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TWO-DIMENSIONAL JACOBIAN CONJECTURE

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The Jacobian Conjecture is formulated as follows:

Jacobian Conjecture. Let $f_1, \ldots, f_n$ be polynomials in $x_1, \ldots, x_n$ over the complex number field C. If the Jacobian $\partial(f_1, \ldots, f_n)/\partial(x_1, \ldots, x_n)$ is a non-zero constant, then $\mathrm{C}[f_1, \ldots, f_n] = \mathrm{C}[x_1, \ldots, x_n]$, or equivalently, the mapping of C^n into itself defined by $(x_1, \ldots, x_n) \to (f_1, \ldots, f_n)$ is biregular.

Once this is proved, one can replace C by any field of characteristic zero.

If $n = 1$, then the conjecture is obviously true, and the case where $n \geq 2$ has been a problem.

The problem was formulated by Keller [K] in 1939. There are some partial answers. Some of partial answers in the two-dimensional case are based on convex polygons of f, g such that $\partial(f, g)/\partial(x, y)$ is a non-zero constant. Therefore, we shall discuss convex polygons of polynomials, and, for the purpose, we begin with some remarks on derivations of $K[x, y]$, where K is a field of characteristic zero.

Then we shall discuss how to apply our results on convex polygons to some known partial answers. Then we try to generalize some results of Abhyankar [A] and then some results of Appelgate-Onishi [AO] are also generalized.

At the end of this article, we try to apply these generalized results to the Conjecture.

Throughout this article, a ring means a commutative ring with identity. K denotes always an algebraically closed field of characteristic zero. For a polynomial f in x, y, we denote by f_x, f_y its partial derivatives.

1. Some remarks on derivations of $K[x, y]$.

First we recall the following theorem.

ZAKS THEOREM (ZAKS [Z]). *If R is a Dedekind subring of a polynomial ring over K and if $K \subseteq R$, then there is a polynomial h such that $R = K[h]$.*

From now on in this section, we restrict our observation to the two-dimensional case.

For each polynomial $g \in K[x, y]$, we have a mapping $f \to \partial(f, g)/\partial(x, y)$. This is a derivation of $K[x, y]$, and we denote this derivation by d_g. d_g is obviously a K-derivation, i,e., d_g is trivial on K.

THEOREM 1.1. *Let d be a non-trivial K-derivation of $K[x, y]$. Then the ring C_d of constants for d is either K or $K[h]$ with an $h \notin K$. In this latter case, $K[h]$ is integrally closed in $K[x, y]$, and $d = (g_1/g_2)d_h$ where $g_i \in K[x, y]$ and g_2 divides both h_x and h_y.*

PROOF: C_d is integrally closed in $K[x, y]$. Therefore by Zaks theorem, we see that C_d is either K or $K[h]$ ($h \notin K$). Since d is a derivation of $K[x, y]$, $d = k_1(\partial/\partial x) + k_2(\partial/\partial y)$ ($k_i \in K[x, y]$), and in the latter case, $dh = 0$, and $k_1 h_x + k_2 h_y = 0$. Let g_2 be the GCM of h_x, h_y and set $g_1 = g_2 k_1/h_y (= -g_2 k_2/h_x)$. Then we have $d = (g_1/g_2)d_h$.

COROLLARY 1.2. *For a polynomial $h \in K[x, y]$, there is a K-derivation d of $K[x, y]$ such that its ring of constants coincides with $K[h]$ if and only if $K[h]$ is integrally closed in $K[x, y]$. (Cf. [NNg])*

COROLLARY 1.3. *If f, g ($\in K[x, y]$) satisfy $\partial(f, g)/\partial(x, y) = 0$, then there is a polynomial h so that f, $g \in K[h]$.*

2. Convex polygons.

Let $f \in K[x,y]$. We denote by $\sup f$ the set of lattice points (i,j) on the plane $\mathbf{R}^2$ such that the coefficient of $x^i y^j$ in f is not zero. The least convex polygon containing $\sup f \cup \{0\}$ is called the convex polygon of f.*)

By a direction, we understand a pair of rational integers p, q such that (i) $p > 0$ or $q > 0$ and (ii) p and q are coprime.

We say that a polynomial f is a (p,q)-form of degree n if it is of the form

$$\sum_{ip+jq=n} a_{ij} x^i y^j \qquad (a_{ij} \in K).$$

The degree n is denoted by $d_{pq}(f)$.

If $f \in K[x,y]$, then f has a decomposition to the sum of (p,q)-forms:

$$f = f_{d_1} + \cdots + f_{d_s} \qquad (d_1 > \cdots > d_s\,;\ \text{each } f_{d_i} \text{ is a non-zero } (p,q)\text{-form of degree } d_i)$$

We call f_{d_1} the leading (p,q)-form of f. Its degree d_1 is called the (p,q)-degree of f and is denoted by $d_{pq}(f)$.

For a (p,q)-form h, $\sup h$ lies on a line. The least line segment containing $\sup h$ is denoted by $\overline{\sup}\ h$ (if $\sup h$ consists of only one point, then $\overline{\sup}\ h$ is a point).

REMARK. In the (p,q)-form decomposition of f above,

(i) if $pq < 0$, the f_{d_s} is the leading $(-p,-q)$-form of f,

(ii) $\sup f$ is in between two lines $px + qy = d_1$ and $px + qy = d_s$, and

(iii) the intersection of $\sup f$ and the line $px + qy = d_i$ coincides with $\sup f_{d_i}$ for each i.

Consider two polynomials and their (p,q)-form decompositions:

$$f(x,y) = f_{d_1} + \cdots + f_{d_s}, \quad g(x,y) = g_{e_1} + \cdots + g_{e_t}.$$

Then $\partial(f,g)/\partial(x,y) = \sum_{ij} \partial\left(f_{d_i}, g_{e_j}\right)/\partial(x,y)$ and each

*)This is often called the Newton polygon of f.

$$\partial\left(f_{d_i}, g_{e_j}\right)/\partial(x,y)$$

is a (p,q)-form of degree $d_i + e_j - p - q$. Therefore we have (cf. [NB], [NNK])

THEOREM 2.1.

(1) *If* $\partial(f,g)/\partial(x,y) = 0$, *then* $\sum_{d_i+e_j=k} \partial\left(f_{d_i}, g_{e_j}\right)/\partial(x,y) = 0$ *for each* k *and*

(2) *if* $\partial(f,g)/\partial(x,y)$ *is a non-zero constant, then the same equalities hold except for the case* $k = p + q$.

