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Hopf algebra for Fuchsian groups

Kyoji Saito (RIMS, Kyoto University)

This is a partial report on the trial of constructing modular functions on the Teichmüller space. The idea is to use the Ising model in stastical mechanics, where the lattice of rank 2 is replaced by the Fuchsian group. The calculations are still on the way and in the present note we report on the the construction of the Hopf algebra and some of its structures.

A more completed note will be published elsewhere.*)

*) 本稿は別途投稿予定の原稿の一部です。

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§ 2 Fuchsian groups

We describe tessellations and the growth functions for Fuchsian groups. The materials are classical, which can be traced back to Fricke and Dehn [ ]. Some results are rather recent [ ] or may be new.

(2.1) The PSL(2,$\mathbb{R}$) acts on the upper half complex plane $\mathbb{H}$ from left by the fractional linear transformation. A discrete subgroup Γ of PSL(2,$\mathbb{R}$) is a Fuchsian surface group, if it acts on $\mathbb{H}$ fixed point freely and the quotient $\Gamma\backslash\mathbb{H}$, denoted by X, is compact. The X is a Riemann surface of genus $g>1$, where the natural projection

(2.1.1) $$u : \mathbb{H} \longrightarrow X$$

is a universal covering map. Conversely, for a given compact Riemann surface X of genus $g>1$, a universal covering map (2.1.1) exists up to an ambiguity of $\Gamma\backslash$PSL(2,$\mathbb{R}$)(= a circle bdle over X).

As is well known, on the surface X one can choose $2g$ oriented circles:

(2.1.2) $$a_1,\ b_1,\ a_2,\ b_2,\ \ldots\ ,\ a_g,\ b_g$$

which intersect in only one point O as in the Fig.1 (cf Fig.8).

The intersection numbers of the circles are easily seen by deforming the Fig.1 to Fig.1' at O.

(2.1.3) $$\langle[a_i],[a_j]\rangle = \langle[b_i],[b_j]\rangle = 0 \quad \text{for } i,j=1,\ldots,g,$$
$$\langle[a_i],[b_j]\rangle = \delta_{ij} \quad \text{for } i,j=1,\ldots,g,$$

where $[a]$ describe the homology class of a in $H_1(X,\mathbb{Z})$.

Cut the surface X along the circles $a_1,\ldots,b_g$(called a canonical dissection) and develop it to a surface Y with boundary. From Fig.1 one sees easily that the boundary of Y is a *connected* circle $a_1b_1a_1^{-1}b_1^{-1}..a_gb_ga_g^{-1}b_g^{-1}$ by looking the interier of Y in the left side. The determinant of the intersection matrix of (2.1.3) is equal to 1 so that $[a_i]$, $[b_i]$ (i=1,...,g) span the homology group $H_1(X,\mathbb{Z})$. Thus Y is homologically trivial and then it is simply connected by classification of surfaces.(See Fig.2.)

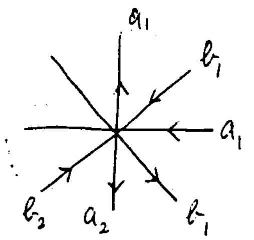


Fig.1 Fig.1'

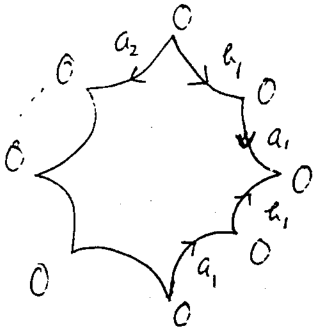


Fig.2

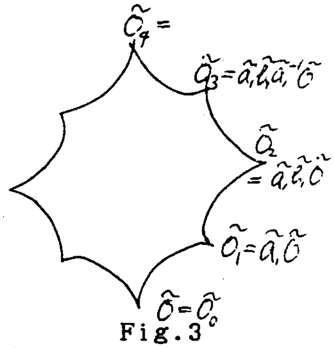


Fig.3

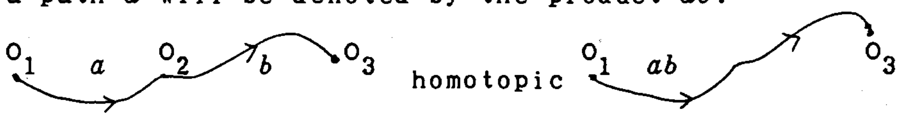
(2.2) The homotopy classes in $\pi_1(X,O)$ represented by the circles $a_1,\ldots,b_g$ will be denoted by

(2.2.1) $$\tilde{a}_1,\ \tilde{b}_1,\ \ldots\ ,\ \tilde{a}_g,\ \tilde{b}_g.$$

Then the fundamental group $\pi_1(X,O)$ is generated by (2.2.1) with a single relation:

(2.2.2) $$\tilde{a}_1\tilde{b}_1\tilde{a}_1^{-1}\tilde{b}_1^{-1}\ \ldots\ \tilde{a}_g\tilde{b}_g\tilde{a}_g^{-1}\tilde{b}_g^{-1} = 1$$

As a convention in this paper, the succession of a path b after a path a will be denoted by the product ab.



Let us choose and fix a point $\tilde{O}$ of $\mathbb{H}$, which is projected to $O\in X$. Then the fundamental group $\pi_1(X,O)$ acts on $\mathbb{H}$ from the left as the covering transformations for(2.1.1).This induces the isomorphism:

(2.2.3) $$\varphi\ :\ \pi_1(X,O) \simeq \Gamma\ .$$

By this φ, we regard (2.2.1) as a generater system for Γ. The change of $\tilde{O}$ to $\gamma\tilde{O}$ for $\gamma\in\Gamma$ induces the change of φ to $\mathrm{ad}(\gamma)\circ\varphi$.

Let us lift the movement on the surface Y along the boundary $a_1,b_1,a_1^{-1},b_1^{-1},\ldots,a_g,b_g,a_g^{-1},b_g^{-1}$ in anti-clockwise direction, to the movement on the surface $\mathbb{H}$ starting from $\tilde{O}$. Since Y is simply connected, it form a polygon of $4g$ vertexes $\tilde{O}_i$ $(i=0,\ldots,4g)$:

(2.2.4) $$\tilde{O}_i := R_i\tilde{O} \qquad (i=0,\dots,4g),$$
with
(2.2.5) $$R_i := \prod_{k=1}^{i} \tilde{c}_i = \tilde{c}_1\tilde{c}_2\dots\tilde{c}_i \qquad (i=0,\dots,4g),$$

where $\tilde{c}_1,\dots,\tilde{c}_{4g}$ is the sequence $\tilde{a}_1,\tilde{b}_1,\tilde{a}_1^{-1},\tilde{b}_1^{-1},\dots,\tilde{a}_g,\tilde{b}_g,\tilde{a}_g^{-1},\tilde{b}_g^{-1}$.

(*I*.e. $R_0:=1$, $R_1:=\tilde{a}_1$, $R_2:=\tilde{a}_1\tilde{b}_1$, $R_3:=\tilde{a}_1\tilde{b}_1\tilde{a}_1^{-1}$, $R_4:=\tilde{a}_1\tilde{b}_1\tilde{a}_1^{-1}\tilde{b}_1^{-1}$,... .., $R_{4g-1}:=\tilde{a}_1\tilde{b}_1\dots\tilde{a}_g^{-1}$, $R_{4g}:=\tilde{a}_1\tilde{b}_1\dots\tilde{a}_g^{-1}\tilde{b}_g^{-1} = 1$.)

The polygon surounds a domain Z in $\mathbb{H}$, which is homeomorphic to Y by the map (2.1.1) (cf. Fig.3). Obviously, Z is a fundamental domain for the action of Γ on $\mathbb{H}$. The i-th edge of Z between $\tilde{O}_{i-1}$ and $\tilde{O}_i$ is denoted by $[\tilde{O}_{i-1}\tilde{O}_i]$.

There exists a unique element of Γ, which transforms the vertex $\tilde{O}_i$ to $\tilde{O}_j$, which is explicitely given by $R_jR_i^{-1}$.

(Note that the homotopy class $\tilde{c}_i$ of the circle on X of the image of edge $[\tilde{O}_{i-1}\tilde{O}_i]$ is given by $R_{i-1}^{-1}R_i$ but not by $R_iR_{i-1}^{-1}$.)

(2.3) The following generator system (2.3.5) of Γ associated to the polygon Z is classical.([ ], See Fig 4.)

For each i with $1\le i\le g$, the elements of Γ bringing $\tilde{O}_{2+4(i-1)}$ to $\tilde{O}_{1+4(i-1)}$ and $\tilde{O}_{3+4(i-1)}$ to $\tilde{O}_{4(i-1)}$ are the same, veryfied from (2.2.5). Let us denoted it by $\tilde{\alpha}_i$.

