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On the Lie-Drach-Vessiot Theory

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Recent years we suceeded in clarifying on a rigorous and comprehensive foundation, principal ideas of Painlevé. Among other things, we proved the irreducibility of the Painlevé equations. It seems therefore a predominant problem in theory of algebraic differential equations is:

Problem A (problem of generalization). Can we realize Lie's dream of infinite dimensional differential Galois theory?

Precisely, Lie(1842-1899) had a dream of generalizing classical Galois theory of algebraic equations to differential equations. The virtual theory would be inifinitie dimensional but he had to begin by constructig finite dimensional theories. In this way he founded theory of Lie groups and Lie algebras. Infinite dimensional theory occupies a small part in his Gesamelte Abhandelungen. The first attempt of realizing Lie's dream was done by a French mathematician Drach(1871-1941). But there are unclear definitions and gaps in proofs in his works. We wonder how these incomplete works were published. Vessiot(1865-1952) spent all his life to complete Drach's works. The works of Vessiot are more accesible than those of Drach but the lack of language, particularly the language of algebraic geometry, makes his works wordy and incomprehensive. After their works the problem is left untouched in spite of its importance. *We propose to realize the dream of Lie by developing an idea of*

Vessiot.

As is well known, Galois theory of differential equations satisfying a finiteness condition was established at the end of the 19th century. A particular case of the theory is Galois theory of ordinary linear differential equations which we call also Picard-Vessiot theory. Kolchin not only made the theory with finiteness condition complete but also constructed a foundation of theory of differential algebra based on the language of algebraic geometry of Weil. In his book published in 1973, he writes: Indeed, since an algebraic equation can be considered as a differential equation in which derivatives do not occcur, it is possible to consider algebraic geometry as a special case of differential algebra. If we are not concerned with mathematical meaning of his words, what he writes is at least logically true. But it is very strange that his strongly normal extension, which should be a generalization of Galois extension, does not include classical Galois extensions.

Problem B (problem of unification). Why is classical Galois extension not strongly normal? Namely, is there a consistent definition which includes both classical Galois extension and strongly normal extension?

We felt unpleasent the incoherence of definitions for years. Pursuing Problem A, we found that Problem B is related with Problem A and *we solve* Problem B *too*.

The infinite dimensional differential Galois theory is expected to have two important applications. The first application is the third proof of the irreducibility of the first Painlevé equation. So far we know two proofs of the irreducibility. The first proof depends on an idea of Nishioka and the proof is done in a framework of Kolchin. The second proof is inspired from Lecons de Stockholm of Painlevé and the analysis of the general solution as a function of initial conditions plays an important role in the second proof.

The third proof is to be done by calculating the Galois group. We notice that their is an anticipating work of Drach published in 1915 on the Galois group of the first Painlevé equation. The paper itself is very interesting but highly incomplete. The first and the second proofs are of negative character. In fact we proved that no solution of the first Painlevé equation is classical. But if we can calculate the Galois group, we understand the nature of the equation better and moreover we can also compare the first Painlevé equation with other equations of higher order.

The second application is deformation of (not necessarily linear) differential equations. Works of Riemann, Fuchs, Schlesinger and Garnier and a recent contribution of the Sato school show the importance of monodromy preserving deformation of linear differential equations. Theta functions and the Painlevé equations are introduced in this framework. A work of Ramis on generation of the differential Galois group of a linear differential equation which has irregular singular points suggests that Galois group preserving deformations of linear differential equations are also natural. Further Drach considered a deformation of non-linear differential equation which preserves his Galois group and showed that the first Painlevé equation describes this deformation. Since his theory is imcomplete, it is very interesting to review his paper using our theory.

§ 1 Classical Galois theory of algebraic equations

Let K be a field. We assume for simplicity $\mathrm{ch}\, K = 0$. Let $f(X) \in K[X]$ bw an irreducible polynomial of degree n. Let $\alpha_1, \alpha_2, \ldots \alpha_n$ be roots of an algebraic equation $f(X)=0$. Let $L = K(\alpha_1, \alpha_2, \ldots \alpha_n)$. Then the Galois group of the algebraic equation $f(X)=0$ is the K-automorphism group $\mathrm{Aut}\, L/K$ (which we denote also by $\mathrm{Aut}_K L$) of the field L. It is therefore more natural to speak of the Galois group of a field extension and we are led to the notion of Galois extension. To explain this , let us recall the definition of a principal homogeneous space.

Definition (1.1). Let G be a group acting on a set X. We say that X is a principal homogeneous space of G if the following condition is satisfied: if x is an element of X, then the map $G \to X$ sending g to gx is bijective.

According to this definition, the empty set is a principal homogeneous space for any group G.

Let L/K be a finite algebraic field extension and $\bar{K}$ an algebraic closure of K. The K-automorphism group $G = \mathrm{Aut}_K L$, which is a finite group, operates on the set $\mathrm{Hom}_K(L, \bar{K})$. Namely for

$g \in \mathrm{Aut}_K L$, $f \in \mathrm{Hom}_K(L, \bar{K})$, we define gf by $f \circ g$. The definition of Galois extension which is convenient for our generalization is the following.

Definition (1.2). A finite algebraic extension L/K is Galois if $\mathrm{Hom}_K(L, \bar{K})$ is a principal homogeneous space of $\mathrm{Aut}_K L$.