Now we have

THEOREM 2.2. ([AO], [NNk], [Nw]) *If* $\partial(f,g)/\partial(x,y) = 0$, *then the convex polygon of* f *is similar to the one of* g *with origin as the center of similarity and with ratio* $\deg f : \deg g$.

PROOF: First we consider the leading $(1,1)$-forms f_m, g_n of f, g, respectively. Then $\partial(f_m, g_n)/\partial(x,y) = 0$, and there is a $(1,1)$-form h such that $f_m = ah^{m'}$, $g_n = bh^{n'}$ $(a,\ b \in K)$. Therefore $\sup f_m$ is similar to $\overline{\sup}\ g_n$ with origin as the center of similarity and with ratio $m : n$. Take one end of each of $\overline{\sup}\ f_m$ and $\overline{\sup}\ g_n$. They are (a,b), (c,d) with $n(a,b) = m(c,d)$. Take one direction (p,q) such that the leading (p,q)-forms f', g' of f, g have terms $x^a y^b$, $x^c y^d$ with non-zero coefficients, respectively. Then, since $\partial(f',g')/\partial(x,y) = 0$, we see that $\overline{\sup}\ f'$ and $\overline{\sup}\ g'$ are similar to each other with origin as the center of similarity. Since the points (a,b) and (c,d) are on $\sup f'$, $\sup g'$, respectively, we see that the ratio is $m : n$. This can be applied to any neighbouring edges successively, and we complete the proof.

3. $\partial(f,g)/\partial(x,y) = 1$.

What we want to prove in this section is the following result (cf. [AO], [NNk]).

THEOREM 3.1. *If* $\partial(f,g)/\partial(x,y)$ *is a non-zero constant and if* $\deg f > 1$, $\deg g > 1$, *then the convex polygon of* f *is similar to that of* g, *with origin as the center of similarity and with ratio* $\deg f : \deg g$.

REMARK. The assumption on the degrees of f, g is important. For instance, $f = x$, $g = y + x^3$ satisfy the Jacobian condition. But, the convex polygon of f is a line segment and that of g is a triangle.

From now on, in this section, f and g are in $K[x,y]$ and $\partial(f,g)/\partial(x, y) = 1$.

Consider the following statement.

PROPOSITION 3.2. *If* $\deg f > 1$, $\deg g > 1$, *then, for any direction* (p,q), $d_{pq}(f) + d_{pq}(g) > p + q$.

If this assertion is proved, in view of Theorem 2.1, (2), our proof of Theorem 2.2 can be applied to prove Theorem 3.1. Thus, in proving Theorem 3.1, it suffices to prove Proposition 3.2. Main idea of our proof of Proposition 3.2 is due to [NNk].

LEMMA 3.3. $(1,0), (0,1) \in \sup f \cup \sup g$. *Furthermore, if* $(0,1) \notin \sup f$, *then* $(1,0) \in \sup f$.

LEMMA 3.4. *If one of* f, g, *say* f, *is a polynomial in* x *only or in* y *only, then* $\deg f = 1$.

Proofs of these two lemmata are straightforward.

COROLLARY 3.5. *If* $\deg f > 1$, *then the convex polygon of* f *is not a line segment.*

PROOF: Assume the contrary. By Lemma 3.3, $\sup f$ contains $(1,0)$ or $(0,1)$. Then f is a polynomial in x only or y only. Then $\deg f = 1$ by Lemma 3.4.

For a polynomial h, we define $t_x(h)$ and $t_y(h)$ by:

$$t_x(h) = \max\{s \mid (s,0) \in \sup h \cup \{(0,0)\}\}$$
$$t_y(h) = \max\{s \mid (0,s) \in \sup h \cup \{(0,0)\}\}$$

PROPOSITION 3.6. *If* $\deg f > 1$, $\deg g > 1$, *then* $t_x(f)$, $t_y(f)$, $t_x(g)$, $t_y(g)$ *are all positive.*

PROOF: It suffices to prove the assertion on $t_x(f)$ only. We may assume that f has no constant term. Assume that $t_x(f) = 0$. Then we have a direction (p,q) such that (i) at least one point of $\sup f$ lies on the line $px + qy = 0$, (ii) $p > 0$, $q < 0$, and (iii) no point of $\sup f$ lies in the area $px + qy > 0$. Lemma 3.3 shows that $(1,0) \in \sup g$. Let f', g' be the leading (p,q)-forms of f, g, respectively. By our choice of (p,q), we have $d_{pq}(f') = 0$, $d_{pq}(g') \geq p > p + q$. Therefore, by Theorem 2.1, (2), we have $\partial(f',g')/\partial(x,y) = 0$. Then there is a

(p,q)-form h such that f', $g' \in K[h]$. In view of the (p,q)-degrees, $d_{pq}(h) > 0$, and therefore f' must be a constant; a contradiction.

PROOF OF PROPOSITION 3.2. We may assume that $p \geq q$. Then $p > 0$. $d_{pq}(f) \geq p \cdot t_x(f) \geq p$, $d_{pq}(g) \geq p \cdot t_x(g) \geq p$. Therefore $d_{pq}(f) + d_{pq}(g) \geq 2p$. Therefore, if $p > q$, then the assertion is proved. Assume now that $p = q$. Then $p = 1$. $d_{pq}(f) = \deg f$, $d_{pq}(g) = \deg g$ and $d_{pq}(f) + d_{pq}(g) = \deg f + \deg g \geq 4 > 2 = p + q$.

4. Some of partial results.

One nice and very difficult partial result is the one due to Moh [Mo]. It says if $\partial(f,g)/\partial(x,y) = 1$ and if both $\deg f$, $\deg g$ are less than 100, then $K[f,g] = K[x,y]$.

It is too difficult to sketch the proof.

One of the partial results which are related to convex polygons is the following theorem given by Magnus [Ma].

We maintain the meaning of f, g ($\partial(f,g)/\partial(x,y) = 1$) also in this section.

MAGNUS THEOREM. *If* $\deg f$, $\deg g$ *are coprime, then* $K[f,g] = K[x,y]$.

Let us see how Theorem 3.1 can be used in proving this result.

