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Resolutions of determinantal ideals

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0 Introduction

Let R be a noetherian commutative ring with unit, m, n and t be positive integers with $t \le m, n$, and S be a polynomial ring $R[x_{ij}]_{\substack{1 \le i \le m \\ 1 \le j \le n}}$ with mn variables. We call the ideal I_t of S generated by t-minors of the matrix (x_{ij}) a *determinantal ideal*. M. Hochster and J. A. Eagon proved that I_t is perfect (*i.e.*, $\mathrm{pd}_S S/I_t = \mathrm{grade}\, I_t$) of codimension $(m-t+1)(n-t+1)$ [12]. The quotient algebra S/I_t is Cohen-Macaulay when R is Cohen-Macaulay. They also proved that S/I_t is a normal domain when R is a normal domain. So if $R = \mathbb{Z}$, the ring of rational integers, then S/I_t is $\mathbb{Z}$-flat, since it is torsion free. With letting each x_{ij} of degree one, S is a graded R-algebra and I_t is homogeneous.

Finding a (graded) minimal free resolution of S/I_t has long been a problem in commutative ring theory (we say that a finite free graded S-complex $\mathbf{F}$ is minimal when the boundary map of $\mathbf{F} \otimes_S K$ is zero, where K is the S-module S/I_1). If we get a minimal free resolution $\mathbf{F}$ of S/I_t when $R = \mathbb{Z}$, then $R \otimes_{\mathbb{Z}} \mathbf{F}$ is a minimal free resolution of $R \otimes_{\mathbb{Z}} S/I_t$. Such a resolution is constructed explicitly in the case $t = 1$ (the Koszul complex), $t = \min(m,n)$ (the Eagon-Northcott complex) [8], and $t = \min(m,n) - 1$ (the Akin-Buchsbaum-Weyman complex) [2].

A. Lascoux constructed the minimal free resolution of S/I_t (for any m, n and t) over a field of characteristic zero explicitly, using representations of general linear groups [20]. In his proof, the complete reducibility of polynomial representations of general linear groups and the Bott's theorem were important. But both of them are false when we consider the case characteristic p. After his result, Akin, Buchsbaum, and Weyman developed ***characteristic-free representation theory*** of general linear groups and constructed the A-B-W resolution.

In section 1, we review the results from characteristic-free representation theory. The main purpose of this section is to introduce Schur complexes. In section 2, we review the Lascoux's approach in the version of (partially) appliable to the case characteristic p. The most important tools are the Bott's theorem and the Kempf's vanishing theorem. We also review some ring-theoretical properties of S/I_t. In section 3, we review three important characteristic-free minimal free resolutions: Koszul complexes, Eagon-Northcott complexes, and Akin-Buchsbaum-Weyman complexes. Section 4 mainly consists of a summary of our original results on the syzygies of S/I_t. In section 5, we briefly mention on Pfaffian ideals and ideals generated by minors of symmetric matrices.

1 Schur complexes

Schur modules, Weyl modules, and Schur complexes are the most important objects in the characteristic-free representation theorey of general linear groups.

Let V be a free R-module of rank n. We denote the tensor (resp. symmetric, exterior) algebra by:

$$TV = \bigoplus_{i\geq 0} T_i V \qquad SV = \bigoplus_{i\geq 0} S_i V \qquad \bigwedge V = \bigoplus_{i\geq 0} \bigwedge^i V$$

Let $\varphi : W \longrightarrow V$ be a map of finite free R-modules. We define $S\varphi \stackrel{def}{=} SV \otimes \bigwedge W$. The algebra $S\varphi$ is graded with the total grading, and we denote the degree i component of $S\varphi$ by $S_i\varphi$. This algebra is a chain complex with the Koszul boundary ∂ given by

$$\partial(\alpha \otimes w_1 \wedge \cdots \wedge w_s) = \sum_{i=1}^{s} (-1)^{i-1} \alpha \cdot \varphi w_i \otimes w_1 \cdots \overset{i}{\check{}} \cdots w_s.$$

Since the boundary preserves degree, $S_i\varphi$ is a subcomplex of $S\varphi$ for any $i \geq 0$. It is well-known that $S\varphi$ is a differential algebra with this structure.