If L/K is a finite algebraic extension, then the automorphism group $\mathrm{Aut}\ L/k$ is a finite group and hence can be considered as a group scheme over K. Let $\mathcal{C}$ be the category of (commutative) K-algebras. We define a functor

$$F_{L/K}: \mathcal{C} \to (\mathrm{Sets})$$

by $F_{L/K}(A) = \mathrm{Hom}_K(L, A) = \mathrm{Hom}_K(\mathrm{Spec}\ A, \mathrm{Spec}\ L)$ for $A \in \mathrm{ob}\ \mathcal{C}$. We have $F_{L/K}(A) = \mathrm{Hom}_A(L \otimes_K A, A)$. Since $G = \mathrm{Aut}\ L/K$ is a finite group , we can consider it as a finite K-group scheme and we can speak of A-valued points of G. Namely, the functor $G_{L/K}: \mathcal{C} \to (\mathrm{Groups})$ is defined by $G_{L/K}(A) = \ G$ for $A \in \mathrm{ob}\ \mathcal{C}$, where the direct sum of copies of the finite group G is extended over the number of coneccted components of $\mathrm{Spec}\ A$. Since the group scheme G opetastes on the K-scheme $\mathrm{Spec}\ L$, the functor $G_{L/K}$ is a group subfunctor of the automorphism functor $\underline{\mathrm{Aut}}_K L$ and hence $G_{L/K}(A)$ is a subgroup of $\underline{\mathrm{Aut}}_K L(A) = \mathrm{Aut}\ L \otimes_K A/A$. Therefore $G_{L/K}(A)$ operates on $F_{L/K}(A)$. In other words the functor $G_{L/K}$ operates on $F_{L/K}$. We know the following result.

Proposition (1.3). For finite algebraic extension L/K of a field K, the following conditions are equivalent.

(1) The extension L/K is Galois.

(2) The K-scheme Spec L is a principal homogeneous space of Aut L/K. Here the finite group Aut L/K is considered as a K-group scheme.

(3) The functor $F_{L/K}$ is a principal hgomogenepous space of the group functor $G_{L/K}$, i.e. $F_{L/K}(A)$ is a principal homogeneous space of $G_{L/K}(A)$ for any $A \in$ ob $\mathcal{C}$.

The equivalence of conditions (2) and (3) is due to definitions. We know that the condition (3) of Proposition (1.2) characterizes the finite group G. To be precise, let H be a finite group hence a K-group scheme or a group functor $H:\mathcal{C} \to$ (Groups). If H operates effectively on the functor $F_{L/K}$ and if $F_{L/K}$ is a principal homogeneous space, then H is isomorphic to G = Aut L/K.

Remark (1.4). We call reader's attention not to confuse the full automorphism functor $\underline{\mathrm{Aut}}_K L = \underline{\mathrm{Aut}}\ L/K$ with a subfunctor G = $\mathrm{Aut}_K L$ (= Aut L/K). We have however $\mathrm{Aut}_K L = \underline{\mathrm{Aut}}_K L(K)$ as finite groups but the finite group Aut L/K is considered as a K-scheme and hence a functor.

Theorem (1.5). Let L/K be a Galois extension. Then there is a 1:1 correspondence between the elements of the following two sets:

(1) The set of subgroups of G = Aut L/K;

(2) The set of subfields of L containing K.

To a subgroup $H \subset G$, the subfield $L^H = \{ x \in L \mid gx = x$ for any $g \in H \}$ corresponds. Conversely, for an intermediate field M

the corresponding subgroup is $\{ g \in G \mid g$ fixes all the elements of $M \}$.

The classical Galois theory is remarkable since it is powerful in applications and it seems we had better not overestimate the theoretical correspondence of two different objects, subgroups and intermediate fields. In fact among applications, we are contented with counting the trisection of the angle, the Delian problems, i.e. the problem of dublication of the cube, construction of the regular polygon of 17 sides by straight ege and compases and the solution by expansion of radicals of algebraic equations.

We have modern Galois theories in characteritic $p > 0$. The main idea is to replace finite groups by finite group schemes or Hopf algebras or more generally by certain algebraic systems. There are a number of theories depending on the choice of algebraic systems. One of the theories gives a generalization of Theorem (1.4) known as Jacobson-Bourbaki correspondence. But corresponding algebraic systems are not always simpler than intermediate fields! For this reason, modern Galois theory are not as effective as the classical Galois theory. Therefore it is more advatageous to give up the 1:1 correspondnce and to consider only a particular type of intermediate fields and try to find a simpler class of algebraic systems corresponding to these intermediate fields.

§ 2 Kolchin theory

All rings that we consider are commutative Q-algebras. Let A be a ring. A map $\delta : A \to A$ is a derivation operator if $\delta(x + y) = \delta x + \delta y$ and $\delta(xy) = (\delta x)y + x\delta y$ for any $x, y \in A$. A differential ring (A, Δ) consists of a ring A and a finite set Δ of derivation oparators such that $\delta_1\delta_2 = \delta_2\delta_1$ for any $x \in A$, $\delta_1, \delta_2 \in \Delta$. An element $x \in A$ is called a constant if $\delta x = 0$ for any $\delta \in \Delta$. The set of all constants of A forms a subring of A which we denote by C_A. When Δ consists of one operator δ, we say that $(A, \Delta) = (A, \delta)$ is an ordinary differential ring. If ther is no danger of confusion, we do not write the set Δ. Kolchin, as Weil did, work in a unversal domain Ω. Namely, we fix the set Δ and (Ω, Δ) is a very big differential field such that the differential algebras which we consider are contained in (Ω, Δ). The following Lemma is fundamental.

Lemma (2.1). Let K be a differential field and C be its subfield of constants. Let A be differential C-algebra consisting only of constants. Then A and K are linearly disjoint over C. Here all the algebras K, C, A are considred as subalgebras of Ω.