PROOF OF MAGNUS THEOREM. If one of $\deg f$, $\deg g$ is one, then we see $K[f,g] = K[x,y]$. Therefore, assuming that both $\deg f$, $\deg g$ are bigger than 1, we shall see a contradiction. Let $m = \deg f$, $n = \deg g$. Consider the leading $(1,1)$-forms f', g' of f, g. Then there is a $(1,1)$-form h such that f', $g' \in K[h]$. Since m and n are coprime, $\deg h$ must be one. Then we may assume that $h = x$. Consider $t_y(f)$, $t_y(g)$. By Theorem 3.1, we have $t_y(f)n = t_y(g)m$, and $t_y(f)$ is divisible by m, which contradicts $0 < t_y(f) < m$.

This result was generalized by Nakai-Baba [NB] to include the following.

NAKAI-BABA THEOREM. *If the GCM of* $\deg f$, $\deg g$ *is 2, then* $K[f,g] = K[x,y]$.

There is one more generalization which is known. Namely,

APPELGATE-ONISHI THEOREM. *Let* f, g ($\in K[x,y]$) *be such that* $\partial(f,g)/\partial(x,y) = 1$.

(1) *If the GCM of* $\deg f$, $\deg g$ *is a prime number, then one of* $\deg f$, $\deg g$ *divides the other.*
(2) *If* $\deg f$ *is the product of two prime numbers, then* $K[f,g] = K[x,y]$.

Their article [**AO**] contains some errors. But the proof can be completed by our Theorem 3.1 and some results of Abhyankar [**A**]. We shall give a proof at the end of the article.

There are several partial answers of different directions. For instance, Abhyankar [**A**, (19.4)] states the following result.

ABHYANKAR THEOREM. *Let* f, g $(\in K[x,y])$ *be assumed that* $\partial(f, g)/\partial(x,y)$ *is a non-zero constant. Then the following 4 statements are equivalent to each other.*

(1) *The assumption implies* $K[f,g] = K[x,y]$ *(Jacobian Conjecture).*

(2) *The assumption implies that* f *has only one point at infinity.*
(3) *The assumption implies that the convex polygon of* f *is a triangle with vertices* $(0,0)$, $(m,0)$, $(0,n)$ *with non-negative integers* m, n. *(Note that this include the case the polygon is a line segment.)*
(4) *The assumption implies that one of* $\deg f$, $\deg g$ *divides the other.*

Abhyankar [**A**, (21, 11)] gives the following result also.

ABHYANKAR THEOREM. *Under the same assumption as above, if it follows that* $K(x,y)$ *is a Galois extension of* $K(f,g)$, *then we have* $K[f,g] = K[x,y]$.

One can see some more partial results in [**BCW**].

In our study, we like to include the following partial result, which is included in the Abhyankar theorem stated above.

THEOREM 4.1. *If the assumption* $\partial(f,g)/\partial(x,y) = 1$ $(f,g \in K[x,y])$ *implies that the convex polygon of* f *is a triangle (or a line segment), then the two-dimensional Jacobian Conjecture is true.*

The proof will be given later, because we need some preliminary results.

5. Leading (p,q)-forms.

In this section, we mean by a polynomial a polynomial in x, y over K, unless the contrary is explicitly stated.

Results in this section are based on Abhyankar [A] and we do not give detailed references (some were generalized by Appelgate-Onishi [AO]).

The following result is easily seen by our Corollaries 1.2, 1.3.

LEMMA 5.1. *If f, g are non-zero (p,q)-forms and if $\partial(f,g)/\partial(x,y) = 0$, then there is a (p,q)-form h such that* (i) *f, $g \in K[h]$,* (ii) *$K[h]$ is integrally closed in $K[x,y]$. In this case, if $d_{pq}(f) \neq 0$, then $d_{pq}(h) \neq 0$ and there are c, $c' \in K$ and non-negative integers s, t such that $f = ch^s$, $g = c'h^t$.*

Two polynomials f, g are said to be (p,q)-related if the leading (p,q)-forms of f, g satisfy the condition in Lemma 5.1.

LEMMA 5.2. *Let f, g be non-zero (p,q)-forms of degree m, n, respectively, then we have:*

(1) $mf = pxf_x + qyf_y, \quad ng = pxg_x + qyg_y$

(2) $px \cdot \partial(f,g)/\partial(x,y) = mfg_y - ngf_y$

$\quad qy \cdot \partial(f,g)/\partial(x,y) = hgf_x - mfg_x$

PROOF: (1) is easily seen by

$$px(x^iy^j)_x + qy(x^iy^j)_y = (pi + qj)x^iy^j.$$

As for (2), we have

$$\begin{aligned} px\partial(f,g)/\partial(x,y) &= \det\begin{pmatrix} pxf_x & f_y \\ pxg_x & g_y \end{pmatrix} \\ &= \det\begin{pmatrix} pxf_x + qyf_y & f_y \\ pxg_x + qyg_y & g_y \end{pmatrix} \\ &= \det\begin{pmatrix} mf & f_y \\ ng & g_y \end{pmatrix} = mfg_y - ngf_y \end{aligned}$$

The second equality is proved similarly.

For non-zero polynomials f, g, we define:

$$\delta_{pq}(f,g) = d_{pq}(fg) - d_{pq}(\partial(f,g)/\partial(x,y)) - p - q$$

LEMMA 5.3. *Under the circumstances, we have* (1) $\delta_{pq}(f,g) \geq 0$

and (2) denoting by f', g' the leading (p,q)-forms of f, g, respectively, we have $\partial(f',g')/\partial(x,y)$ coincides with the leading (p,q)-form of $\partial(f,g)/\partial(x,y)$ or 0 according to $\delta_{pq}(f,g)$ is 0 or not.

PROOF: (1) is clear. (2) follows from the fact that $d_{pq}(f,g)-p-q$ -degree part of $\partial(f,g)/\partial(x,y)$ is $\partial(f',g')/\partial(x,y)$.

LEMMA 5.4. *Let f, g be polynomials such that $d_{pq}(f) \neq 0$ and f and $\partial(f,g)/\partial(x,y)$ are (p,q)-related. Then there is a polynomial h such that f and $\partial(f,h)/\partial(x,y)$ are (p,q)-related and $\delta_{pq}(f,h)=0$*

PROOF: If $\delta_{pq}(f,g)=0$, then we may take g as h. Supposing that $\delta_{pq}(f,g)>0$, it is enough to show that there is a polynomial h such that f and $\partial(f,h)/\partial(x,y)$ are (p,q)-related and $\delta_{pq}(f,h)<\delta_{pq}(f,g)$.

