the same as in</u> (1.3). <u>Assume moreover that</u> S <u>is not a half Inoue surface, and</u> $C^2<0$. <u>Then there are line bundles</u> $L_j \in H^1(S,O_S^*)$, $E_j=c_1(L_j)\in H^2(S,\mathbb{Z})$, $(1\leqq j\leqq n)$, $n=b_2(S)$ <u>such that</u>

(1.4.1) E_j $(1\leqq j\leqq n)$ <u>is a $\mathbb{Z}$ basis of</u> $H^2(S,\mathbb{Z})$,

(1.4.2) $K_S L_j = -1$, $L_j L_k = -\delta_{jk}$,

(1.4.3) $C = -(L_{r+1}+\ldots+L_n)$, $K_S = L_1+\ldots+L_n$ <u>for some</u> $1\leqq r\leqq n-1$.

(1.4.4) $c_1(H_j) = E_{j,1}-(E_{j,2}+\ldots+E_{j,k_j})$, $E_{j,k}$ <u>being one of</u> E_i $(1\leqq i\leqq n)$, <u>all distinct for</u> j <u>fixed</u>.

(1.5) Remark. If $C^2=0$, $c_1(C)=0$, but $C \neq 0$ in $H^1(S,O_S^*)$. If S is a half Inoue surface, then $C^2<0$, $c_1(C)=-c_1(K_S)=-(E_1+\ldots+E_n)$, r=0, but $C\neq -K_S$. In fact, K_S+C is a line bundle of order two. See [5,§9].

§2 Dual graphs of curves

(2.1) Definition. A VII_0 surface S with $b_2>0$ is said to be *special* if S has at least b_2 rational curves.

It is still unknown whether there are un-special VII_0 surfaces with $b_2>0$.

We notice that the maximum number of (possibly singular) rational curves on a VII_0 surface is $b_2(=b_2(S))$, the second Betti number of S) (Ma. Kato, see [5,(3.5)] for the proof.)

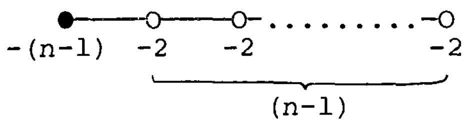
Any special VII_0 surface has a cycle of rational curves.

Any (minimal) surface with a global spherical shell is special.

Let C be a cycle of rational curves. *A branch* D *of* C is by definition a reduced connected divisor with CD = 1 having no common irreducible components with C.

As a simplest case, we shall prove the following

(2.2) Theorem. *If a specail* VII_0 *surface* S *has a rational curve* C *with a node having branches, then the dual graph of* b_2 *rational curves is*



where $n=b_2(S)$, ● *(resp.* O) *denotes* C *(resp. a nonsingular* $\mathbb{P}^1$).

Proof. By (1.4), $C = -(L_{r+1}+\ldots+L_n), K_S=L_1+\ldots+L_n$, $1\leq r\leq n-1$. By the assumption, there is a branch of C. So we take a non-singular rational curve D with CD=1. We assume $c_1(D)=a_1E_1+\ldots+a_nE_n$. Since $2+K_SD+D^2=0$, we have $\sum_i (a_i^2+a_i) = 2$. Hence there is a unique i such that $a_i^2+a_i=2$ and except i, $a_j^2+a_j=0$.

By $CD=1$, we have $a_{r+1}+\dots+a_n=1$. Therefore up to permutation, $(a_{r+1},\dots,a_n)=(1,0,\dots,0)$, and $a_j=0$ or -1 for $j\leqq r$. Since $D^2\leqq -2$ by the minimality of S, we may assume $a_1=-1$. Since $L_1C=L_1(-L_{r+1}-\dots-L_n)=0$, there is a flat line bundle F on S such that $L:=L_1+F$ is trivial on C, namely, $L\otimes O_C=O_C$ (see [5,(2.14)]).

(2.2.1) Lemma. <u>Let</u> N <u>be a line bundle on</u> S <u>with</u> $c_1(N)=b_1E_1+\dots+b_nE_n$. <u>Assume</u> $b_1+\dots+b_n>0$. <u>Then</u> $H^0(S,N)=0$.

<u>Proof</u>. Assume the contrary. Then there is an effective divisor D on S such that $c_1(D)=b_1E_1+\dots+b_nE_n$. Then by (1.1.2), $K_SD\geqq 0$. However $K_SD=(L_1+\dots+L_n)(b_1L_1+\dots+b_nL_n)=-(b_1+\dots+b_n)<0$, which is absurd. □

(2.2.2) Lemma. $H^q(S,L)=0$ <u>for any</u> q.