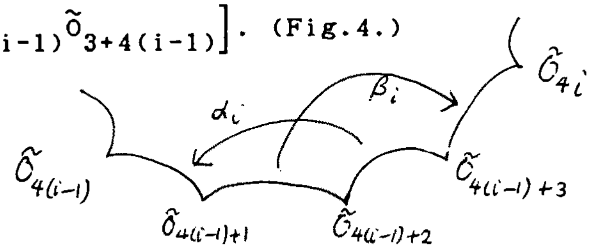
(2.3.1) $$\tilde{\alpha}_i := R_{1+4(i-1)}\, R_{2+4(i-1)}^{-1} = R_{4(i-1)}\, R_{3+4(i-1)}^{-1}.$$
$$= R_{4(i-1)}\, \tilde{a}_i\, \tilde{b}_i^{-1}\, \tilde{a}_i^{-1}\, R_{4(i-1)}^{-1} .$$

Samely, the elements of Γ bringing $\tilde{O}_{1+4(i-1)}$ to $\tilde{O}_{4+4(i-1)}$ and $\tilde{O}_{2+4(i-1)}$ to $\tilde{O}_{3+4(i-1)}$ are the same, denoted by $\tilde{\beta}_i$.

$$(2.3.2) \qquad \tilde{\beta}_i := R_{4+4(i-1)} R^{-1}_{1+4(i-1)} = R_{3+4(i-1)} R^{-1}_{2+4(i-1)}.$$
$$= R_{4(i-1)}\, \tilde{a}_i\, \tilde{b}_i\, \tilde{a}_i^{-1}\, \tilde{b}_i^{-1}\, \tilde{a}_i^{-1}\, R^{-1}_{4(i-1)} .$$

These mean that $\tilde{\alpha}_i$ is an element of PSL(2,ℝ), which transforms the edge $\left[\tilde{O}_{2+4(i-1)}\tilde{O}_{3+4(i-1)}\right]$ to the edge $\left[\tilde{O}_{1+4(i-1)}\tilde{O}_{4(i-1)}\right]$ and that $\tilde{\beta}_i \in$ PSL(2,ℝ) transforms the edge $\left[\tilde{O}_{1+4(i-1)}\tilde{O}_{2+4(i-1)}\right]$ to the edge $\left[\tilde{O}_{4+4(i-1)}\tilde{O}_{3+4(i-1)}\right]$. (Fig.4.)

Fig.4



Using (2.3.1) and (2.3.2), one calculates easily

$$(2.3.3) \qquad \tilde{\alpha}_i\tilde{\beta}_i\tilde{\alpha}_i^{-1}\tilde{\beta}_i^{-1} = R_{4(i-1)}\, \tilde{b}_i^{-1}\tilde{a}_i^{-1}\tilde{b}_i\tilde{a}_i\, R^{-1}_{4(i-1)} = R_{4(i-1)}R^{-1}_{4i},$$

so that finally one obtains a relation (cf. (2.4.7)):

$$(2.3.4) \qquad \tilde{\alpha}_1\tilde{\beta}_1\tilde{\alpha}_1^{-1}\tilde{\beta}_1^{-1} \ldots \tilde{\alpha}_g\tilde{\beta}_g\tilde{\alpha}_g^{-1}\tilde{\beta}_g^{-1} = 1 .$$

Since one can easily solve (2.3.1) and (2.3.2) to obtain an expression of $\tilde{a}_i$ and $\tilde{b}_i$ by means of $\tilde{\alpha}_1,\ldots,\tilde{\beta}_g$ (for an explicite form, see (2.4.4) and (2.4.4)'), we conclude that:

Assertion *The Fuchsian group* Γ *is generated by the system:*

$$(2.3.5) \qquad \tilde{\alpha}_1,\ \tilde{\beta}_1,\ \tilde{\alpha}_2,\ \tilde{\beta}_2,\ \ldots\ ,\ \tilde{\alpha}_g,\ \tilde{\beta}_g$$

where $\tilde{\alpha}_i$ *and* $\tilde{\beta}_i$ *are elements of* PSL(2,ℝ) *bringing edges of the polygon Z from* $[\tilde{O}_{2+4(i-1)}\tilde{O}_{3+4(i-1)}]$ *to* $[\tilde{O}_{1+4(i-1)}\tilde{O}_{4(i-1)}]$ *and*

$[\tilde{O}_{1+4(i-1)}\tilde{O}_{2+4(i-1)}]$ *to* $[\tilde{O}_{4+4(i-1)}\tilde{O}_{3+4(i-1)}]$ *respectively.* (See Fig.4.)

The system (2.3.5) *satisfies the single relation* (2.3.4).

(2.4) Let us give another interpretation of the system (2.3.5). For the canonical dissection (2.1.2), let us define the "dual" canonical dissection of X as a system of $2g$ circles on X :

(2.4.1) $$\alpha_1,\ \beta_1,\ \alpha_2,\ \beta_2,\ \dots\ ,\ \alpha_g,\ \beta_g$$

intersecting in only one point Ω as in the Fig.5 (in an opposite ordering than the Fig.1) such that the circles in system (2.1.2) and in system (2.4.1) intersects in only the following cases:

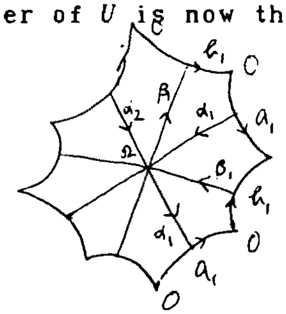
a) a_i and α_i intersects normally at a point with the sign

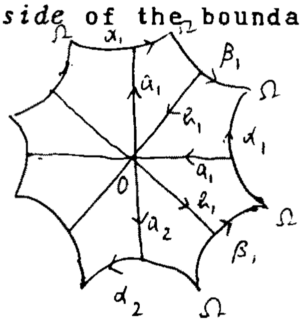
(2.4.2) $$\langle[\alpha_i],[a_i]\rangle = 1 \qquad \text{for } i=1,\dots,g.$$

b) b_i and β_i intersects normally at a point with the sign

(2.4.2)' $$\langle[\beta_i],[b_i]\rangle = -1 \qquad \text{for } i=1,\dots,g.$$

The existence and the uniqueness (up to the isotopy) of such dual dissection (2.4.1) may be clear from the Fig.5 drawn on Y. By this new cut system (2.4.1), the surface X is developped to a simply connected surface, say U (Fig.7). We remark that the interier of U is now the *right side* of the boundary move of the cut.





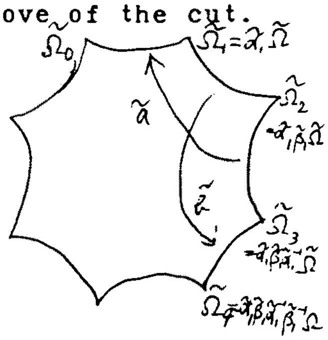


Fig. 5 Fig. 6 Fig. 7

Now the point Ω is inside the surface Y, which is homeomorphc to Z by (2.1.1) and one can choose uniquely (after a choice of $\tilde{O}$) a point $\tilde{\Omega}$ in $Z \subset \mathbb{H}$ which is projected to Ω in X. Then the fundamental group $\pi_1(X,\Omega)$ acts on $\mathbb{H}$ from the left as covering transformations for (2.1.1) so that one obtains an isomorphism:

$$\phi : \pi_1(X,\Omega) \simeq \Gamma . \tag{2.4.3}$$

It is a straightforward consideration to show that:

Assertion 1. *By the isomorphism* ϕ, *the homotopy classes in* $\pi_1(X,\Omega)$ *of the circles* α_i and β_i $(i=1,\ldots,g)$ (2.4.1) *are mapped to the generater system* $\tilde{\alpha}_i$ and $\tilde{\beta}_i$ $(i=1,\ldots,g)$ (2.3.5) *of the* Γ *respectively. By this identification, we shall regard* $\tilde{\alpha}_i$ *and* $\tilde{\beta}_i$ *as homotopy classes in* $\pi_1(X,\Omega)$.

Let V be the polygon in $\mathbb{H}$ surrounded by the vertexes

$$\tilde{\Omega}_i := \mathfrak{R}_i \tilde{\Omega} \qquad (i=0,\ldots,4g), \tag{2.4.4}$$

where

$$\mathfrak{R}_i := \prod_{k=1}^{i} \tilde{\gamma}_k = \tilde{\gamma}_1 \ldots \tilde{\gamma}_i \qquad (i=0,\ldots,4g), \tag{2.4.5}$$

and $\tilde{\gamma}_1,\ldots,\tilde{\gamma}_{4g}$ is the sequence $\tilde{\alpha}_1,\tilde{\beta}_1,\tilde{\alpha}_1^{-1},\tilde{\beta}_1^{-1},\ldots,\tilde{\alpha}_g,\tilde{\beta}_g,\tilde{\alpha}_g^{-1},\tilde{\beta}_g^{-1}$. V is homeomorphic to U by the map (2.1.1), which gives another fundamental domain for Γ.