Let f', g' be the leading (p,q)-forms of f, g, respectively. Lemma 5.3 shows that $\partial(f',g')/\partial(x,y)=0$, and therefore f and g are (p,q)-related, and there is a polynomial k such that $f'=ck^s$, $g'=c'k^t$ (Lemma 5.1). Then there is $c''\in K$ such that $c''\neq 0$ and f^t and $c''g^s$ have the same leading (p,q)-forms. Set $h=f^t-c''g^s$. Then $\partial(f,h)/\partial(x,y)=-\partial(f,c''g^s)/\partial(x,y)=-c''sg^{s-1}\partial(f,g)/\partial(x,y)$. Therefore f and $\partial(f,h)/\partial(x,y)$ are (p,q)-related.

$$\begin{aligned}
d_{pq}(\partial(f,h)/\partial(x,y)) &= d_{pq}(g^{s-1})+d_{pq}(\partial(f,g)/\partial(x,y)) \\
&= d_{pq}(g^{s-1})+d_{pq}(fg)-p-q-\delta_{pq}(f,g) \\
&= d_{pq}(fg^s)-p-q-\delta_{pq}(f,g) \\
&> d_{pq}(fh)-p-q-\delta_{pq}(f,g)
\end{aligned}$$

This shows that $\delta_{pq}(f,h)<\delta_{pq}(f,g)$.

COROLLARY 5.5. *Let f be a polynomial such that $d_{pq}(f)\neq 0$. Assume that there is a polynomial g such that f and $\partial(f,g)/\partial(x,y)$ are (p,q)-related. Then there are (p,q)-forms h, k such that the leading (p,q)-form f' of f is ch^s $(c\in K)$ and $\partial(h,k)/\partial(x,y)$ is $c'h^t$ with $c'\in K$ and a non-negative integer t.*

PROOF: By Lemma 5.4, we may assume that $\delta_{pq}(f,g)=0$. Let g' be the leading (p,q)-form of g. Then by Lemma 5.3, $\partial(f',g')/\partial(x,y)$ is the leading (p,q)-form of $\partial(f,g)/\partial(x,y)$. Therefore, there is a (p,q)-form h such that leading (p,q)-forms of f, $\partial(f,g)/\partial(x,y)$ are ch^s, $c'h^u$

(c, $c' \in K$; s, u are non-negative integers). Since $d_{pq}(f) \neq 0$, we have $s > 0$. Now we have :

$$\begin{aligned} c'h^u &= \partial(f', g')/\partial(x, y) = \partial(ch^s, g')/\partial(x, y) \\ &= csh^{s-1}\partial(h, g')/\partial(x, y). \end{aligned}$$

Thus $u \geq s - 1$, and we have the assertion with $t = u - s + 1$.

The following lemma is due to **[AO]**.

LEMMA 5.6. *Let f be an irreducible (p, q)-form and $g = h/k$ with (p, q)-forms h, k. Assume that*

(1) *$d_{pq}(g) = d_{pq}(h) - d_{pq}(k)$ is not zero and*
(2) *in the local ring $R = K[x, y]_{(f)}$, $\partial(f, g)/\partial(x, y)$ is divisible by f.*

Then g is divisible by f in R.

PROOF: By the assumption, we have

$$\begin{pmatrix} f_x & -f_y \\ qy & px \end{pmatrix} \begin{pmatrix} g_y \\ g_x \end{pmatrix} \equiv \begin{pmatrix} 0 \\ ng \end{pmatrix} \qquad (\text{mod } fR).$$

Multiplying matrix $\begin{pmatrix} px & f_y \\ -qy & f_x \end{pmatrix}$ from the left, we have

$$\begin{pmatrix} f_y ng \\ f_x ng \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (\text{mod } fR).$$

Since f is irreducible, we see that $g \in fR$.

LEMMA 5.7. *Let f, g be (p, q)-forms with $d_{pq}(g) \neq 0$. If $\partial(f, g)/\partial(x, y)$ is divisible by f, then every irrducible factor of f divides g.*

PROOF: Let h be an irreducible factor of f and let $f = h^i f_1$ (f_1 not divisible by h). Then $\partial(f, g)/\partial(x, y) \equiv ih^{i-1} f_1 \cdot \partial(h, g)/\partial(x, y)$ (mod h^i). Thus $\partial(h, g)/\partial(x, y) \equiv 0 \quad (\text{mod } h)$. Thus, by Lemma 5.6, h divides g.

LEMMA 5.8. *Let f, g be (p, q)-forms such that $\partial(f, g)/\partial(x, y) = f^N$ with a natural number N bigger than 1. If $p+q > 0$ and if $d_{pq}(f) \geq 0$, then f divides g.*

PROOF: Let h be an irreducible factor of f and let $f = h^i f_1$, $g = h^j g_1$ with f_1, g_1 not divisible by h. If $i \leq j$, then h^i divides g. Assume that $i > j$.

$$\begin{aligned} f^N &= \partial(f,g)/\partial(x,y) = \partial(h^i f_1, h^j g_1)/\partial(x,y) \\ &= h^{i+j}\partial(f_1,g_1)/\partial(x,y) + ih^{i+j-1} f_1 \partial(h,g_1)/\partial(x,y) \\ &\quad - jh^{i+j-1} g_1 \partial(h,f_1)/\partial(x,y) \end{aligned}$$

Since $i > j$, we see that $if_1\partial(h,g_1)/\partial(x,y) - jg_1\partial(h,f_1)/\partial(x,y)$ is divisible by h. Consider $k = g_1^i/f_1^j$ in the ring $R = K[x,y]_{(h)}$. We have

$$\begin{aligned} &\partial(h,k)/\partial(x,y) \\ =&(g_1^{i-1}/f_1^{j+1})(if_1\partial(h,g_1)/\partial(x,y) - jg_1\partial(h,f_1)/\partial(x,y)) \end{aligned}$$

and this is divisible by h. We shall show $d_{pq}(k) \neq 0$.

$$d_{pq}(k) = i \cdot d_{pq}(g_1) - j \cdot d_{pq}(f_1) = id_{pq}(g) - jd_{pq}(f).$$

On the other hand, $Nd_{pq}(f) = d_{pq}(f) + d_{pq}(g) - p - q$, we have $d_{pq}(g) = (N-1)d_{pq}(f) + p + q$. Since $N > 1$ and $p + q > 0$, we have $d_{pq}(g) > d_{pq}(f)$. Since $i > j$, we see that $d_{pq}(k) > 0$. Now, by Lemma 5.7, k is divisible by h, which is impossible. Thus $i \leq j$. Since this is true for every irreducible factor of f, we see that f divides g.