Let us study ordinary differential rings. What follows holds for general differential fields but we limit ourselves to ordinary case since this case is substaintial and the generalization is easy. Kolchin introduced the notion of strong morphism.

Definition (2.2). Let L be an (ordinary) differential field and hence L is a subfield of Ω. A (differential) morphism $f:L \to \Omega$ is strong if the following two conditions are satisfied:

(1) f fixes all the constants of L;

(2) The composite field $f(L).L$ is generated over L by constants.

The following interpretation of the Definition(2.2) seems more natural.

Proposition (2.2.1). Notation being as in Definition (2.2). The following conditions are equivalent.

(1) The morphism f is strong.

(2) The morphism f induces an C_Ω-automorphism of the composite field $L.C_\Omega$.

Let us understand the definition by examples.

Example (2.3). Let $L = Q(x, e^x)$ with derivation $\delta = d/dx$. (L, δ) is a differential field. Denoting e^x by y, we have

(2.3.1) $\delta y = y$.

We show that any (differential) $Q(x)$-morphism $f:L \to \Omega$ is strong. In fact $C_L = Q$ and f fixes rational numbers. The image $w = f(y)$ should satisfy

(2.3.2) $\delta w = w$.

On the other hand $\delta(w/y) = ((\delta w)y - w\delta y)/y^2 = 0$ by (2.3.1) and (2.3.2). Nanely $w/y = c$ is a constant and we have $w = yc$. Hence $w = f(y) \in L.\ C_\Omega$ and $f(L).L$ is generated over L by constnants.

The above argument works more generally for a linear differential equation and gives another example.

Example (2.4). Let $(K(D), \delta)$ be a differential field of all meromorphic functions over a domain D of $\mathbb{C}$, δ being the derivation d/dx with respest to the coordinate x of $\mathbb{C}$. Let K be a differential subfield of $K(D)$. Let $A = (a_{ij}) \in M_n(K)$ and $Y = (y_{ij}) \in GL_n(K(D))$ satisfying a differential equation $Y' = A\,Y$, where Y' denotes (δy_{ij}). Let $L = K(y_{ij})_{1 \leq i,j \leq n}$ be the field obtained by the adjunction of the y_{ij}'s to K. Then L is a differential field and any (differential) K-morphism $L \to \Omega$ is strong.

Lemma (2.1) gives the following

Proposition (2.5). Let L be a differential field and $f: L \to \Omega$ is a morphism. If f fixes all constants of L, then the following conditions are equivalent.

(1) f is strong, i.e. the composite field $f(L).L$ is generated over L by constants.

(2) The composite field $f(L).L$ being denoted by M, we have $M = L.C_M$.

The following notion of strongly normal extension is introduced by Kolchin as a generalization of classical Galois extension.

Definition (2.6). Let L/K be an (ordinary) differential field extension which is finitely generated over K as an abstract field extension. We say that L/K is strongly normal if any K-morphism $f:L \to \Omega$ is strong.

Examples (2.3) and (2.4) are strongly normal extensions as we have seen above. The following phenomena are unpleasant. First, since a strong morphism leaves constants invariant, if L/K is strongly noamal, then $C_L = C_K$. Therefore a Galois extension $Q(\sqrt{-1})/Q$ in Ω is not strongly normal. Secondly, an extension $Q(x, e^x)/Q(x, e^{3x})$ is strongly normal by Example (2.3) but this extension is not Galois. The first phenomenon is more disagreable than the second. We can avoid these defects replacing the universal domain by tensor products.

Let (A, δ) be a differential ring and (R, δ_1), (S, δ_2) be differential A-algebras. Then $R \otimes_A S$ is a differential algebra. Here the differential operator $\delta : R \otimes_A S \to R \otimes_A S$ is defined by putting $\delta(r \otimes s) = (\delta r) \otimes s + r \otimes (\delta s)$ for $r \in R$, $s \in S$ and extending it additively.

Bialynicki-Birula adopted the following characterization as a definition.

Proposition (2.7). Let L/K be a differential field extension. We assume that (1) as an abstract field extension, L/K is finitely generated over K and regular, (2) $C_L = C_K$. Then the following conditions are equivalent.

(1) L/K is strongly normal.

(2) The differential field $Q(L \otimes_K L)$ is generated over $i_2(L)$ by constants, where i_2 denotes the morphism $L \to L \otimes_K L$, $x \mapsto 1 \otimes x$ for $x \in L$.

This interpretation due to Bialynicki-Birula is liberated from the universal domain but our definition is different. Let (R, δ) be a differential ring. We have an embedding $i:(R, \delta) \to (R^\natural[[T]], d/dT)$, $x \mapsto \sum_{n=0}^{\infty} \frac{1}{n!}(\delta^n x)T^n$ for $x \in R$. Here we denote by $R^\natural$ an abstract ring R to emphasize that we forget the differential operator δ. The morphism i is a diferential homomorphism, i.e. $i(\delta(x)) = \frac{d(i(x))}{dT}$ for any $x \in R$.