Assertion 2. *The* $\tilde{a}_i \in \Gamma$ ((2.2.1)) *transforms the edges of the polygon* V *from* $\left[\tilde{\Omega}_{2+4(i-1)}\tilde{\Omega}_{3+4(i-1)}\right]$ *to* $\left[\tilde{\Omega}_{1+4(i-1)}\tilde{\Omega}_{4(i-1)}\right]$ *and* $\tilde{b}_i \in \Gamma$ *transforms* $\left[\tilde{\Omega}_{1+4(i-1)}\tilde{\Omega}_{2+4(i-1)}\right]$ *to* $\left[\tilde{\Omega}_{4+4(i-1)}\tilde{\Omega}_{3+4(i-1)}\right]$

for i=1,...,g (cf Fig.8) *so that one has a representation*:

(2.4.6) $\quad \tilde{a}_i := \mathfrak{R}_{1+4(i-1)}\, \mathfrak{R}^{-1}_{2+4(i-1)} = \mathfrak{R}_{4(i-1)}\, \mathfrak{R}^{-1}_{3+4(i-1)}$.

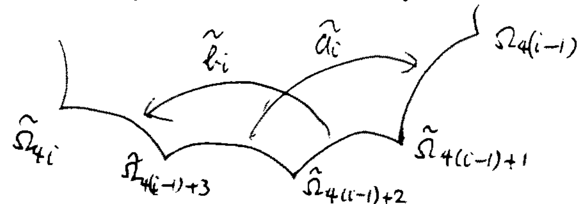
(2.4.6)' $\quad \tilde{b}_i := \mathfrak{R}_{4+4(i-1)}\, \mathfrak{R}^{-1}_{1+4(i-1)} = \mathfrak{R}_{3+4(i-1)}\, \mathfrak{R}^{-1}_{2+4(i-1)}$.

There is a relation between R_i(2.2.4) and $\mathfrak{R}_i$(2.4.5) as follows.

(2.4.7) $\quad R_{4i}\mathfrak{R}_{4i} = 1 \quad$ for i=1,...,g.

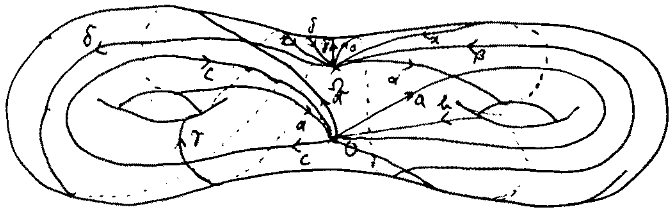
(This follows imediately from (2.3.3) by an induction on i.)

Fig. 8



A cannonical dissections on a surface of genus 2 and its dual canonical dissection is drawn in the Fig. 9.

Fig. 9



(2.5) Let us call the correction of polygons γZ ($\gamma \in \Gamma$) on $\mathbb{H}$ a tesselation T for the cut system (2.1.2). Precisely, T consists of the data of a correction of the faces, edges, and vertexes:

(2.5.1)
$$T_2 := \{ \gamma Z : \gamma \in \Gamma \},$$
$$T_1 := \{ \gamma[\tilde{O}_i \tilde{O}_{i+1}] : \gamma \in \Gamma,\ i=0,\dots,4g-1 \},$$
$$T_0 := \{ \gamma\tilde{O} : \gamma \in \Gamma \},$$

such that one has a locally finite cell decomposition:

(2.5.2) $$\mathbb{H} = \bigcup_{F\in T_2}\mathring{F} \ \cup \bigcup_{E\in T_1}\mathring{E} \ \cup \bigcup_{V\in T_0} V \quad ,$$

($\mathring{F}$ and $\mathring{E}$ denote the interier of F and E respectively), which is invariant by the action of Γ, inducing the system (2.1.2).

Two faces γZ and $\delta Z \in T_2$ are adjacent(ie. have a common edge) if and only if either $\gamma^{-1}\delta$ or $\delta^{-1}\gamma$ is a generater in (2. .). (Remark that $\delta\gamma^{-1}$ brings γZ to δZ but not $\gamma^{-1}\delta$.) Two vertexes $\gamma\tilde{O}$ and $\delta\tilde{O}$ are adjascent so that $[\gamma\tilde{O},\delta\tilde{O}]$ is an edge in T_1 if and only if either $\gamma^{-1}\delta$ or $\delta^{-1}\gamma$ is a generater in (2. .).

Suppose that the faces γZ and δZ and the vertexes $\gamma'\tilde{O}$ and $\delta'\tilde{O}$ are adjucent through an edge as in the Fig. .
Then one has

(2.5.3) $$\gamma^{-1}\delta = \alpha_i \quad \Leftrightarrow \quad \gamma'^{-1}\delta' = a_i,$$

(2.5.4) $$\gamma^{-1}\delta = \beta_i \quad \Leftrightarrow \quad \gamma'^{-1}\delta' = b_i^{-1},$$

for $i=1,\ldots,g$. Fig.

A face $\gamma Z \in T_2$ and a vertexe $\delta\tilde{O} \in T^0$ are adjascent if and only if $\gamma^{-1}\delta \in \{ R_{4i} : i=1,\ldots,g\}$ or equivalently $\delta^{-1}\gamma \in \{ \mathfrak{R}_{4i} : i=1,\ldots,g\}$, where the arrangement of the vertexes is the same as the Fig. .

(2.5.6) $$\gamma\mathfrak{R}_i\tilde{Z} \in T_2 \quad (i=0,1,\ldots,4g)$$

and edges

(2.5.7) $$\gamma[\tilde{O},\tilde{c}_i\tilde{O}\ ] \in T_1 \quad (i=1,\ldots,4g)$$

are joining in a clockwise cyclic ordering. (For $\tilde{c}_i = R_{i-1}^{-1}R_i$, recall (2.2.5).)

Remark 1. Similar to T, one may consider a tessellation $T^*=(T_2^*,T_1^*,T_0^*)$ for the cut system (2.4.1). Then T and T^* are dual in the sence that there are one to one correspondences:

(2.5.8)
$$F \in T_2 \longleftrightarrow V^* \in T_0^* \quad \text{with} \quad V^* \in F$$
$$E \in T_1 \longleftrightarrow E^* \in T_1^* \quad \text{with} \quad E \quad E^*,$$
$$V \in T_0 \longleftrightarrow F^* \in T_2^* \quad \text{with} \quad V \in F^*.$$

(Here the notation " " means "intersect transverally".)
Thus the data of T and T^* are equivalent so that we shall treat mainly T in the present paper.

2. The combinatorial data of T is nothing but the Caylay graph in combinatorial group theory for the group Γ with the generater system (2.2.1). (Cf. [ ].)

(2.6) Word problem for the group Γ.

The word problem for is a problem to give a finite processes to determine whether two words written by the letters in give the same element in Γ. We recall a solution due to Dehn [, ,].

Assertion (Dehn) *Any nonempty word W in the generater system representing the identity element in* Γ *can be shortened by one of two processes:*

i) *delete a word* cc^{-1} *from* W,

ii) *for a cyclic permutation* $R'=A\cdot B^{-1}$ *of* R *with* $\ell(A)>\ell(B)$, *replace* A *in* W *by* B.

(2.7) Dihedral group action on Γ.

The following is a straightforward consideration.

Assertion *The set of automorphisms of* Γ *preserving the set* $G:=\{\alpha_1,\alpha_1^{-1},\beta_1,\beta_1^{-1},\ldots,\alpha_g,\alpha_g^{-1},\beta_g,\beta_g^{-1}\}$ *is isomorphic to a dihedral group with two generators* φ, ϕ *and three relations:*

(2.7.1) $$D_g = \langle\ \varphi,\ \phi \mid \varphi^g=1,\ \phi^2=1,\ (\phi\varphi)^2=1\ \rangle$$
$$= \{\varphi^0,\varphi^1,\varphi^2,\ldots,\varphi^{g-1},\ \phi,\ \phi\varphi,\ \phi\varphi^2,\ldots,\ \phi\varphi^{g-1}\}.$$

Here φ *is a cyclic rotation:*

(2.7.2) $$\varphi(\alpha_i) := \varphi(\alpha_{i+1}) \qquad \textit{for } i=1,\ldots,g \textit{ with } \alpha_{g+1}:=\alpha_1,$$
$$\varphi(\beta_i) := \varphi(\beta_{i+1}) \qquad \textit{for } i=1,\ldots,g \textit{ with } \beta_{g+1}:=\beta_1,$$

and ϕ *is an orientation reversing homomorphism:*

(2.7.3) $$\phi(\alpha_i) := \beta_{g-i+1} \qquad \textit{for } i=1,\ldots,g,$$
$$\phi(\beta_i) := \alpha_{g-i+1} \qquad \textit{for } i=1,\ldots,g.$$

Proof The dihedral group above obviously preserves G. Let ψ be an automorphism of Γ preserving G. In the free group generated by G, the element $R:=\psi(\alpha_1)\psi(\beta_1)\psi(\alpha_1^{-1})\psi(\beta_1^{-1})\ldots\psi(\alpha_g^{-1})\psi(\beta_g^{-1})$ is conjugate to either $S:=\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\ldots\alpha_g^{-1}\beta_g^{-1}$ or S^{-1}. Since R is an element of length $4g$, we see that R is a cyclic permutation of either S or S^{-1}.

(2.8) The growth functions.

As a consequence of the solution of the word problem, we can determine the growth function and related functions [,].