REMARK. As is stated in [AO], $i < j$ for each h.

COROLLARY 5.9. *Under the assumption in Corollary 5.5, if furthermore $d_{pq}(f) > 0$ and $p + q > 0$, then the (p,q)-form k can be chosen so that $\partial(h,k)/\partial(x,y) = h^s$ with $s = 0$ or 1.*

As a corollary to this, we have

PROPOSITION 5.10. *Let f, g be polynomials such that $\partial(f,g)/\partial(x,y) = 1$. Let (p,q) be a direction such that $p + q > 0$ and $d_{pq}(f) > 0$. Then there are (p,q)-forms h, k such that (1) the leading (p,q)-forms f', g' of f, g are of the form ch^s, $c'h^t$ (c, $c' \in K$) and (2) $\partial(h,k)/\partial(x,y) = h^u$ with $u = 0$ or 1.*

PROOF: By the definition, f and $\partial(f,g)/\partial(x,y)$ are (p,q)-related. Therefore, Corollary 5.9 shows our assertion.

We shall pay attention to such a pair h, k in Proposition 5.10.

THEOREM 5.11. *Let h, k be (p,q)-forms such that $\partial(h,k)/\partial(x,y) = 1$. Then one of h, k is of degree 1. Consequently, $K[h,k] = K[x,y]$.*

PROOF: By the assumption, we have $d_{pq}(h) + d_{pq}(k) = p + q$. We may assume that the points $(1,0)$, $(0,1)$ are on the convex polygons of h, k, respectively. Since h, k are (p,q)-forms, every term of h has (p,q)-degree p. The similar is true for k.

(1) Assume first that $p \geq q \geq 0$. Then k is of degree 1, or a polynomial in y only, hence $\deg k = 1$ by Lemma 3.4.

(2) Assume now that $p > 0$, $q < 0$. Then every term of k must have negative (p,q)-degree. Then k is divisible by y. If $\deg h > 1$, $\deg k > 1$, then by Theorem 3.1, we see that the convex polygons of h, k are similar, hence both have vertices in the positive area of both x-axis and y-axis, which is impossible. Therefore one of $\deg h$, $\deg k$ must be 1.

LEMMA 5.12. *Let h, k be (p,q)-forms such that $\partial(h,k)/\partial(x,y) = h$. Then, (1) k must be reducible, provided that k has no constant term, and (2) there is no irreducible polynomial f whose square divides k.*

PROOF: If k is irreducible, then h must be of the form ck^i $(c \in K)$ by Lemma 5.7. Then the Jacobian must vanish; a contradiction. Thus (1) is proved. As for (2), assume that $k = f^i k_1$, $h = f^j h_1$ $(i \geq 2$, $j \geq 0$; k_1, h_1 not divisible by $f)$. Then the Jacobian must be divisible by f^{j+1}; a contradiction.

THEOREM 5.13. *Let h, k be as in Lemma 5.12. Assume furthermore that $pq > 0$. Then k is the product of irreducible (p,q)-forms k_1, k_2 and h is of the form $ck_1^i k_2^j$ $(c \in K$; i, j are nonnegative integers such that $i \neq j)$.*

PROOF: By the condition on the Jacobian, we have $d_{pq}(k) = p + q$. We may assume that $p \geq q > 0$.

(1) If $p = q$, then $\deg k = 2$, and k is the product of two linear forms k_1, k_2 by Lemma 5.12. Therefore, we see the assertion in this case by Lemma 5.7.

(2) Assume that $p > q > 0$. Let $k = k_1 k_2$ (by Lemma 5.12). $d_{pq}(k_i) \geq q$, $d_{pq}(k_1) + d_{pq}(k_2) = p + q$ implies that one of $d_{pq}(k_i)$ is

q. Thus we may assume that $d_{pq}(k_1) = p$ and $d_{pq}(k_2) = q$. Then $k_1 = cx + c'y^{p/q}$ ($0 \neq c \in K$; $c' \in K$ and if $c' \neq 0$, then $q = 1$), and $k_2 = c''y$ with $c'' \in K$. Therefore k_1 and k_2 are irreducible, and the assertion is proved by Lemma 5.7.

COROLLARY 5.14. *Under the circumstances, we have* $K[k_1, k_2] = K[x, y]$.

PROOF: Immediate from the proof above.

COROLLARY 5.15. *Let* $f, g \in K[x, y]$ *be such that* $\partial(f, g)/\partial(x, y) = 1$. *Then the convex polygon* N_f *of* f *has no edge* $\overline{\sup}\ f'$ *such that* f' *is the leading* (p, q)*-form of* f *with* $pq > 0$ *and the edges* (a, b), (a', b') *of* $\overline{\sup}\ f'$ $(a < a',\ b > b')$ *are such that* $ab \neq 0$, $a'b' \neq 0$.

PROOF: Assume that such an edge exists. By Corollary 5.9, there are (p, q)-forms h, k such that $\partial(h, k)/\partial(x, y) = 1$ or h, and such that $f' = ch^s$ with $c \in K$. By Lemma 5.13, h can have at most two different irreducible factors. By the assumption, h is divisible by x and y. Thus $f' = c'x^iy^j$ ($c' \in K$) and $\overline{\sup}\ f'$ is a point, not an edge.

Lemma 5.16. *Let* h, k *be* $(1,0)$*-forms such that* $\partial(h,k)/\partial(x,y) = h$ *and let* x^ay^b *be the leading* $(1,1)$*-form of* h. *Then* h, k *can be written as* $h = x^a h_1(y)$, $k = xk_1(y)$.

(1) *If* $\deg k_1 = 1$, *then* $h_1 = (y-c)^b$ *with* $c \in K$, *disregarding the constant factor, and* $b \neq a$.

(2) *If* $\deg k_1 \neq 1$, *then* $b = a(\deg k_1)$. *Furthermore, there is a root* c *of* $h_1(y)$ *whose multiplicity is at least* a. *However, no root of* $h_1(y)$ *has multiplicity exactly* a.