<u>Proof</u>. By (2.2.1), $H^0(S,L)=0$. By (2.2.1) and Serre duality, $H^2(S,L)=H^0(S,K_S-L)^*=H^0(S,L_2+\dots+L_n-F)^*=0$ (because $1\leqq r\leqq n-1<n$). By Riemann-Roch, $h^1(S,L)=-\chi(S,L)=(K_SL-L^2)/2-\chi(S,O_S)=-\chi(S,O_S)$ $=-(p_g-q+1)=-(0-1+1)=0$. □

Consider the exact sequence $0\to O_S(L-C)\to O_S(L)\to O_C(L)\to 0$. Taking $L\otimes O_C=O_C$ into account, we have an exact sequence

$$\begin{aligned}0 &\to H^0(S,L-C)\to H^0(S,L)\to H^0(C,O_C)\\ &\to H^1(S,L-C)\to H^1(S,L)\to H^1(C,O_C)\\ &\to H^2(S,L-C)\to H^2(S,L)\to 0.\end{aligned}$$

Therefore $h^0(S,K_S-L+C)=h^2(S,L-C)=h^1(C,O_C)=1$. Note that $K_S-L+C=L_1+\dots+L_n-(L_1+F)-(L_{r+1}+\dots+L_n)=L_2+\dots+L_r-F$. Hence

(2.2.3) Lemma. $r=1$.

<u>Proof</u>. By (2.2.1), $H^0(S,K_S-L+C)=0$ if $r\geqq 2$. □

Since $c_1(D)=E_{r+1}-(E_1+\dots)$, we have $c_1(D)=E_2-E_1$. We also have $C=-(L_2+\dots+L_n)$. Let D' and D" be the other rational curves on S. Then both D' and D" are nonsingular rational curves as was explained in the introduction. By (1.4.4), we have

$$c_1(D')=E_j-(E_{j_1}+\dots+E_{j_s}),$$
$$c_1(D'')=E_k-(E_{k_1}+\dots+E_{k_t})$$

for some j, j_h, k, k_h. By $CD'\geqq 0, DD'\geqq 0$, we have $j\neq 1,2$ and $s=1$ or $s=2$. Similarly $k\neq 1,2$, $t=1$ or 2 and moreover $j\neq k$. By the assumption that S is special, there are (n-2) rational curves D_j $(3\leqq j\leqq n)$ on S. By the above observation, we can index

$$c_1(D_j) = E_j-E_{I_j}$$

for a subset I_j of $\{1,\dots,n\}\backslash\{j\}$. By $D_j^2\leqq -2$, I_j is nonempty.

(2.2.4) Lemma. $\# I_j = 1$.

We omit the proof. See [8].

Assuming (2.2.4), it is not difficult to see that $c_1(D_j) =E_j-E_{j-1}$ $(2\leqq j\leqq n)$, where $D_2=D$. The proof is done simply by $D_jD_k\geqq 0, CD_j\geqq 0$ for $j\neq k$. (For instance, if n=3, $c_1(D_3)\neq E_3-E_1$ because $D_2D_3=(E_2-E_1)(E_3-E_1)=-1$ is absurd. Hence $c_1(D_3)=E_3-E_2$.) Thus we complete the proof of (2.2). q.e.d.

After preparing various lemmas, we are able to prove

(2.3) Theorem. <u>Let</u> S <u>be a special</u> VII_0 <u>surface with</u> C <u>a unique cycle of rational curves having branches</u>. <u>Then the dual graph of</u> b_2 <u>rational curves is the same as one of those of minimal surfaces with global spherical shells</u>.

Any graph in (2.3) is given explicitly. In (2.3), C^2 is negative because $c_1(C)=0$, hence C has no branches, if $C^2=0$.

(2.4) Theorem. *Let S be a special VII_0 surface with C a unique cycle of rational curves. Assume that $C^2<0$ and that C has no branches. Then S is isomorphic to a half Inoue surface.*

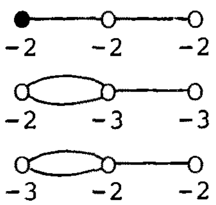
(2.5) Corollary. *Let S be a special VII_0 surface with C a cycle of rational curves. Then either $C^2=b_2(C)-b_2(S)$, or $C^2=-b_2(C)=-b_2(S)$ and S is a half Inoue surface.*

In (2.5), if S has two cycles of rational curves, the assertion is a part of the strange duality.

(2.6) Problem. Give a direct proof of (2.5).

(2.7) Example. Let S be a special VII_0 surface with $b_2=3$, and a unique cycle with branches. Then there are three cases;

dual graph



homological expressions of curves (with numbering from the left)

$C_1=-E_2-E_3$, $D_2=E_2-E_1$, $D_3=E_3-E_2$

$C_1=E_1-E_2$, $C_2=E_2-E_1-E_3$, $D_3=E_3-E_1-E_2$

$C_1=E_1-E_2-E_3$, $C_2=E_2-E_1$, $D_3=E_3-E_2$,

where ● (resp. O) stands for a rational curve with a node (resp. a nonsingular rational curve).

§3 A theorem.

(3.1) Definition. Let S be a VII_0 surface with $b_2>0$. The Dloussky invariant Dl(S) of S is

$$Dl(S) = -\sum D^2 + \#(\text{rational curves with a node})$$

where D ranges over all the irreducible curves on S.

(3.2) Theorem. <u>Let</u> S <u>be a</u> VII_0 <u>surface with</u> $b_2>0$. <u>Then</u>

$$Dl(S) \leqq 3b_2(S),$$

<u>equality holding if and only if</u> S <u>is an Inoue surface with</u> $b_2>0$.

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