Let Γ be as before a Fuchsian surface group and fix a generator system (2. .). For each $\gamma\in\Gamma$, we define the length by

(2.8.1) $\ell(\gamma) := \inf\{n : \gamma = \gamma_1 \cdots \gamma_n,\ \text{either } \gamma_i \text{ or } \gamma_i^{-1} \in \quad \}$.

Then the growth function of Γ relative to   is defined to be

(2.8.2) $a_n := \#\{ \gamma \in \Gamma : \ell(\gamma) = n \}$,

whose generating function, called a growth power siries for Γ, is also defined to be

(2.8.3) $$P(t) := \sum_{n=0}^{\infty} a_n t^n = \sum_{\gamma \in \Gamma} t^{\ell(\gamma)} .$$

The $P(t)$ is calculated by Cannon, Wagreich [   ].

(2.8.4) $$P(t) = \frac{(1+t)(1-t^{2g})}{1 - (4g-1)t + (4g-1)t^{2g} - t^{2g+1}}$$

For a later purpose, we give a proof of the formula including some auxiliary functions.

For a face $\gamma Z \in T^2$, the length $\ell(\gamma Z)$ is defined to be $\ell(\gamma)$.

I.e. $\ell(\gamma Z) := \ell(\gamma)$.

For an edge $E \in T^1$ and a vertex $V \in T^0$, length are defined to be:

$$\ell(E) := \inf\{ n : {}^{\exists}F \in T^2 \text{ s.t. } \ell(F) = n \text{ and } E \subset \overline{F} \},$$

$$\ell(V) := \inf\{ n : {}^{\exists}F \in T^2 \text{ s.t. } \ell(F) = n \text{ and } V \in \overline{F} \}.$$

The number of elements of length n are defined as

f_n := the number of faces of length n = a_n,

e_n := the number of edges of length n,

and v_n := the number of vertexes of length n,

whose generating functions are denoted by

$f(t) := \sum f_n t^n$,

$e(t) := \sum e_n t^n$

and $v(t) := \sum v_n t^n$.

A face of length n is said to be overlapping if it is adjacent to two faces of length n-1. Then we put,

o_n := the number of overlapping faces of length n

$o(t) := \sum o_n t^n$.

Under these notations, we show the following recursion relations.

$$f_n = (4g-1)f_{n-1} - o_{n-1} - o_n \quad , \text{ for } n\geq 2,$$

$$e_n = (4g-1)f_n - o_n \quad , \text{ for } n\geq 1,$$

$$v_n = (4g-2)f_n - o_n \quad , \text{ for } n\geq 1,$$

$$o_n = v_{n-2} \quad , \text{ for } n\geq 0.$$

(2.8.) $$f(t) = \frac{(1+t)(1-t^{2g})}{1-(4g-1)t+(4g-1)t^{2g}-t^{2g+1}}$$

(2.8.) $$e(t) = \frac{4g\ (1-t^{2g})}{1-(4g-1)t+(4g-1)t^{2g}-t^{2g+1}}$$

(2.8.) $$v(t) = \frac{4g\ (1-t)}{1-(4g-1)t+(4g-1)t^{2g}-t^{2g+1}}$$

(2.8.) $$o(t) = \frac{4g\ t^{2g}(1-t)}{1-(4g-1)t+(4g-1)t^{2g}-t^{2g+1}}$$

The roots of the denominater polynomial lie on the unit circle except for a pair of real roots ω and ω^{-1}, where the bigger one, say ω^{-1}, is aproximately given by

$$4g-1-(4g-1)^{-2(g-1)} < \omega^{-1} < 4g-1$$

§3. Configuration algebra

Most of the results are varid for finitely generated infinite groups, or even for infinite graphs with certain homogenety.

(3.1) Configurations in Γ.

Due to the tessolation T on $\mathbb{H}$ (2. .), the group Γ as a set has a structure of an oriented colored graph. Namely, two element γ and δ of Γ are connected by an edge if and only if $\gamma^{-1}\delta$ belongs to the set of the generators (2. .). The edge carries a color with an orientation according to the elements $\gamma^{-1}\delta$ in

A subset S of Γ inherits the oriented colored graph structure from Γ. Two subsets S and $\mathbb{T}$ are said to have the same configuration type, if both have the same oriented colored graph structure (This means that if $S= \underset{i\in I}{\ } S_i$ and $\mathbb{T} = \underset{j\in J}{\ } \mathbb{T}_j$ are decompositions to their connected components, then there is a bijection $\varphi: I \to J$ and a map $\phi: I \longrightarrow \Gamma$ such that $\mathbb{T}_{\varphi(i)} = \phi(i)S_i$ for $^{\forall} i\in I$.)

By a configuration type S, we shall simply mean such a colored graph of a subset S of Γ. In this paper, we shall consider only finite configulations, ie. S with $\#S<\infty$. Let us fix notations:

(3.1.1) $\quad$ Conf := the set of all finite configuration types,

(3.1.2) $\quad$ Conf_0:= { $S\in$ Conf : S is connected.} .

Note. As a convention, we shall not include the void set ϕ in Conf. When it is convienent to include ϕ as a configuration, we shall use a notation: $\mathrm{Conf}\cup\{\phi\}$.

The set Conf naturally carries an abelian semigroup structure by taking the disjoint union $S\ T$ as the product of S and $T\in$Conf. Thus $\mathrm{Conf}\cup\{\phi\}$ is naturally identified with the free abelian semigroup generated by Conf_0, where ϕ is the unit element.

(3.2) Configuration algebra

Let us consider the algebra,

(3.2.1) $$\mathbb{Z}\Big[\ \mathrm{Conf}\ \Big]$$

over the semigroup Conf, (i.e. the algebra generated by the set Conf with the relations $S\cdot T = S\ \ T$ for $^{\forall}S,\ T\in$ Conf.), which may be regarded as the polynomial algebra freely generated by Conf_0.

The algebra carries a graded algebra structure by taking the cadinarity $\#(S)$ of a configuration as the degree of S, due to the aditivity:

(3.2.2) $$\#(S\cdot T) = \#(S) + \#(T)\ .$$

The adic topology on the algebra is defined by taking the sequence of ideals

(3.2.3) $$\mathcal{J}_n := \text{the ideal gnerated by } S\in\mathrm{Conf} \text{ with } \#(S)\geq n$$

as the fundamental system of neighbourhoods of 0. The formal completion $\mathbb{Z}\Big[\Big[\ \mathrm{Conf}\ \Big]\Big] := \varprojlim_n \mathbb{Z}\Big[\mathrm{Conf}\Big]/\mathcal{J}_n$ of the algebra is denoted by

(3.2.4) $$R(\Gamma)$$

and will be called *the configuration algebra.* The augumentation ideal $\varprojlim_n \mathcal{J}_1/\mathcal{J}_n$ of the configuration algebra will be denoted by

(3.2.5) $$R(\Gamma)_+$$

For an algebra $\mathbb{A}$ with a unit element, the extension of the coefficients is dnoted as

(3.2.6) $$R(\Gamma,\mathbb{A}) := R(\Gamma)\otimes_{\mathbb{Z}}\mathbb{A}$$

(3.2.6)' $$R(\Gamma,\mathbb{A})_+ := R(\Gamma)_+\otimes_{\mathbb{Z}}\mathbb{A}\quad .$$

An elment A of $R(\Gamma,\mathbb{A})_+$ is uniquely expressed by an infinite sum:

$$A = \sum_{S\in\mathrm{Conf}} A(S)S$$

for $A(S)\in\mathbb{A}$ ($S\in\mathrm{Conf}$).

(3.3) Exponential and logarithmic maps.

Let $\varphi\in\mathbb{A}[[t]]$ be a formal power series in t with coefficients in $\mathbb{A}$. Then one may define a map:

$$\varphi : R(\Gamma,\mathbb{A})_+ \longrightarrow R(\Gamma,\mathbb{A})$$

by substituting $f\in R(\Gamma,\mathbb{A})_+$ in the variable t. The map is equivariant with any algebra automorphism of $R(\Gamma,\mathbb{A})$.

As the particular case, let us consider the exponential map

(3.3.1) $$M \in R(\Gamma,\mathbb{Q})_+ \longrightarrow \exp(M) -1 \in R(\Gamma,\mathbb{Q})_+$$

and the logarithm map

(3.3.2) $$A \in R(\Gamma,\mathbb{Q})_+ \longrightarrow \log(1+A) \in R(\Gamma,\mathbb{Q})_+$$

where $\exp(M) := \sum_{n=0}^{\infty} \frac{1}{n!} M^n$ and $\log(1+A) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} A^n$.

The maps are well defined and inverse of each other due to the standard argument for complete algebras. Hence they give a bijection of $R(\Gamma,\mathbb{Q})$ to itself.

The additivity: $\exp(M_1+M_2) = \exp(M_1)\exp(M_2)$ and the multiplicativity: $\log((1+A_1)(1+A_2)) = \log(1+A_1) + \log(1+A_2)$ are standard.