Proof. Set $n = \deg k_1$. By Lemma 5.7, n cannot be 0. Assume first that $n = 1$. Then $h = x^a k_1^{\ b}$ *(disregarding the constant factor)*. If $a = b$, then the Jacobian of h, k vanishes, and $a \neq b$.

Assume now that $n \neq 1$. Then $(a+b)+(1+n) - 2 > a+b$. Therefore $(a+b+n-1)$-degree part $\partial(x^ay^b, xy^n)/\partial(x,y)$ must vanish. Thus $b = an$. By Lemma 5.7, every root of $h_1(y)$ is a root of $k_1(y)$. k_1 has n roots and h_1 has an roots. Therefore, some root, say c, has multiplicity at least a. If the multiplicity is exactly a, then by the transformation $x-c \to x$, we may assume that $c = 0$. Then the least degree parts of h, k are x^ay^a, xy. Then the Jaco-

bian of h,k has no term of degree 2 , a contradiction.

Deg k can be big, as one sees by the following example:

EXAMPLE 5.17. Let a, t, u be natural numbers such that $u \neq a$ and set $v = ta + a - tu$. Then $a \neq v$. With a nonzero element c of K, we set $k_1 = cy^t + (a-v)^{-1}$, $h = x^a y^v k_1^u$ and $k = xyk_1$.

In this case, b in Lemma 5.16 is $v + tu = a(t+1)$. We can see easily that $\partial(h,k)/\partial(x,y) = h$.

Now we consider the case where $p < 0$. Let h,k be (p,q)-forms such that $\partial(h,k)/\partial(x,y) = h$. Set $r = -p$. Then (p,q)-forms of degree 0 are polynomials in $T = x^q y^r$ Therefore, h, k are written as $h = x^c y^d h_1(T)$ $(h_1(0) \neq 0)$, $k = xy\, k_1(T)$. Set $m = \deg h_1(T)$, $n = \deg k_1(T)$. Then the leading (1,1)-form of h is $x^a y^b$ with $a = c+mq$, $b = d+mr$. Let a_1, b_1 be natural numbers such that $a : b = a_1 : b_1$ and such that a_1, b_1 are coprime.

Lemma 5.18. *Under the circumstances, assume that* $p+q \neq 0$.

(1) *If* $n = 0$, *then* $m = 0$.

(2) *Assume that* $m \neq 0$. *Then* $n \neq 0$ *and, by setting* $s = a_1 p + b_1 q$,

(i) *we have* $1+nq = a_1 e$, $1-np = b_1 e$, $p+q = se$, $a_1 - b_1 = sn$,

(ii) *if furthermore* $p+q > 0$, *then* $q > p+q \geq s > 0$, $a_1 > b_1$.

Proof. (1) follows from Lemma 5.7. Assume that $m \neq 0$. Then $n \neq 0$ by (1). Top ends of sup h and sup k are (a,b) and $(1+nq, 1+nr)$, respectively. By the Jacobian condition, the Jacobian of $x^a y^b$, $x^{1+nq} y^{1+nr}$ must vanish. Thus $1+nq = a_1 e$, $1+nr = b_1 e$ with a natural number e. Therefore $pa_1 e + qb_1 e = p+q$, $a_1 e - b_1 e = n(p+q) = nse$. Therefore $a_1 - b_1 = se$. Now we have (ii).

Remark. Interchanging p and q, we can apply results above to the case where $q < 0$. For instance, corresponding result to (ii) above is: $p > p+q \geq a_1 p + b_1 q$, $b_1 > a_1$.

6. Proof of Theorem 4.1.

Assume that the condition $\partial(f,g)/\partial(x,y) = 1$ implies that the convex polygon N_f of f is a triangle having vertices $(a,0)$, $(0,b)$, $(0,0)$ with nonnegative integers a, b. Under this assumption, we are to

prove the two-dimensional Jacobian Conjecture by double induction on (m, n), where m and n are determined as follows. First choose one of f, g, say f so that $\deg f \leq \deg g$. Then take a, b as above for N_f. Then $m = \max\{a, b\}$ and $n = \min\{a, b\}$.

We may assume that $m = a$. Since the polygon N_f is a triangle, we have $m = \deg f$.

Take a direction (p, q) so that $p : q = b : a$. Then the leading (p, q)-form f' of f has its support on the edge of N_f (here, by the edge, we mean an edge not on x- or y-axis). By Proposition 5.10, there are (p, q)-forms h, k such that $\partial(h, k)/\partial(x, y) = h^u$ with $u = 0$ or 1 and $f' = ch^s$ with $c \in K$.

(1) The case where $u = 0$, $a = b$. Then $(p, q) = (1, 1)$ and $\deg h = \deg k = 1$, $K[h, k] = K[x, y]$. Then, in terms of h, k, the degree of f does not change and the triangle of f in terms of h, k has only one point $(a, 0)$ on the old edge of N_f. Therefore the other vertex of the new triangle must be $(0, b')$ with $b' < b$.

(2) The case where $u = 0$, $b < a$. Since $\overline{\sup}\, h$ is parallel to the edge of N_f, h is not linear, hence k is linear by Theorm 5.11, and $K[h, k] = K[x, y]$. If we express f in terms of h, k, then the degree becomes $s = a/(\deg h)$. Hence the new m is smaller than the old m.

(3) The case where $u = 1$, $a = b$. Then $(p, q) = (1, 1)$ and $k = k_1 k_2$ with linear forms k_i such that $K[k_1, k_2] = K[x, y]$. Furthermore, $f' = ch^s$, $h = c' k_1^i k_2^j$ (c, $c' \in K$). In terms of k_1, k_2, the highest degree part of f is $c'' k_1^{si} k_2^{sj}$ ($c'' \in K$). Since the convex polygon must be a triangle, one of i, j must be 0. We may assume that $j = 0$. The original edge has ends $(a, 0)$, $(0, a)$ and the new triangle has edges $(a, 0)$, $(0, b')$ with $b' < a$.

(4) The remaining case, namely, $u = 1$ and $b < a$. Our proof of the case (3) can be adapted, because $K[k_1, k_2] = K[x, y]$ by Corollary 5.14.