Lemma (2.8). The mosphism $i:R \to R^\natural[[T]]$ is universal in the following sense. Let $f:(R, \delta) \to (A[[T]], d/dT)$ be a differential morphism. Then there exists a unique abstract algebra morphism $\bar{f}$: $R^\natural \to A$ such that $\tilde{f} \circ i = f$. Here we denote by $\tilde{f}$ the morphism $R^\natural[[T]] \to A[[T]]$ induced by $\bar{f}:R^\natural \to A$. Namely , we have

$$\mathrm{Hom}(R^\natural, A) \simeq \mathrm{Hom}((R, \delta), (A[[T]], d/dT)).$$

Let now L/K be a differential field extension which is finitely generated over K as an abstract field extension. Let $C(K^\natural)$ be the category of K -algebras. We have the morphism $i:K \to K^\natural[[T]]$. Intuitively we would define a functor

$$F_{L/K}:C(K^\natural) \to (\mathrm{Sets})$$

by $F_{L/K}(A) = \mathrm{Hom}_K((L, \delta), (A[[T]], d/dT))$. The right hand side consists of, by definition, differential morphisms $f:L \to A[[T]]$

of which the restriction $f|K$ coincides with the composite $K \xrightarrow{i} K'[[T]] \to A[[T]]$, where the last morphism is induced from the K'-algebra structure of A. Notice that if the field consists only of constants, then the functor $F_{L/K}$ coincides with the functor $F_{L/K}$ introduced in § 1.

The group functor $G_{L/K}$ in § 1 has also an analogue:
We would define

$$G_{L/K}: C(C_K) \to (\text{Groups})$$

by $G_{L/K}(A) = \mathrm{Aut}\,(L \otimes_C A)/(K \otimes_C A)$ for $A \in C(C_K)$, where we denote C_K by C to avoid the complicated notation in the tensor product. The functor $G_{L/K}$ is defined on the category $C(C_K)$ which contains $C(K')$ as a full subcategory. Therefore we can speak of the restriction of $G_{L/K}$ to $C(K')$.

The precise definitions of $F_{L/K}$ and $G_{L/K}$ are given by using a model of L/K. Namely let L^0 be a differential K-algebra which is finitely generated as an abstract K-algebra such that the quotient field of L^0 is L. It is easy to show that such a differential K-algebra exists. We have to consider pseudo-morphisms in the sense of E.G.A. IV, §20. In our language the strongly normal extension is characetrized as follows.

Proposition (2.9). Let L/K be a differential field extension such that L is finitely generated over K as an abstract field. If $C_L = C_K$, then the following conditions are equivalent.

(1) L/K is strongly normal.

(2) The functor $F_{L/K}$ is a principal homogeneous space of the group functor $G_{L/K}$ restricted to $C(K')$.

Kolchin proved that if L/K is strongly normal, then $G_{L/K}$ is representable by a C_K-group scheme of finite type. Therefore $G_{L/K}|$ $C(K\ )$ is a K-group scheme of finite type. More generally we have

Conjecture (2.10). Let L/K be a differential field extension which is finitely generated as an abstrct field extension. In this situation the group functor $F_{L/K}$ is representable by a C_K-group scheme of finite type if and only if C_L/C_K is algebraic.

If the field L consists only of constsants, then the functor $G_{L/K}$ coincides with $\underline{\mathrm{Aut}}_K L$. Hence it does not give the correct functor $G = \mathrm{Aut}_K L$. We need a definition.

Definition (2.11). Let G be a group scheme finite type over a field k. We say that G is split over k if all connected components of G are absolutely irreducible. We adopt the following

Definition (2.12). Let L/K be a differential field extension which is finitely generated over K as an abstract field extension. If there exists a split C_K-group scheme G of finite type such that the functor $F_{L/K}$ is a principal homogeneous space of G_K, then we say that the extension L/K is automorphic.

As we have seen in § 1, Galois extension is automorphic. For differential field extension L/K which is finite algebraic, L/K is automorphic if and only if L/K is Galois as an abstract field extension. If $C_L = C_K$, automorphic extension is strongly normal. Conversely if C_K is algebraically closed, strongly normal extension is automorphic. We have

Theorem (2.13). Let L/K be an automorphic extension with a C_K-group scheme G. Then there is a 1:1 correspondence between the elements of the following two sets.

(1) C_K-subgroup schemes of G split over C_{L^H}.

(2) Differential subfields of L such that L is automorphic over M.

We have the following result.

Proposition (2.14). If L/K is an automorphic extension with a C_K-group scheme G, then the group scheme G is uniquely determined up to isomorphism.

§ 3 Lie-Drach-Vessiot theory

Let X be a complex manifold. Traditionally since Lie, a system P of differential equation for sections of the projection $p_1 : X \times X \to X$ onto the first factor is called a Lie pseudo-group if the following conditions are satisfied.

(1) The identity Id_X is a solution of P, i.e. the map $x \to (x, x)$ is a solution of P.

(2) Let $s:U \to V$ be an isomorphism of two open sets of X. If s is a solution of P, more precisely if the map $U \to U \times V$ sending $x \in U$ to $(x, s(x))$ is a solution of P, then $s^{-1}:V \to U$ is a solution of P.

(3) Let U, V and W be three open sets of X and $s:U \to V$, $t:V \to W$ isomorphisms. If s and t are solutions of P, then $t \circ s$ is a solution of P.

Examples (3.1.1). Let $X = \mathbf{C}$ and we consider the differemtial equation $dy/dx = 1$, where x is the coordinate of $\mathbf{C}$. Then the general solution is $y = x + c$.

(2) Let $X = \mathbf{P}^1$ and we consider the Schwarzian differential equation $y^{(3)}/y' - \frac{3}{2}(y''/y')^2 = 0$, where the differentiation is done with respect to a inhomogeneous coordinate x of $\mathbf{P}^1$. Then the general solution is $y = \frac{ax + b}{cx + d}$.