Formula. *Let* $A := \sum_{S\in\mathrm{Conf}} \mathrm{A}(S)S$ *and* $M := \sum_{S\in\mathrm{Conf}} \mathrm{M}(S)S$ *be elements of* $R(\Gamma,\mathbb{Q})_+$ *such that* $1+A = \exp(M)$ *(or equivalently,*

$M=\log(1+A)$). *Then the coefficients of A and M are related by the transformations:*

$$(3.3.5) \qquad A(S) = \sum_{S=k_1S_1 \cdots k_mS_m} \frac{1}{k_1!\cdots k_m!} M(S_1)^{k_1} \cdots M(S_m)^{k_m}$$

and

$$(3.3.6) \qquad M(S) = \sum_{S=k_1S_1 \cdots k_mS_m} \frac{(k_1+\cdots+k_m-1)!(-1)^{k_1+\cdots+k_m-1}}{k_1!\cdots k_m!} A(S_1)^{k_1}\cdots A(S_m)^{k_m}.$$

Here the summations are over all possible decompositions:

$$(3.3.7) \qquad S = k_1S_1 \quad \cdots \quad k_mS_m$$

such that $S_i \in$ Conf $(i=1,\ldots,m)$ *are pairwisely different.*

(Here we use a convention $kS := \underbrace{S \quad \cdots \quad S}_{k\text{-copies}}$ for $k \in \mathbb{N}$ and $S \in$ Conf.)

Proof Recall that the configuration algebra may be regarded as the algebra of formal power serieses of the set of variables Conf_0. Even the set Conf_0 is infinite, the standard arguments for exponential and logarithmic functions are valid so that we omitt the proof. □

Remark We shall often use the notations A and M to indicate the relation $1+A=\exp(M)$. In spite of original attempt to indicate "additive" and "multiplicative" by A and M, our use of them are cofused as avbove.

(3.4) Representation of $R(\Gamma)$ in a power series.

To a configuration type $S \in \mathrm{Conf} \cup \{\phi\}$, let us associate a vector

$$E(S) = (e_1,\ldots,e_{2g}) \in \mathbb{N}_0^{2g},$$

called the exponent of $\mathcal{S}$ by

(3.4.1) $$e_{2i-1} := \#\{\gamma\in\Gamma\backslash S:\ ^{\exists}\delta\in S \text{ s.t. } \gamma^{-1}\delta=\alpha_i \text{ or } \alpha_i^{-1}\},$$

$$e_{2i} := \#\{\gamma\in\Gamma\backslash S:\ ^{\exists}\delta\in S \text{ s.t. } \gamma^{-1}\delta=\beta_i \text{ or } \beta_i^{-1}\},$$

where S is a subset of Γ of the type $\mathcal{S}$. $E(\mathcal{S})$ depends only on the type $\mathcal{S}$ but not on S. The exponent has the following additivity:

(3.4.2) $$E(\mathcal{S}_1\ \ \mathcal{S}_2) = E(\mathcal{S}_1) + E(\mathcal{S}_2)\ .$$

Let us introduce $2g$ indetermintes $X = (X_1,\ldots,X_{2g})$ and put

(3.4.3) $$X^{E(\mathcal{S})} := X_1^{e_1}\cdot X_2^{e_2}\cdot\ldots\cdot X_{2g}^{e_{2g}}\ .$$

The additivity (3.4.2) implies that the correspondence $\mathcal{S} \longrightarrow X^{E(\mathcal{S})}$ is a semigroup homomorphism from Conf to the set of monomials in X, so that one has a well defined ring homomorphism:

(3.4.4) $$\mathbb{Z}\left[\mathrm{Conf}\right] \longrightarrow \mathbb{Z}[X_1,X_2,\ldots,X_{2g}]$$

We shall regard that the polynomial ring $\mathbb{Z}[X]$ carries the adic topology defined by the maximal ideal $(X)=(X_1,\ldots,X_{2g})$. To show that (3.4.4) is continuous w.r.t. the toplogies, we prepare a Lemma. Let us denote by $|E(\mathcal{S})|$ the sum $\sum_{i=1}^{2g} e_i$.

Lemma *For* $\mathcal{S}\in$Conf, *one has*

(3.4.5) $$|E(\mathcal{S})| > 4(g-1)\cdot\#(\mathcal{S}).$$

Corollary *The homomorphism* (3.4.4) *is continuous w.r.t. the adic topology. Hence it induces a ring homomorphism between* *the formal completions:*

(3.4.6) $$R(\Gamma) \longrightarrow \mathbb{Z}[[X_1, X_2, \ldots, X_{2g}]] \; .$$

Proof of the Lemma. Due to the additivities (3.2.3) and (3.4.2), we have only to consider the case S is connected.

Since any vertex of Γ as a graph has $4g$ edges, we have

*) $$|E(S)| = 4g\ \#(S) - 2\ \#\{\text{edges on } S\} \; .$$

The following Euler's relation on a graph is well known.

**)

$$\deg(S) - \#\{\text{edges on } S\} + \#\{\text{linearly independent cycles on } S\} = n(S),$$

where $n(S)$ denotes the number of connected components of S.

Combining the relations *) and **),

(3.4.7) $$|E(S)| = (4g-2)\cdot\#(S) + 2\ n(S) - 2\ \#\{\text{linearly independent cycles on } S\}.$$

Let us show an inequality:

(3.4.8) $$\#\{\text{linearly independent cycles on } S\} < \#(S),$$

by induction on $\#(S)$. There is no cycles on S sofar $\#(S)<4g$. Let S be a subset of Γ, whose configuration type is S ($\neq\phi$). Since $\#S=\#(S)<\infty$, there is a point $\gamma_0 \in S$ such that $\ell(\gamma_0)=\max\{\ell(\gamma):\gamma\in S\}$. Then the vertex of S corresponding to γ_0 is adjacent to at most two other vertexes of S (cf (. .)). Then it is easy to show that one can find a basis of cycles on S such that at most one cycle of them passes the vertex γ_0. This means that the # of linearly independent cycles on $S\setminus\{\gamma_0\}$ is non less than the # of linearly independent cycles on S minus 1.

(3.4.7) and (3.4.8) imply (3.4.5). □

Remark 1. The map (3.4.6) is not a graded homomorphism.

2. The map (3.4.6) is an open map, since one has an estimate: $4g\cdot\#(S) \geq |E(S)|$ in the other direction than (3.4.5).

(3.5) Recall the dihedral group D_{2g} (2.7.1) action on Γ as automorphism of the group preserving the set of generators G. The action of D_{2g} on Γ preserves the graph structure up to a change of the colors and orientations. Hence the action induces an action on the semigroup Conf. Since the action is continuous w.r.t. the adic topology, it induces an automorphism of the configuration algebra:

(3.5.1) $$D_{2g} \times R(\Gamma) \longrightarrow R(\Gamma)$$

On the other hand the dihedral group D_{2g} acts on the monomials in $X=(X_1,\ldots,X_{2g})$ by

(3.5.2) $$\varphi(X_i) := X_{i+2} \quad (i=1,\ldots,2g) \text{ (here } X_{2g+i}:=X_i)$$

and

(3.5.3) $$\phi(X_i) := X_{2g-i+1} \quad (i=1,\ldots,2g).$$

Comparing (2.7.2), (2.7.3) and (3.5.2), (3.5.3), one sees:

For any $\alpha\in D_{2g}$ and $S\in$ Conf, one has

$$X^{E(\alpha(S))} = \alpha(X^{E(S)}) .$$

Hence the actions of D_{2g} on the configuration algebra and on the power series in X is equivariant with the homomorphism (3.4.6).

(3.5.4)
$$\begin{array}{ccc} D_{2g} \times R(\Gamma) & \longrightarrow & R(\Gamma) \\ \downarrow & & \downarrow \\ D_{2g} \times \mathbb{Z}[[X]] & \longrightarrow & \mathbb{Z}[[X]] \end{array}$$

§ 4. Covering coefficients and the Hopf algebra

We introduce a Hopf algebra structure on the configuration algebra, which plays an important roll in (6. .).

(4.1) Covering coefficients.

We introduce a concept of covering coefficients

(4.1.1)
$$\binom{S_1 \cdots S_m}{S} \quad \in \mathbb{N}$$

for configuration types $S_1,\ldots,S_m$ and $S \in \mathrm{Conf} \cup \{\phi\}$ as follows.

i) Fix a subset S of Γ, whose configuration type is S.

ii) Put

(4.1.2)
$$\left[\begin{matrix} S_1 \cdots S_m \\ \mathrm{S} \end{matrix}\right] := \{(\mathrm{S}_1,\ldots,\mathrm{S}_m) : \mathrm{S}_i \ (i=1,\ldots,m) \text{ is a subset of S s.t.}$$
a) the configuration type of S_i is S_i. b) $\bigcup_{i=1}^{m} \mathrm{S}_i = \mathrm{S}.\}$

iii) Show that the cardinarity of $\left[\begin{matrix} S_1 \cdots S_m \\ \mathrm{S} \end{matrix}\right]$ depends only on the type S but not on the choice of S. (The verification is left.)

iv) Put

(4.1.3)
$$\binom{S_1 \cdots S_m}{S} := \#\left[\begin{matrix} S_1 \cdots S_m \\ \mathrm{S} \end{matrix}\right] .$$

(4.2) Imediate from the definition, we have the followings, whose proofs are omitted.

i) The coefficient $\binom{S_1 \cdots S_m}{S}$ is invariant under the action of the symmetric group $\mathfrak{S}_m$ by the permutation of S_i's.

ii) If $S_i=\phi$ for $1\leq{}^{\exists}i\leq m$, then the term can be eliminated. I.e.

$$\binom{S_1 \cdots S_{i-1}\ \phi\ S_{i+1} \cdots S_m}{S} = \binom{S_1 \cdots S_{i-1} S_{i+1} \cdots S_m}{S}. \tag{4.2.1}$$

iii) For S and $T \in \mathrm{Conf}\cup\{\phi\}$,

$$\binom{T}{S} = \begin{cases} 1 & \text{if } S = T, \\ 0 & \text{else,} \end{cases} \tag{4.2.2}$$

iv)

$$\binom{S_1 \cdots S_m}{\phi} = \begin{cases} 1 & \text{if } \cup S_i=\phi, \\ 0 & \text{else.} \end{cases} \tag{4.2.3}$$

(4.3) The following formula on the coefficients is important.