7. An approach to the Jacobian Conjecture.

By our proof given above, it suffces to prove that the assumption $\partial(f, g)/\partial(x, y) = 1$ implies that the convex polygon N_f of f is a triangle having vertices $(a, 0)$, $(0, b)$, $(0, 0)$ with nonnegative integers a, b.

Therefore we assume that there are polynomials f, g such that

$\partial(f,g)/\partial(x,y) = 1$ and the convex polygon N_f of f is not a triangle. Then there is a direction (p,q) such that $pq > 0$ and the leading (p,q)-form f' of f is such that $\overline{\sup}\ f'$ has an end (a,b) with $ab \neq 0$. By Corollary 5.15, we may assume that the other end of $\overline{\sup}\ f'$ is $(a',0)$ or (a,b). We apply Theorem 5.13 to f' and we obtain h and $k = k_1 k_2$. Since f' is divisible by x, we have $k_1 = x$, $k_2 = y + cx^{p/q}$ in $(a',0)$ case. If we express f in terms of k_1, k_2, then the new polygon has (a,b) as a vertex, and the corresponding new edge has less value of a' (as far as $q > 0$). Considering the edge of the other direction, too, we see that:

Keeping the vertex (a,b) to be a vertex, we can modify variables x, y so that the convex polygon N_f is contained in the tetragon having vertices $(0,0)$, $(a,0)$, (a,b), $(0,b)$.

By the symmetry of x, y, we may assume that $a \geq b$. Considering the $(1,1)$-leading form of f and applying Theorem 5.13, we see that $a \neq b$. Therefore $a > b$. Then b is not a multiple of a, and we can apply Lemma 5.16 to see that the situation is reduced to the case where (a,b) is the unique point of N_f lying on the line $x = a$.

Then we apply the remark to Lemma 5.18, and we see that there is no edge of N_f corresponding to a direction (p,q) such that $q < 0$, $p+q > 0$, because $a > b$. Thus we reduced to the case N_f is contained in the tetragon having vertices $(0,0)$, $(a-b,0)$, (a,b), $(0,b)$.

If a is not a multiple of b, then we can do the same to the edge of the other side, and we can reduce to the case that the convex polygon N_f is contained in a tetragon having vertices $(0,0)$, $(a-b,0)$, (a,b), $(0,c)$, where c is a positive rational number less than b.

If a is a multiple of b, then we can consider the leading $(0,1)$-form f'', which is of the form $y^b F(x)$. Then, taking a c in K such that c is a root of $F(x)$ with multiplicity a' bigger than b; the existence is seen by applying Lemma 5.16 to $(0,1)$-direction. Then we write f in terms of $x-c$, y. Then (a',b) is a vertex of N_f.

Thus, our conclusion is the following:

THEOREM 7.1. *If there is an f as above, then by a suitable choice of x', y' such that $K[x',y'] = K[x,y]$, the convex polygon of f in*

terms of x', y' is as follows:

It has one vertex (a, b) such that $a > b > 0$ and $a + b =$ (degree of f in terms of x', y') and the polygon N_f is contained in the tetragon having vertices $(0,0)$, $(a-b, 0)$, (a, b), $(0, b)$. Furthermore, $(0, b)$ is not a point of N_f.

If a is not a multiple of b, then (a, b) is the unique point of the polygon lying on the line $y = b$.

If a is a multiple of b, then N_f may have another vertex (a', b) $(a > a' > b)$ on the line $y = b$.

Aiming to apply this result to some special cases considered by Appelgate-Onishi [**AO**], we add a corollary to this. Namely, consider f, g such that $\partial(f, g)/\partial(x, y) = 1$ such that both $\deg f$ and $\deg g$ are bigger than 1. Then Theorem 3.1 shows that the convex polygons N_f, N_g are similar, and we see that there is a convex polygon W such that $N_f = mW$, $N_g = nW$ with natural numbers m, n ($m : n = \deg f : \deg g$ and m, n are coprime). Let us call this W the basic polygon for f, g.

COROLLARY 7.2. *Under the circumstances, the vertex (a, b) in Theorem 7.1 satisfies the condition that, setting $a = ma_1$, $b = mb_1$, we have a_1, b_1 are natural numbers which are not coprime.*

PROOF: Assume that a_1, b_1 are coprime. First, consider the case $b_1 = 1$. Since $(0, b)$ is not on N_f, W has no vertex on y-axis except for $(0,0)$, which contradicts Proposition 3.6. Assume now that $b_1 \neq 1$. Then a is not a multiple of b. Take the edge of N_f having an end at (a, b) and extending towards y-axis. Let (p, q) be the corresponding direction to this edge $(q > 0)$. By Theorem 7.1, p is negative. W has an edge parallel to the edge and having (a_1, b_1) as an end. Since W has a vertex lying on y-axis besides $(0,0)$, $a_1 p + b_1 q \geq q$. Then, Lemma 5.18 shows us a contradiction.

THEOREM 7.3. (APPELGATE-ONISHI [**AO**]) *Let f, g be polynomials such that $\partial(f, g)/\partial(x, y) = 1$.*

(1) *If GCM of $\deg f$, $\deg g$ is a prime number, say p^*, then $K[f, g] = K[x, y]$.*

(2) *If $\deg f$ is the product of two prime numbers p^*, q^* (p^* may be equal to q^*), then $K[f, g] = K[x, y]$.*

PROOF: (2) follows from (1) easily by induction on $\deg g$. Namely, GCM of $\deg f$, $\deg g$ is one of 1, a prime factor of $\deg f$, $\deg f$. The first case follows from Magnus theorem and the second case follows from (1). As for the third case, since f, g are $(1,1)$-related, there are $c\ (\in K)$ and r (natural number) such that $g - cf^r$ has degree less than $\deg g$. Thus the proof is completed by induction on $\deg g$. Therefore, it remains to prove (1). First, consider the leading $(1,1)$-form f' of f. Let (a,b), (a',b') be two ends of $\overline{\sup}\ f'$.

(i) The case where $b = a' = 0$: The convex polygon N_f of f is then a triangle. Take $(1,1)$ forms h, k given by Theorem 5.13, with respect to f'. $k = k_1 k_2$ with linear forms k_1, k_2. If $h = k_1^i k_2^j$ with $ij \neq 0$, then in terms of k_1, k_2, we reduce to the case $a' = a \neq 0$, $b' = b \neq 0$. In the other case, we can reduce to the case $a' = a = \deg f$, $b' = b = 0$.