So the iterated multiplication

$$(\star) \qquad T_i\varphi = \varphi \otimes \cdots \otimes \varphi \xrightarrow{m} S_i\varphi$$

is a chain map (note that $\varphi : W \longrightarrow V$ is a chain complex of length one!). We denote the *graded dual* of $S\varphi^*$ by $\bigwedge \varphi$. Namely, we define $\bigwedge^i \varphi \stackrel{def}{=} (S_i\varphi^*)^*$, and $\bigwedge \varphi = \bigoplus_i \bigwedge^i \varphi$. By definition, $\bigwedge \varphi$ is a graded chain complex, and we have $\bigwedge^1 \varphi = (S_1\varphi^*)^* = (\varphi^*)^* = \varphi$. Taking the dual of $(\star)$ for the map φ^*, we obtain a chain map

$$\Delta : \textstyle\bigwedge_i \varphi \longrightarrow (\varphi^* \otimes \cdots \otimes \varphi^*)^* \cong T_i\varphi.$$

For a chain complex A and $n \geq 0$, the symmetric group $\mathfrak{S}_n$ acts on $T_nA = A^{\otimes n}$. The action of $\sigma \in \mathfrak{S}_n$ is given by

$$\sigma(a_1 \otimes \cdots \otimes a_n) = (-1)^{\sum_{i<j, \sigma i > \sigma j} \deg(a_i)\cdot\deg(a_j)} a_{\sigma^{-1}1} \otimes \cdots \otimes a_{\sigma^{-1}n}.$$

Let $\lambda = (\lambda_1, \ldots, \lambda_s)$ be a partition of n. The element $\sigma(\lambda) \in \mathfrak{S}_n$ is defined as follows. For example, if $\lambda = (5, 3, 2)$, then $\sigma(\lambda)$ is the unique permutation which maps the Young diagram Y to Y':

$$Y = \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & & \\ 9 & 10 & & & \end{matrix} \implies Y' = \begin{matrix} 1 & 4 & 7 & 9 & 10 \\ 2 & 5 & 8 & & \\ 3 & 6 & & & \end{matrix}$$

so that $\sigma(\lambda)(2) = 4$ and $\sigma(\lambda)(6) = 10$. We don't give the precise definition here. For a partition $\lambda = (\lambda_1, \ldots, \lambda_s)$, the *transpose* $\tilde{\lambda} = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_t)$ of λ is the partition given by $\tilde{\lambda}_i = \#\{j \mid \lambda_j \geq i\}$. For example, if $\lambda = (5, 3, 2)$ as above, then $\tilde{\lambda} = (3, 3, 2, 1, 1)$. The Schur map

$$d_\lambda : \textstyle\bigwedge_\lambda \varphi \stackrel{def}{=} \bigwedge^{\lambda_1} \varphi \otimes \cdots \bigwedge^{\lambda_s} \varphi \longrightarrow S_{\tilde{\lambda}}\varphi \stackrel{def}{=} S_{\tilde{\lambda}_1}\varphi \otimes S_{\tilde{\lambda}_t}\varphi$$

is defined to be the composite map

$$\textstyle\bigwedge^{\lambda_1} \varphi \otimes \cdots \bigwedge^{\lambda_s} \varphi \xrightarrow{\Delta \otimes \cdots \otimes \Delta} T_{\lambda_1}\varphi \otimes \cdots \otimes T_{\lambda_s}\varphi = T_n\varphi \xrightarrow{\sigma(\lambda)}$$
$$T_n\varphi = T_{\tilde{\lambda}_1}\varphi \otimes \cdots T_{\tilde{\lambda}_t}\varphi \xrightarrow{m \otimes \cdots \otimes m} S_{\tilde{\lambda}_1}\varphi \otimes \cdots \otimes S_{\tilde{\lambda}_t}\varphi.$$

We define the *Schur complex* of φ with respect to the partition λ to be the image of d_λ. This definition contains two important definitions on modules.

If $W = 0$, then $L_\lambda\varphi$ depends only on the module V, and concentrated in degree zero. We define the *Schur module* $L_\lambda V$ of V with respect to the partition λ to be the degree zero component of $L_\lambda(0 \to V)$. If $V = 0$, then $L_\lambda\varphi$ is concentrated in degree n. We define the *Weyl module* $K_\lambda V$ of W with respect to the partition λ to be the degree n component of $L_\lambda(W \to 0)$. If R is a field of characteristic zero, and if $\tilde{\lambda}_1 \leq \operatorname{rank} W$, then $K_\lambda V$ is an irreducible polynomial representation of $\mathrm{GL}(W)$. If R contains the rationals, then $K_{\tilde{\lambda}} W \cong L_\lambda W$ as a $\mathrm{GL}(W)$-module, but not in general.