Assertion. *For* $S_1,\ldots,S_m,T_1,\ldots,T_n \in \mathrm{Conf}\cup\{\phi\}$, *one has*

$$\sum_{U\in \mathrm{Conf}\cup\{\phi\}} \binom{S_1 \cdots S_m}{U} \binom{U\ T_1 \cdots T_n}{S} = \binom{S_1 \cdots S_m T_1 \cdots T_n}{S}. \tag{4.3.1}$$

Proof. Consider the map

$$\binom{S_1 \cdots S_m T_1 \cdots T_n}{S} \longrightarrow \bigcup_{U\in \mathrm{Conf}} \binom{U\ T_1 \cdots T_n}{S}$$

$$(\mathbb{S}_1,\ldots,\mathbb{S}_m,\mathbb{T}_1,\ldots,\mathbb{T}_n) \longrightarrow (\textstyle\bigcup_{i=1}^{m}\mathbb{S}_i,\mathbb{T}_1,\ldots,\mathbb{T}_n) \quad .$$

The fiber over a point $(\mathbb{U},\mathbb{T}_1,\ldots,\mathbb{T}_n)$ is bijective to the set $\binom{S_1 \cdots S_m}{\mathbb{U}}$ so that one has the bijection

$$\binom{S_1 \cdots S_m T_1 \cdots T_n}{\mathbb{S}} \simeq \bigcup_{U\in \mathrm{Conf}} \binom{S_1 \cdots S_m}{\mathbb{U}} \binom{U\ T_1 \cdots T_n}{\mathbb{S}} \quad . \ \square$$

Note The summation index U in (4.3.1) runs over all Conf. But in fact the sum is finite, since the coefficients $\binom{S_1 \cdots S_m}{U}$ for large U vanish. In the following of the paper, we shall meet the same phenomenon, on which we shall not mention each time.

(4.4) Another important formula for the covering coefficients is the following decomposition formula.

Assertion *For* $S_1,\ldots,S_m$, U, $V\in$ Conf∪(ϕ), *one has a formula:*

(4.4.1) $$\binom{S_1\cdots S_m}{U\ V} = \sum_{S_1=R_1\ T_1}\cdots\sum_{S_m=R_m\ T_m}\binom{R_1\cdots R_1}{U}\binom{T_1\cdots T_m}{V}.$$

Here R_i *and* T_i *run over* Conf∪(ϕ) *for all possible decomopositions of* S_i (i=1,...,m).

Proof Consider the map

$$\binom{S_1\cdots S_m}{\mathbb{U}\ \mathbb{V}} \longrightarrow \bigcup_{S_1=R_1\ T_1}\cdots\bigcup_{S_m=R_m\ T_m}\binom{R_1\cdots R_1}{\mathbb{U}}\times\binom{T_1\cdots T_m}{\mathbb{V}},$$

$$(S_1,\ldots,S_m) \longrightarrow (S_1\cap\mathbb{U},\ldots,S_m\cap\mathbb{U}) \times (S_1\cap\mathbb{V},\ldots,S_m\cap\mathbb{V}).$$

One checks easily that the map is bijective. □

(4.5) The Hopf algebra structure on the cofiguration algebra.

As consequences of (4.2)-(4.4), we give an Abelian Hopf algebra structure on the Configuration algebra.

First, for a positive integer $m\in\mathbb{N}$, let us define a map Φ_m defined on $U\in$ Conf∪(ϕ) as

(4.5.1) $$\Phi_m(U) := \sum_{S_1\in\mathrm{Conf}\cup(\phi)}\cdots\sum_{S_m\in\mathrm{Conf}\cup(\phi)}\binom{S_1\cdots S_m}{U}\ S_1\otimes\cdots\otimes S_m$$

which takes values in the completed tensor product $\hat{\otimes}_m R(\Gamma)$ of m-copies of the configuration algebra. Due to (4.2) iv),

$$\Phi_m(\phi) = 1.$$

The map Φ_m is multiplicative in the sence that

(4.5.2) $$\Phi_m(U\ V) = \Phi_m(U)\ \Phi_m(V)$$

for U, V∈Conf∪{ϕ}. This can be directly shown from (4.4.1).

Hence Φ_m induces an algebra homomorphism:

(4.5.3) $$\Phi_m : R(\Gamma) \longrightarrow \hat{\otimes}_m R(\Gamma)$$

$$U \longrightarrow \Phi_m(U)$$

which is denoted by the same Φ_m. The symmetric group $\mathfrak{S}_m$ acts on the m-tensor product by permutating letters $1\otimes..\overset{\bullet}{.}\otimes1\otimes S_i\otimes1\otimes..\otimes1$ (i=1,...,m). Then the image of Φ_m lies on the $\mathfrak{S}_m$-invariant subalgebra, due to (4.2) i).

We define now a coproduct structure on the configuration algebra by Φ_2, which is abelian due to the remark above.

(4.6) Associativity.

The formula (4.3.1) implies the relation:

(4.6.1) $$(1\otimes...\otimes1\otimes\Phi_m)\circ\Phi_{n+1} = \Phi_{m+n}$$

for m, n ∈ℕ. Particularly, this implies the relation:

$$(\Phi_2\otimes1)\circ\Phi_2 = (1\otimes\Phi_2)\circ\Phi_2$$

which is the associativity of the coproduct of the Hopf algebra structure. It is also clear from (4.6.1), that Φ_m for m≥2 are described as the compositions of Φ_2.

(4.7) Unit element.

The augumentation map for the algebra, i.e. the map

(4.7.1) $\mathrm{aug} : R(\Gamma) \longrightarrow \mathbb{Z}$, $(S\in\mathrm{Conf} \longrightarrow 0,\ \phi \longrightarrow 1)$

gives the identity element of the Hopf structure. This means

(4.7.2) $(\mathrm{aug}\cdot\mathrm{id})\circ\Phi_2 = \mathrm{id}$.

I.e. the composition of the maps: $R(\Gamma) \xrightarrow{\Phi_2} \hat{\otimes}_2 R(\Gamma) \xrightarrow{\mathrm{aug}\cdot 1} R(\Gamma)$ becomes the identity due to (4.2) ii) and iii) as follows.

$$\text{Image of } S = \sum_{T,U\in\mathrm{Conf}\cup\{\phi\}} \binom{T\ U}{S} T\cdot\mathrm{aug}(U) = \sum_{T\in\mathrm{Conf}\cup\{\phi\}} \binom{T\ \phi}{S} T = S.$$

(4.8) The involution map ι.

Assertion There exists uniquely an automorphism

(4.8.1) $\iota : R(\Gamma) \longrightarrow R(\Gamma)$

of the configuration algebra such that

i) *ι is involutive, i.e.* $\iota^2=\mathrm{id}$.

ii) *The composition of the maps*: $R(\Gamma) \xrightarrow{\Phi_2} \hat{\otimes}_2 R(\Gamma) \xrightarrow{\iota\cdot 1} R(\Gamma)$ *is the augumentation map* (4.7.1). I.e.

(4.8.2) $(\iota\cdot\mathrm{id})\circ\Phi_2 = \mathrm{aug}$.

Proof. For an $n\in\mathbb{N}$, let us denote by Conf(n) the sub semigroup of Conf generated by $S\in\mathrm{Conf}_0$ with $\#(S)\leq n$. We denote also by $R(n) := \mathbb{Z}[[\ \mathrm{Conf}(n)]]$ the complete subalgebra of $R(\Gamma)$ generated by Conf(n). We remark that the restriction $\Phi_2(n) := \Phi_2|_{R(n)}$ maps $R(n)$ to $R(n)\hat{\otimes}R(n)$.

Let us show by induction on n the existence and the uniqueness of an involutive automorphism $\iota(n)$ of $R(n)$ such that

i) $\iota(n)|_{R(n-1)} = \iota(n-1)$,

ii) $(\iota(n)\cdot 1)\circ\Phi_2(n)$ = the augumentation map for $R(n)$,

iii) $\iota(n)\left(\mathcal{I}_m\cap R(n)\right) \subset \mathcal{I}_m\cap R(n)$ for $m\in\mathbb{N}$.