(ii) The case where $a' = a \neq 0$, $b' = b \neq 0$: $a = ma_1$, $b = mb_1$ with $a_1 + b_1 = p^*$, where $m = (\deg f)/p^*$. Then a_1, b_1 are coprime. Applying the proof of Theorem 7.1 to this case, we have a contradiction by Corollary 7.2.

(iii) The case where $\overline{\sup}\ f$ is not a point, but not the case (i): By Corollary 5.15, either $b = 0$ or $a' = 0$. By the symmetry on x, y, we may assume that $b = 0$, $a' \neq 0$. Take $(1,1)$-forms h, k given by Theorem 5.13, with respect to f'. Then we can reduce to the case (ii).

(iv) The remaining case: We may assume that $a' = a = \deg f$, $b' = b = 0$. Let (a^*, b^*) be such that $b^* > 0$ and such that the line segment combining (a^*, b^*) and $(a, 0)$ is an edge of N_f. We can write $a^* = ma_2$, $b^* = mb_2$, $a = ma_1$. Consider first the case where $a^* \neq 0$. Let (p,q) be the direction corresponding to this edge $(p : q = b^* : (a - a^*) = b_2 : (a_1 - a_2))$. Consider the leading (p,q)-form f'' of f, and we take h, k given by Theorem 5.13. Then we can reduce to the case where (a^*, b^*) is a vertex of the new polygon of f. $a_2 + qb_2 = p^*$ because $a = \deg f = mp^*$. Therefore a_2, b_2 are coprime. Therefore, applying the proof of Theorem 7.1, we obtain a contradiction. Thus $a^* = 0$, and N_f is a triangle. b^* is a multiple of m. By Theorem 5.13 and its proof, we see that the corresponding direction (p,q) to the edge is such that $p = 1$. Therefore b^* is a factor of $a = mp^*$. Therefore $b^* = m$. Now, we apply Theorem 5.13 to reduce $\deg f$, keeping $(0, b^*)$ to be a vertex. Since every coordinate of every vertex of N_f must be

a multiple of m, any of other vertices must have 0 as its y-coordinate. Therefore, N_f is always a triangle, and we obtain the case where $\deg f = m$. Then $\deg g$ is prime to $\deg f$, and we have the assertion by Magnus theorem.

As another application of our observation, we prove:

Theorem 7.4. *Let f, g be polynomials such that $\partial(f,g)/\partial(x,y) = 1$ and let D be the G.C.M. of* $\deg f$, $\deg g$.

(1) *If none of* $\deg f$, $\deg g$ *divides the other then* $D \geq 9$.

(2) *If $D \leq 8$, then $K[f,g] = K[x,y]$ and one of* $\deg f$, $\deg g$ *divides the other.*

Proof. Since (2) follows from (1), it suffices to prove (1). Let m, n be as in Theorem 7.1. In the reduction process of the proof of Theorem 7.1, $m : n$ does not change and $\deg f = mD$ does not increase. Therefore we may assume that the polygon N_f of f is as in the conclusion of Theorem 7.1.

Let W be the basic polygon for f, g. By Theorem 7.1, W has a vertex (a,b) (not the same as (a,b) in Theorem 7.1) such that $a > b$ and such that there is an edge E of W such that (1) the corresponding direction (p,q) is such that $p < 0$ and (2) (a,b) is an end of E. Since $a > b$, we have $p+q > 0$ by Proposition 3.6. Now we apply Lemma 5.18 to this (a,b). Since E is not a point, m in Lemma 5.18 is not 0. Thus we adapt the symbols used in Lemma 5.18.

(I) $a_1 e < a$.

Indeed, the edge E is parallel to the line going through $(1,1)$ and $(a_1 e, b_1 e)$. If $a_1 e \geq a$, then the line containing E meets with y-axis at a point $(0,y)$ with $y < 1$ and W cannot have a proper vertex on y-axis.

(II) a,b are not coprime.

This follows from (I).

(III) $b \geq 3$.

By (II), we have $b > 1$. Assume now that $b = 2$. Then, by (I), we have $b_1 = e = 1$. Then $1-np = b_1 e = 1$, and $p = 0$, which is not the case.

Since (a,b) is a point of W, we have $a+b \leqq D$. If $b \geqq 4$, then $a > b$ implies $a+b \geqq 9$. Assume that $b = 3$. Then by (II), a is a multiple of 3. Since $a > b$, we have $a \geqq 6$, and $a+b \geqq 9$. Thus $D \geqq 9$.

REFERENCES

[A] S. S. Abhyankar, "Expansion techniques in algebraic geometry," Tata Inst. Fund. Research, Bombay, 1977.

[AM] S. S. Abhyankar & T. T. Moh, *Embeddings of a line in the plane*, J. Reine Angew. Math. **276** (1975), 149–166.

[AO] H. Appelgate & H. Onishi, *The Jacobian Conjecture in two variables*, J. Pure Appl. Alg. **37** (1985), 215–227.

[BCW] H. Bass, E. H. Connell & D. Wright, *The Jacobian Conjecture: Reduction of degree and formal expansion of inverse*, Bull. A.M.S., New Ser. **7** (1982), 287–330.

[K] O. H. Keller, *Ganze Cremona-Transformationen*, Monats. Math. Physik **47** (1939), 299–306.

[Ma] A. Magnus, *On polynomial solutions of a differential equation*, Math. Scan. **3** (1955), 255–260.

[Mo] T. T. Moh, *On the Jacobian Conjecture and the configuration of roots*, J. Reine Angew. Math. **340** (1983), 140–212.

[NB] Y. Nakai & K. Baba, *A generalization of Magnus' theorem*, Osaka J. Math. **14** (1977), 403–409.

[NNg] A. Nowicki & M. Nagata, *Rings of constants for k-derivations in* $k[x_1, \ldots, x_n]$, J. Math. Kyoto Univ. **28** (1988), 111–118.

[NNk] A. Nowicki & Y. Nakai, *On Appelgate-Onishi's lemma*, J. Pure Appl. Alg. **50** (1988), 195–207.

[Nw] A. Nowicki, *On the Jacobian equation* $J(f,g) = 0$ *for polynomials in* $k[x,y]$, Nagoya Math. J. **100** (1988), 151–157.

[Z] A. Zaks, *Dedekind subrings of* $k[x_1, \ldots, x_n]$ *are rings of polynomials*, Israel J. Math. **9** (1971), 285–289.