(3) Let $X = \mathbf{C}^2$ and x_1, x_2 be the usual coordinate system on $\mathbf{C}^2$. We consider the differential equation $D(y_1, y_2)/D(x_1, x_2) = 1$. We can not write the general solution but this is a Lie pseudo-group.

The Examples (3.1.1) and (3.1.2) are really algebraic groups but Example(3.1.3) offers a typical example of Lie pseudo-group. To our end, Lie pseudo-group can be interpreted as a functor. To explain our definition of Lie pseudo-group, we need a preparation. Let k be a field of characteristic 0 and $AL(k)$ the category of all Artinian local k-algebras. For an integer $n \geq 0$, we define a groups functor $\mathcal{G}_n:AL(k) \to (\mathrm{Groups})$ by setting $\mathcal{G}_n(A) = ((y_1, y_2, \ldots, y_n) \mid y_i \in A[[t_1, t_2, \ldots, t_n]]$ with $y_i(t) \equiv t_i \bmod m$ for $1 \leq i \leq n$,

$\mathfrak{m}$ being the maximal ideal of A) for $A \in \mathrm{ob}\ AL(k)$. The group law in $\mathcal{G}_n(A)$ is the composition of series, i.e. for $y = (y_1, y_2, \ldots, y_n)$ and $z = (z_1, z_2, \ldots, z_n) \in \mathcal{G}_n(A)$, we define yz by $y \circ z = (y_1(z), y_2(z), \ldots, y_n(z))$, which is in $\mathcal{G}_n(A)$. $Id = (t_1, t_2, \ldots, t_n)$ is an identity in $\mathcal{G}_n(A)$, i.e. $Idy = yId = y$ for any $y \in \mathcal{G}_n(A)$. To prove that $\mathcal{G}_n(A)$ is a group, it is sufficient to show that any element of $\mathcal{G}_n(A)$ has a right inverse. Let us assume for simplicity $n = 1$. Let $y \in \mathcal{G}_1(A)$. $y(t) = a_0 + a_1 t + a_2 t^2 + \ldots$ with $a_i \equiv 0 \mod \mathfrak{m}$ if $i \neq 1$, and $a_1 \equiv 1 \mod \mathfrak{m}$. First, let us look for a series $z(t) = b_0 + b_1(t - a_0) + b_2(t - a_0)^2 + \ldots$ such that $zy = t$. This is done since if we assume $zy = t$, we can determine the coefficients $b_0, b_1, \ldots \in A$ succesively. Moreover we have $b_i \equiv 0 \mod \mathfrak{m}$ if $i \neq 1$, and $b_1 \equiv 1 \mod \mathfrak{m}$. The series $z(t)$ is in fact in $A[[t]]$. In fact, the constant term of $z(t)$ is $b_0 + b_1(-a_0) + b_2(-a_0)^2 + \ldots$ which is in A since $(-a_0)^q = 0$ for a sufficiently large integer q. Similarly the coefficient of t^p of $z(t)$ is determined in A for $p \geq 1$.

Definition (3.2). Let k be a field of characteristic 0. Let $y_1, y_2, \ldots, y_n$ be differential indeterminates over $(k[[t_1, t_2, \ldots, t_n]], (d/dt_1, d/dt_2, \ldots, d/dt_n))$. Let I be an differential ideal of $k[t]\{Y\} = k[[t_1, t_2, \ldots, t_n]]\{y_1, y_2, \ldots, y_n\}$. If a functor $H: C(k) \to (\mathrm{Sets})$, $A \to \{(y_1, y_2, \ldots, y_n) \in \mathcal{G}_n(A) \mid F(y_1, y_2, \ldots, y_n) = 0$ for any $F \in I\}$ is a subgroup functor of $\mathcal{G}_n$, the we say that H is a Lie pseudo-group functor or simply a Lie functor.

Definition (3.3). A morphism of Lie pseudo-group functors is a morphism of group functors.

This definition is simpler than the traditional definition using prolongations.

Let L/K be an ordinary differential field extension which is finitely generated as an abstract field extension. Let us assume $L = K\langle y\rangle$ since the general case is treated with a little modification. Let $F(y, y', \ldots, y^{(n)}) = 0$ with $F(Y) \in K(Y)$ be the differential equation satisfied by y such that $\partial F/\partial Y^{(n)} \neq 0$ and the degree of F in $Y^{(n)}$ is the smallest. Then tr.d.$[L: K] = n$ and $\{y, y', \ldots, y^{(n-1)}\}$ is a transcendental base of L/K. Hence if we put $y_i = y^{(i-1)}$, then $\partial/\partial y_i : K(y_1, y_2, \ldots, y_n) \to K$ is a derivation for $1 \leq i \leq n$. We can extend the $(\partial/\partial y_i)$'s to derivations of $K\langle y\rangle = L \to L$ which we denote by δ_i. Hence L is a differential algebra with respect to $\{\delta_1, \delta_2, \ldots, \delta_n\}$ whci we denote by $L^{\#}$. The δ_i's define a derivation operators of of $L^{\#}[[t]]$ commuting with $\partial/\partial t$, i.e. differentiations of coefficients. We denoting these operators again by δ_i $(1 \leq i \leq n)$, $L^{\#}[[t]]$ is a differential algebra with $\Delta = \{\partial/\partial t, \delta_1, \delta_2, \ldots, \delta_n\}$. As we explained in § 2, we have the canonical embedding $i: L \to L^{\#}[[T]]$, t being a variable. We have a commutative diagram

$$\begin{array}{ccc} L & \longrightarrow & L^{\#}[[t]] \\ | & & | \\ K & \longrightarrow & K^{\natural}[[t]]. \end{array}$$