For n=0, $R(0)=\mathbb{Z}$ and $\iota(0)=\mathrm{id}_{\mathbb{Z}}$. Suppose the hypothese is shown for an $n\geq 0$. We want to determine $\iota(n+1)(S)$ for $S\in\mathrm{Conf}_0$ with $\#(S)=n+1$ by solving the equation:

$$(\iota(n+1)\cdot 1)\circ\Phi_2(n+1)(S) = 0$$

More explicitely, the equation is rewritten by the use of (4.5.1)

$$\iota(n+1)(S)\left(\sum_{V\in\mathrm{Conf}\cup\{\phi\}}\binom{S\ V}{S}V\right)$$
$$+\left(\sum_{\substack{U,V\in\mathrm{Conf}\cup\{\phi\}\\ U,V\neq S}}\binom{U\ V}{S}\iota(n)(U)\ V\right)$$
$$+\left(\sum_{\substack{U\in\mathrm{Conf}\cup\{\phi\}\\ U\neq S}}\binom{U\ S}{S}\iota(n)(U)\right)S$$
$$=0$$

For simplicity, one may write the equation as

$$\iota(n+1)(S)\left(1+\mathcal{A}(S)\right)+\mathcal{B}(S)+\left(1+\iota(n)\left(\mathcal{A}(S)-S\right)\right)S=0, \tag{4.8.2}$$

where

$$\mathcal{A}(S)-S:=\sum_{\substack{V\in\mathrm{Conf}\\ V\neq S}}\binom{S\ V}{S}V \qquad \in R(n)\cap\mathcal{I}_1, \tag{4.8.3}$$

and

$$\mathcal{B}(S):=\sum_{\substack{U,V\in\mathrm{Conf}\cup\{\phi\}\\ U,V\neq S}}\binom{U\ V}{S}\iota(n)(U)\ V \quad \in R(n)\cap\mathcal{I}_{n+1}. \tag{4.8.3*}$$

Since $\mathcal{A}(S)\in\mathcal{I}_1$, the inverse of $1+\mathcal{A}(S)$ $(=\sum_{m\geq 0}(-\mathcal{A}(S))^m)$ exists in $R(n+1)$ so that (4.8.2) is solved as

$$\iota(n+1)(S):=\frac{-1}{1+\mathcal{A}(S)}\left(\mathcal{B}(S)+\left(1+\iota(n)\left(\mathcal{A}(S)-S\right)\right)S\right) \tag{4.8.4}$$

Extend (4.8.4) multiplicatively to Conf(n+1) so that one obtains an algebra homomorphism from $\mathbb{Z}\left[\mathrm{Conf}(n+1)\right]$ to $R(n+1)$. Since the right hand side of (4.8.4) belongs to $\mathcal{I}_{n+1}$, the map is continuous w.r.t. the adic topology so that it can be completed to a map from $R(n+1)$ to itself, which we denote $\iota(n+1)$. The properties i), ii) and iii) for $\iota(n+1)$ are shown in the construction. The uniqueness of $\iota(n+1)$ follows from the unique solution (4.8.4) for the equation (4.8.2).

What remains to show is the identity: $\iota(n+1)^2(S)=S$ for $S\in\mathrm{Conf}_0$ with $\#(S)=n+1$. Let us apply $\iota(n+1)$ to the equality (4.8.2), where we shall denote $\iota(n)$ simply by ι.

*) $\iota(n+1)^2(S)\left(1+\iota(\mathcal{A}(S)-S)+\iota(n+1)(S)\right)+\iota\mathcal{B}(S)+\left(1+\mathcal{A}(S)-S\right)\iota(n+1)(S)=0$

Here we have $\iota\mathcal{B}(S)=\mathcal{B}(S)$ due to the symmetry (4.2)i) and the fact $\iota(n)^2=\mathrm{id}$ (i.e. the induction hypothesis on $\iota(n)$).

Taking the difference: *)-(4.8.2), one has

$$\left(\iota(n+1)^2(S)-S\right)\left(1+\iota(\mathcal{A}(S))\right) = 0.$$

Since $\iota(\mathcal{A}(S))\in\mathcal{I}_1$ so that $1+\iota(\mathcal{A}(S))$ is invertible in $R(n+1)$, this implies the involutivity of $\iota(n+1)$. □

Remark 1. The coproduct Φ_2 (4.5.3) and the augmentation map (4.7.1) are defined already on the polynomial ring $\mathbb{Z}\left[\ \mathrm{Conf}\ \right]$, whereas the involution map ι (4.8.1) cannot be defined in the polynomial ring and it was nescesarry to extend the algebra to that of power serieses as above.

Precisely, for the definition of ι, it is enough to extend the polynomial algebra $\mathbb{Z}\left[\ \mathrm{Conf}\ \right]$ to its localization by the set

$$\mathfrak{M}:=\{1 + \mathcal{A}(S):\ S\in\mathrm{Conf}_0\}$$

(i.e. to the algebra of meromorphic functions, whose poles are at most products of powers of elements of $\mathfrak{M}$).

2. In §5, the functions $\mathscr{A}(S)$ ($S\in$Conf) are re-introduced and investigated more precisely. Particularly we shall show that

$$\left(1 + \iota(\mathscr{A}(S))\right)\cdot\left(1 + \mathscr{A}(S)\right) = 1$$

for $S\in$Conf (5.3.1). This relation may be understood as an inductive definition for ι for (4.8.3) replacing (4.8.2).

§5. The growth functions for configurations.

(5.1) Growth functions for configuration types.

For S and $T \in$ Conf$\cup\{\phi\}$, we define $A(S,T)\in \mathbb{N}$ as follows.

i) Fix a subset of $\mathbb{T}$ of Γ, whose configuration type is T.

ii) Put

(5.1.1)

$$\mathbb{A}(S,\mathbb{T}) := \#\{\mathrm{S}: \mathrm{S} \text{ is a subset of } \mathbb{T}, \text{ whose configuration is } S\}.$$

iii) Show that the cardinarity of $\mathbb{A}(S,\mathbb{T})$ depends only on the types S and T but not on the choice of $\mathbb{T}$. (The proof left.)

iv) Put

(5.1.2) $$A(S,T) := \#\mathbb{A}(S,\mathbb{T}).$$

We shall call $A(S,T)$ the growth function (or gr-function for short) for the configurations. Offcause $A(\phi,T)=1$ for $T\in$Conf$\cup\{\phi\}$.

Using the growth function, let us introduce a series

(5.1.3) $$\mathscr{A}(T) := \sum_{S\in \mathrm{Conf}} A(S,T)S$$

whose investigation is the central problem of the paper. Since

(5.1.4) $$1 + \mathscr{A}(T) = \sum_{S\in 2^T} [S],$$

where we denote by [S] the configuration type of a set S, one has a product formula:

(5.1.5) $$(1 + \mathscr{A}(T_1\cdot T_2)) = (1 + \mathscr{A}(T_1))(1 + \mathscr{A}(T_2))$$

follows imediate from (5.1.4).

Remark 1. By comparing the definitions, we see imediately

$$A(S,T) = \binom{T\ S}{T}$$

for S and $T \in \mathrm{Conf}\cup\{\phi\}$. Hence the two definitions (4.8.3) and (5.1.3) for $\mathscr{A}(T)$ coincides.

Remark 2. A first aproximation of the growth function is

(5.1.6) $$A(S,T) \le \frac{1}{k_1!\ldots k_m!}\ \#(T)^{n(S)},$$

where $k_1,\ldots,k_m$ are given by the irreducible decomposition of S:

(5.1.7) $$S = k_1S_1 \ \ldots\ \ k_mS_m$$

such that $S_i\in\mathrm{Conf}_0$ are pairwisely different and $n(S) = \sum_{i=1}^{m} k_i$.

Proof One has an embedding

$$A(S,\mathbb{T}) \longrightarrow \prod_{i=1}^{m}\left(\left(\prod^{k_i} A(S_i,\mathbb{T})\right)_0 / \mathfrak{S}_{k_i}\right)$$

by associating to each $S\in A(S,\mathbb{T})$ its irreducible decompostion. Here $\left(\prod^{k} A\right)_0 := \{(a_1,\ldots,a_k)\in \prod^{k} A : a_i\text{'s are pairwisely different}\}$, on which the symmetric group $\mathfrak{S}_k$ is acting freely.

Let $S_i\subset\Gamma$ be a set of the configuration type S_i. Since S_i is connected, $\mathbb{A}(S_i,\mathbb{T}) = \{\gamma S_i : \gamma\in\Gamma \text{ and } \gamma S_i \subset \mathbb{T}\} \subset \bigcap_{\gamma\in S_i} \mathbb{T}\gamma^{-1}$. This implies $A(S_i,T) := \#\mathbb{A}(S_i,\mathbb{T}) \leq \#(T)$. □

(5.2) Product expansion formula for growth functions.

Let us show a very important expansion formula (5.2.1) for products of growth functions.

Lemma *Let* $S_1,\ldots,S_m$ $(m\geq 1)$ *and* T *be given configuration types in* $\mathrm{Conf}\cup\{\phi\}$. *Then,*

$$(5.2.1) \qquad \prod_{i=1}^{m} A(S_i,T) = \sum_{S\in \mathrm{Conf}\cup\{\phi\}} \binom{S_1 \cdots S_m}{S} A(S,T) .$$

Proof. Let $\mathbb{T}$ be a subset of Γ, whose configuration type is T and let us denote by $2^{\mathbb{T}}$ the set of all subsets of $\mathbb{T}$.

Consider a map

$$(\mathrm{S}_1,\ldots,\mathrm{S}_m) \in \prod_{i=1}^{m} \mathbb{A}(S_i,\mathbb{T}) \longmapsto \mathrm{S} := \bigcup_{i=1}^{m} \mathrm{S}_i \in 2^{\mathbb{T}} ,$$

whose fiber over S is $\binom{S_1 \cdots S_m}{\mathrm{S}}$ so that one has the decomposition

$$\prod_{i=1}^{m} \mathbb{A}(S_i,\mathbb{T}) \simeq \bigcup_{\mathrm{S}\in 2^{\mathbb{T}}} \binom{S_1 \cdots S_m}{\mathrm{S}} .$$

By counting the cardinarity of the both sides, one obtains the formula. □

Remark The expansion formula (5.2.1) is essentially reduced to the case m=2, since the higher cases are shown by induction on m as follows. Multiply $A(S_{m+1},T)$ to (5.2.1) and apply the formua for m=2.

$$\prod_{i=1}^{m+1} A(S_i,T) = \sum_{S\in\mathrm{Conf}\cup\{\phi\}} \binom{S_1\cdots S_m}{S} A(S,T)A(S_{m+1},T)$$

$$= \sum_{S\in\mathrm{Conf}\cup\{\phi\}} \binom{S_1\cdots S_m}{S} \sum_{U\in\mathrm{Conf}\cup\{\phi\}} \binom{S\ S_{m+1}}{U} A(U,T)$$

Using (4.3.1),

$$= \sum_{U\in\mathrm{Conf}\cup\{\phi\}} \binom{S_1\cdots S_m S_{m+1}}{U} A(U,T) \ .$$

□

(5.3) A very important consequence of (5.2.1) is the following.

Lemma For any configuration type $T\in\mathrm{Conf}\cup\{\phi\}$ and $m\in\mathbb{N}$, *one has*

(5.3.1) $$(1+\mathscr{A}(T))\otimes\ldots\otimes(1+\mathscr{A}(T)) = \Phi_m(1+\mathscr{A}(T)).$$

Proof Define an infinite series of m variables $S_1,\ldots,S_m$:

*) $$\sum_{S_1\in\mathrm{Conf}\cup\{\phi\}} \cdots \sum_{S_m\in\mathrm{Conf}\cup\{\phi\}} \left(\prod_{i=1}^{m} A(S_i,T)\right) S_1\otimes\ldots\otimes S_m$$

in the complete tensor $\hat{\otimes}_m R(\Gamma)$ of m-copies of the Configuration algebras. By definitions of $\mathscr{A}(T)$ (5.1.3), this equals to

**) $$(1+\mathscr{A}(T))\otimes\ldots\otimes(1+\mathscr{A}(T)) \qquad \text{(a tensor of m-copies)}$$

On the other hand, by a use of the product expantion formula (5.2.1), *) is expressed as

$$\sum_{S_1\in\mathrm{Conf}\cup\{\phi\}} \cdots \sum_{S_m\in\mathrm{Conf}\cup\{\phi\}} \left(\sum_{S\in\mathrm{Conf}\cup\{\phi\}} \binom{S_1\cdots S_m}{S} A(S,T)\right) S_1\otimes\ldots\otimes S_m$$

Recalling the definition of the map Φ_m (4.5.1), this equals to

$$\sum_{S\in\mathrm{Conf}\cup\{\phi\}} \Phi_m(S)\ A(S,T)$$

$$= \Phi_m\left(\sum_{S\in\mathrm{Conf}\cup\{\phi\}} S\ A(S,T)\right)$$

***) $= \Phi_m(1+\mathscr{A}(T))$

Comparing ***) with **), we obtain the relation. □

(5.4) Another consequence of the (5.2.1) is the following.

(5.4.1) $$\Big(1 + \iota(\mathscr{A}(T))\Big)\Big(1 + \mathscr{A}(T)\Big) = 1,$$

for any configuration type $T \in \mathrm{Conf} \cup \{\phi\}$.

Proof Apply (5.3.1) to the $(\iota\cdot 1)\circ\Phi_2(T) = \mathrm{aug}(T)$ (4.8.1). □

(5.4) The logarithmic growth function.

As we saw in (5.1.4), the function $A(S,T)$ is polynomial growth in $\#T$ of degree $n(S)$. For our purpose, we need to know not only its leading coefficient but its precise lower terms. This is achieved by introducing another function $M(S,T) \in \mathbb{Q}$ for S, $T \in \mathrm{Conf}$, which we call the logarithmic growth function,as follows.

Take the logarithm of the series $\mathscr{A}(T)$ (5.1.3) (cf (3.3.4)),

(5.4.1) $$\mathscr{M}(T) := \log(1+\mathscr{A}(T)) ,$$

in $R(\Gamma,\mathbb{Q})$. The coefficients of its expansion

(5.4.2) $$\mathscr{M}(T) = \sum_{S \in \mathrm{Conf}} M(S,T)S .$$

are the logarithmic growth functions (or log gr-functions), taking their values in $\mathbb{Q}$. We formaly put

(5.4.3) $$M(\phi,T) := 0 \qquad \text{for} \quad T \in \mathrm{Conf}.$$

For a sake of completeness, let us recall the relations (3.3.5) and (3.3.6) and apply it to the coefficients of $\mathscr{A}(T)$ and $\mathscr{M}(T)$.

(5.4.4)

$$A(S,T) = \sum_{S=k_1S_1 \cdots k_mS_m} \frac{1}{k_1!\cdots k_m!} M(S_1,T)^{k_1} \cdots M(S_m,T)^{k_m},$$

(5.4.5)

$$M(S,T) = \sum_{S=k_1S_1 \cdots k_mS_m} \frac{(k_1+\cdots+k_m-1)!(-1)^{k_1+\cdots+k_m-1}}{k_1!\cdots k_m!} A(S_1,T)^{k_1} \cdots A(S_m,T)^{k_m}.$$

Remark 1. In fact, $\mathcal{A}(T)$ is a finite sum. Nevertheless $\mathcal{M}(T)$ is an infinite series in the algebra, whose support is in T.

2. For a connected $S \in \mathrm{Conf}_0$, we have

(5.4.6) $$A(S,T) = M(S,T).$$

(Directly seen from (5.4.4) or (5.4.5).

(5.5) What is surprizing is that the polynomial relations (5.2.1) imply *linear relations* on the log gr-function.

Lemma Let configuration types $T, S_1, \ldots, S_m \neq \phi$ ($m \geq 2$) *be given. Then,*

(5.5.1) $$\sum_{S \in \mathrm{Conf} \cup \{\phi\}} \binom{S_1 \cdots S_m}{S} M(S,T) = 0 .$$

Proof Substitute $1 + \mathcal{A}(T)$ by $\exp(\mathcal{M}(T)$ in the formula (5.3.1) so that one obtains a relation:

$$\exp(\mathcal{M}_1(T) + \cdots + \mathcal{M}_m(T)) = \Phi_m(\exp(\mathcal{M}(T)))$$

where $\mathcal{M}_i(T) := 1 \otimes \cdots \otimes \overset{i\text{-th}}{\mathcal{M}(T)} \otimes \cdots \otimes 1$. Since $\Phi_m(\exp(\mathcal{M}(T)) = \exp(\Phi_m(\mathcal{M}(T)))$, we obtain the relation:

$$(5.5.2) \qquad \mathcal{M}_1(T)+\ldots+\mathcal{M}_m(T) = \Phi_m(\mathcal{M}(T)).$$

Now develop the both sides of (5.5.2) in the series of variables $S_1,\ldots,S_m$ and compare the coefficients. Since the left hand side does not have a cross term in the variables $S_1,\ldots,S_m$, the corresponding coefficients in the right hand side should vanish. Combining (5.2.6) and (5.4.3), this means the formula (5.5.1).□

Remark 1. The formula (5.5.1) is essential in the case m=2 with $S_i \neq \phi$ (i=1,2), since the cases of m≥3 are reduced to that case by a use of (4.2.4) as follows.

$$\begin{aligned}\sum_S \binom{S_1\cdots S_m}{S} M(S,T) &= \sum_S \Big(\sum_{U\in\mathrm{Conf}\cup\{\phi\}} \binom{S_1\cdots S_{m-1}}{U}\binom{U\ S_m}{S}\Big) M(S,T) \\ &= \sum_{U\in\mathrm{Conf}} \binom{S_1\cdots S_{m-1}}{U}\Big(\sum_S \binom{U\ S_m}{S} M(S,T)\Big) \\ &\quad + \binom{S_1\cdots S_{m-1}}{\phi}\Big(\sum_S \binom{\phi\ S_m}{S} M(S,T)\Big) \\ &= 0+0=0 .\end{aligned}$$

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