Let us denote by $(\mathcal{L}, \Delta)$ the differential algebra of $(L^{\#}[[t]], \Delta)$ generated by $i(L)$ and L. Similarly $(\mathcal{K}, \Delta)$ is the differential

subalgebra of $(L^{\#}[[t]], \Delta)$ generated by $i(K)$ and $L^{\#}$.
Since $\{\delta_1, \delta_2, \ldots, \delta_n\}$ kills $i(K)$, $i(K)$ and $L^{\#}$ are Δ-invariant so that $\mathcal{X} = i(K)L^{\#}$. Since the ring of constants of $(L^{\#}[[t]], \Delta)$ is $K^{\natural}$, $C_{\mathcal{X}}$, $C_{\mathcal{L}} = K^{\natural}$. We have the canonical embedding $\iota : (\mathcal{L}, \Delta) \to \mathcal{L}^{\natural}[[T, U_1, U_2, \ldots, U_n]], (\partial/\partial T, \partial/\partial U_1, \partial/\partial U_2, \ldots, \partial/\partial U_n))$ introduced in § 2. Let us see what happens in Example (2.3.1).

Example (3.3.1). We have $L = Q(x, e^x) = K\langle e^x\rangle$ with $K = Q(x)$. Let us take e^x as y. By the canonical morphism $i : L \to L^{\natural}[[t]]$, x is sent to $x + t$ and e^x to ye^t in $L^{\natural}[[t]]$. In fact, $i(y)$ $= \sum_{i=0}^{\infty} \frac{1}{n!} \frac{d^n y}{dx^n} t^n = \sum_{i=0}^{\infty} \frac{1}{n!} yt^n = y \sum_{i=0}^{\infty} \frac{1}{n!} t^n = ye^t$. Since $\mathcal{X}$ is generated by $i(K)$, L with $\partial/\partial t$ and $\partial/\partial y$, $\mathcal{X} = i(K).L$. $\mathcal{L}$ is generated over $\mathcal{X}$ by $i(y) = ye^t$ with $\partial/\partial t$ and $\partial/\partial y$. Therefore $Q(\mathcal{L}) = Q(\mathcal{X}[e^t])$.

Lemma (3.3). The abstract algebra $\mathcal{L}$ is uniquely determined by L/K or is independent of the choice of the generator y of L/K.

We define a functor

$\mathcal{F}_{L/K} : Al(Q(\mathcal{L}^{\natural})) \to (Sets)$,

by $\mathcal{F}_{L/K}(A) = \{f \in Hom_{\mathcal{X}}((\mathcal{L}, \Delta), (A[[T, U_1, U_2, \ldots, U_n]], (\partial/\partial T, \partial/\partial U_1, \partial/\partial U_2, \ldots, \partial/\partial U_n)))|$ The reduction $\bar{f} : \mathcal{L} \to A[[T, U]] \to A/m[[T, U]]$ coincides with $\iota : \mathcal{L} \to \mathcal{L}^{\natural}[[T, U]] \to A/m[[T, U]]\}$ for $A \in$ ob $AL(Q(\mathcal{L}))$. In other words, $\mathcal{F}_{L/K}$ is the functor of all infinitesimal deformations of $\iota : \mathcal{L} \to \mathcal{L}^{\natural}[[T, U]]$.

Lemma (3.4). $\mathcal{F}_{L/K}(A) = \mathrm{Hom}_{\mathcal{X}}(\mathcal{L}, A)$ for any $A \in \mathrm{ob}\, AL(Q(\mathcal{L}^{\natural}))$.

Since $\mathcal{L}$ depends only on the extension L/K by Lemma (3.3), Lemma (3.4) shows that the functor $\mathcal{F}_{L/K}: AL(Q(\mathcal{L}^{\natural})) \to (\mathrm{Sets})$ is independent of the choice of the generator y of L/K. Let us analyze Example (2.3)

Example (3.4.1). As we have seen in (3.3.1) for L/K of Example (2.3), $Q(\mathcal{L}) = Q(\mathcal{X}[e^t])$. We consider now $\iota: \mathcal{L} \to \mathcal{L}^{\natural}[[T, U]]$. $\iota(e^t) = e^t e^T$ by definition. For $A \in \mathrm{ob}AL(Q(\mathcal{L}))$, a deformation $f: \mathcal{L} \to A[[T, U]]$ is determined by $f(e^t)$. Since $\partial e^t/\partial t = e^t$ and $\partial e^t/\partial y = 0$, we should have $\partial f(e^t)/\partial T = f(e^t)$ and $\partial f(e^t)/\partial y = 0$ and hence $f(e^t)\iota((e^t)^{-1}) = f(e^t)\iota(e^t)^{-1}$ is in A and is a unit: $f(e^t) = c\iota(e^t)$ with $c \in A$, $c \equiv 1 \bmod m$). Therefore $\mathcal{F}_{L/K}(A) = (c \in A \mid c \equiv 1 \bmod m)$ for $A \in \mathrm{ob}\, AL(Q(\mathcal{L}^{\natural}))$.

Now let us define the infinitesimal counterpart of the functor $G_{L/K}$ of § 2. Let L/K be an ordinary differential field extension which is finitely generated as an abstract field extension. Let $K\langle y\rangle = L$ and we take $y_1, y_2, \ldots, y_n$ as above. Let us denote by L^{*} the differential field $(L^{\natural}, (\delta_1, \delta_2, \ldots, \delta_n))$. We have the canonical embedding

(3.5) $j: L^{*} \to$

$$(L^{\natural}[[U_1, U_2, \ldots, U_n]], (\partial/\partial U_1, \partial/\partial U_2, \ldots, \partial/\partial U_n)).$$

Therefore we may identify $\partial/\partial U_i$ with δ_i for $1 \le i \le n$. For $A \in \mathrm{ob}\, AL(L^{\natural})$, we have the natural inclusion $L^{\natural}[[U]] \subset A[[U]]$ and hence by (3.5) $(L^{\natural}[[U]], (\delta_1, \delta_2, \ldots, \delta_n))$ is an L^{*} - algebra. On the other hand, we have $\mathcal{L} \subset L^{\#}[[t]]$. We have a canonical morphism $L^{\#} \to (L^{\natural}[[U]], \{\partial/\partial U_1, \partial/\partial U_2, \ldots, \partial/\partial U_n\})$ and hence we have $\mathcal{L} \subset L^{\#}[[t]] \subset (L^{\natural}[[U]][[t]] = L^{\natural}[[t,U]], \{\partial/\partial t, \partial/\partial U_1, \partial/\partial U_2, \ldots, \partial/\partial U_n\})$.

Lemma (3.6). The natural differential morphism $\mathcal{L} \otimes_{L^{\#}} A[[U]] \to A[[t, U]]$, $w \otimes h(U) \to wh(U)$ for $w \in \mathcal{L}$, $h(U) \in A[[U]]$ is an injection for $A \in \mathrm{ob}\, AL(L^{\natural})$.

We define a group functor

$\mathcal{G}_{L/K}: AL(L^{\natural}) \to (\mathrm{Groups})$,

by $\mathcal{G}_{L/K}(A) = \{f \in \mathrm{Aut}_{\mathcal{X} \otimes_{L^{\#}} A[[U]]}(\mathcal{L} \otimes_{L^{\#}} A[[U]]), \{\partial/\partial t, \partial/\partial U_1, \partial/\partial U_2, \ldots, \partial/\partial U_n\}) \mid f \equiv \mathrm{Id} \bmod m$, m being the maximal ideal of $A\}$ for $A \in \mathrm{ob} AL(L\ )$.

Let us study Example (2.3).

Example (3.6.1). For L/K in Example (2.3), by Lemma(3.6) $\mathcal{L} \otimes_{L^{\#}} A[[U]]$ is isomorphic to a subalgebra of $A[[t,U]]$ for $A \in \mathrm{ob}\, AL(L^{\natural}))$ which contains e^t by Example (3.3.1). $\mathcal{X} \otimes_{L^{\#}} A[[U]]$ - automorphism f of $\mathcal{L} \otimes_{L^{\#}} A[[U]]$ is determined by the image $f(e^t)$. The argument of Example (3.4.1) shows that we must have $f(e^t) = ce^t$, $c \in A$ with $c \equiv 1 \bmod m$. Conversely if we put $f(e^t) = ce^t$ with $c \in A$ satisfying $c \equiv 1 \bmod m$, then f determines an automophism. Thereofre we have $\mathcal{G}_{L/K}(A) = \{c \in A \mid c \equiv 1 \bmod m\}$ for $A \in \mathrm{ob}\, AL(L^{\natural}))$.

By Lemma (3.3), the riang $\mathcal{L}$ is uniquely determined and independent of the choice of a generator. Hence $AL(Q(\mathcal{L}'))$ depends only on the extension L/K.

Lemma (3.7). The functor $\mathcal{G}_{L/K}:AL(Q(\mathcal{L}')) \to (\text{Groupes})$ is independent of the choice of a generator.

The following question is an analogue of Conjecture (2.10).

Conjecture (3.8). Is $\mathcal{G}_{L/K} : AL(L') \to (\text{Groupes})$ a Lie functor?

We have the canonical morphism $\mathcal{L} \to \mathcal{L}'[[T, U]]$ and the inclusion $\mathcal{L}'[[U]] \to \mathcal{L}'[[T, U]]$. Therefore we have a canonical morphism of taking product as in Lemma (3.6) $\mathcal{L} \otimes_{L^*} \mathcal{L}'[[U]] \to \mathcal{L}'[[T, U]]$. Similarly we have a natural morphism $\mathcal{L} \otimes_{L^*} A[[U]] \to A[[T, U]]$ of taking product for $A \in \text{ob } AL(L')$.

Lemma(3.9). The natural differential morphism $\mathcal{L} \otimes_{L^*} A[[U]] \to A[[T, U]]$ is injective for $A \in \text{ob}AL(Q(\mathcal{L}))$.

The composite $\mathcal{L} \to \mathcal{L} \otimes_{L^*} A[[U]] \to A[[t, U]]$ of canonical maps coincides with $\iota:\mathcal{L} \to \mathcal{L}'[[T, U]]$. Let $f \in \mathcal{G}_{L/K}(A)$, then the composite $\mathcal{L} \to \mathcal{L} \otimes_{L^*} A[[U]] \xrightarrow{f^{-1}} \mathcal{L} \otimes_{L^*} A[[U]] \to A[[T, U]]$ is a infinitesimal deformation of ι, i.e. is in $\mathcal{F}_{L/K}(A)$, where the last and the second morphisms are canonical. Therefore $\mathcal{G}_{L/K}|AL(Q(\mathcal{L}'))$

operates on $\mathcal{F}_{L/K}$.

Lemma (3.10). The following conditions for L/K are equivalent.

(1) $(\mathcal{G}_{L/K}|AL(Q(\mathcal{L}^\natural)), \mathcal{F}_{L/K})$ is a principal homogeneous space.

(2) The morphism of the operation $\mathcal{G}_{L/K}|AL(Q(\mathcal{L}^\natural) \times \mathcal{F}_{L/K} \to \mathcal{F}_{L/K}$ is surjective.

Proposition (3.11). If the equivalent conditions of Lemma(3.8) are satisfied, then $\mathcal{G}_{L/K}|AL(Q(\mathcal{L}^\natural))$ is a Lie functor.

The following definition is consistent with the definitions of Galois extension and automorphic extension.

Definition (3.12). If L/K satisfies the following conditions, we say that L/K is infinitesimally automorphic.

(1) $(\mathcal{G}_{L/K}|AL(Q(\mathcal{L}^\natural)), \mathcal{F}_{L/K})$ is a principal homogeneous space.

(2) The group functor $\mathcal{G}_{L/K}:AL(L^\natural) \to (Groups)$ is a Lie functor defined over K . Namely there exists a Lie functor $H:AL(K^\natural) \to (Groups)$ such that $H|AL(K^\natural) \to (Groups)$ such that $H|AL(L^\natural)$ is isomorphic to $\mathcal{G}_{L/K}$.

Conjectural Lemma (3.13). Let L/K be an ordinary differential field extension which is finitely generated as an abstract field extension. Then there exists a canonical group functor $H:AL(K^\natural) \to (Groups)$ such that $H|AL(L^\natural)$ is isomorphic to $\mathcal{G}_{L/K}$.

Let us explain what we mean bycondition (2) of Definition (3.12) and Conjectural Lemma (3.13). Let L^0 be a differential K-algebra of L such that the $K^\natural$-algebra L^0 is of finite type and L is the quotient field of L^0. It is well-known and easy to show that we can find such an L^0. Let $L^{0\natural} = L[y_1, y_2, \ldots, y_m]$ such that $(y_1, \ldots, y_n)$ form a transcendental base of L/K. We can construct an algebra $\mathcal{L}^0$ similarly as $\mathcal{L}$. Namely, we can introduce derivation operators δ_i of L^0 as for L/K. We denote by L^{0*} the differential algebra $(L^{0\natural}, (\delta_1, \delta_2, \ldots, \delta_n))$. Let $\mathcal{L}^0$ be $\Delta = (\partial/\partial t, \delta_1, \delta_2, \ldots, \delta_n)$-subalgebra of $L^*[[t]]$ generated by $i(L^0)$, L^{0*} and $\mathcal{X}^0$ Δ-subalgebra generated by $i(K)$ and L^{0*}.

We assume that $K^\natural$ is algebraically closed. We denote the K-scheme Spec L^0 by X. Let v:Spec K → X be a K-rational point. We may identify v with the $K^\natural$-morpohism $\tilde{v}:L^0 \to K^\natural$ of $K^\natural$-algebras. Therefore we have a Δ-morphism $L^\natural[[t, U]] \to K^\natural[[t, U]]$. We have therefore a morphism $i_v:\mathcal{L}^0 \to K^\natural[[t,U]]$. We can consider deformations of i_v which form a functor $AL(K^\natural) \to (Sets)$. As we have $L^{0*} \to L^{0\natural}[[U_1, U_2, \ldots, U_n]] \to K^\natural[[U_1, U_2, \ldots, U_n]]$ which we denote by $\bar{v}$. We can define a group functor

$$\mathcal{G}_{v:L^0 \to K}(A) = \{ f \in \mathrm{Aut}_{\mathcal{X}^0 \otimes_{L^0} A[[U]]}\ \mathcal{L}^0 \otimes_{L^{0*}} A[[U]] \mid f \equiv \mathrm{Id} \bmod$$

m, m being the maximal ideal of A} for $A \in$ ob $AL(K^\natural)$. Here $A^\natural[[U]]$ is an L^{0*}-algebra by $L^0 \xrightarrow{\bar{v}} K^\natural[[U]] \to A[[U]]$.

What precisely Conjectural Lemma (3.13.1) means is the following. Since $AL(L^\natural)$ is a subcategory of $AL(K^\natural)$, we can speak of restrction.

Conjectural Lemma (3.13.1). There exists a non-empty

Zariski-Open set U of X such that $\mathcal{G}_{v:L^0 \to K^\natural}$ is independent of the K-valued point v and $\mathcal{G}_{v:L^0 \to K^\natural} | AL(L^\natural)$ is canonically isomorphic to $\mathcal{G}_{L/K}$.

The condition (2) of Definition (3.14) means that the conclusion of Conjectural Lemma (3.13.1) is satisfied.

Let us study Example (2.3.1).

Example (3.14). By (3.4.1) and (3.6.1), $Q(x, e^x)/Q(x)$ is infinitesimally automorphic.

The arguments used to study Example (2.3.1) shows the following.

Proposition (3.15). Let L/K be a strongly normal extension. Let G be the Galois group which is a group scheme over C_K. Then L/K is infinitesimally automorphic and $\mathcal{G}_{L/K}(A) = (w \in G(A) | w \equiv 1 \bmod m$, m being the maximal ideal of $A)$ for $A \in$ ob $AL(L^\natural)$.

We expect a similar result for an automophic extension and hence the proof is related with Conjecture (2.10).

Conjecture (3.17). The same conclusion as Proposition (3.10) holds also for an automorphic extension.

Our final result is yet conjectural.

Conjenctural Theorem (3.12). Let L/K be an infinitesimally automorphic extension and M an intermediate field between L and K such that M/K is infinitesimally automorphic. Then there exists a surjective morphism of Lie pseudo-group functors $\mathcal{G}_{L/K} \to \mathcal{G}_{M/K}$.

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