Theorem 1.1 (Akin-Buchsbaum-Weyman [3]) *The complex $L_\lambda\varphi$ is a finite free complex (i.e., a complex of finite length with its each term finite free). If S is an R-algebra, then there is a natural isomorphism $L_\lambda(S \otimes_R \varphi) \cong S \otimes_R L_\lambda\varphi$.*

These notion can be naturally extended to the case base scheme is not affine. Namely, instead of using the base ring R, we fix a base scheme X and consider a map of vector bundles $\varphi : W \longrightarrow V$. Then we can define the Scure complex $L_\lambda\varphi$ of φ, and it is a finite complex of vector bundles. By the theorem, L_λ is compatible with taking the inverse image with respect to a morphism of schemes. The following results are fundamental in the characteristic-free representation theory.

Theorem 1.2 (Akin-Buchsbaum-Weyman [3]) *Let X be a scheme and $0 \longrightarrow W \xrightarrow{\psi} V \xrightarrow{\varphi} U \longrightarrow 0$ be an exact sequence of vector bundles on X. Then there is a quasi-isomorphism $K_\lambda W \longrightarrow L_\lambda\varphi$, and there is a quasi-isomorphism $L_\lambda\psi \longrightarrow L_\lambda U$ for any partition λ.*

Theorem 1.3 (Akin-Buchsbaum-Weyman [3], Boffi [4]) *Let X be a scheme, $\varphi : W \longrightarrow V$ be a map of vector bundles on X, and λ be a partition. Then for $k \geq 0$, the degree k component of $L_\lambda\varphi$ admits a filtration of bundles whose associated graded object is $\bigoplus_{\mu,\nu} c^\lambda_{\mu,\nu} L_\mu V \otimes K_\nu W$, where μ and ν runs all partitions and $c^\lambda_{\mu,\nu}$ is the Littlewood-Richardson coefficient.*

Theorem 1.4 ([11]) *Let X be a scheme, $\varphi : W \longrightarrow V$ be a map of vector bundles, and E a vector bundle on X. Then, for any $k \geq 0$, the symmetric power $S(\varphi \otimes E)$ admits a filtration of complexes of bundles whose associated graded object is $\bigoplus_\lambda L_\lambda\varphi \otimes L_\lambda E$, where λ runs over all partitions of k.*

Note that L_λ is a functor from the category of maps of bundles to the category of complexes of bundles. Namely, if $\alpha : \varphi \longrightarrow \psi$ is a chain map (regarding the maps φ and ψ as complexes of length one), there is a functorial map $L_\lambda\alpha : L_\lambda\varphi \longrightarrow L_\lambda\psi$. If λ is a single-rowed partition (n), then we have $L_\lambda = \bigwedge^n$ by definition. If $\lambda = (1,1,\ldots,1)$ (n times 1), then we have $L_\lambda = S_n$.

2 Geometric background

Though our approach is (purely) algebraic (or combinatorial), there is some indispencable geometric background about this topic mainly due to A. Lascoux [20]. In this section the base ring $R = K$ is a field. Let V and W be vector spaces over K of dimension m and n respectively. The symmetric algebra $S = S(V \otimes W)$ can be identified with the polynomial ring $K[x_{ij}]$. We set $X = \operatorname{Spec} S$. The generic matrix (x_{ij}) corresponds to the universal map of bundles $u : V \longrightarrow W^*$ on $X = \operatorname{Hom}(V, W^*)$. Thus, the *determinantal variety* $Y = \operatorname{Spec} S/I_t$ is the zero of the map $\bigwedge^t u : \bigwedge^t V \longrightarrow \bigwedge^t W^*$, where I_t is the determinantal ideal generated by $t \times t$ minors of (x_{ij}). We denote the grassmanian of $(t-1)$-quotients of V by G and we set $\tilde{G} \stackrel{def}{=} X \times_K G$. We denote the universal $(t-1)$-quotient bundle and the universal $(m-t+1)$-subbundle on G by Q and R, respectively. There is a tautological exact sequence

$$0 \longrightarrow R \xrightarrow{i} V \xrightarrow{p} Q \longrightarrow 0$$

on G. The composite map $R \xrightarrow{i} V \xrightarrow{u} W^*$ on $\tilde{G}$ determines a cosection $s : R \otimes W \longrightarrow \mathcal{O}_{\tilde{G}}$. We denote the zero of s by Z. Since u is a generic map, it is easy to see that Z is non-singular variety, and the Koszul complex

$$\cdots \longrightarrow \textstyle\bigwedge^i(R \otimes W) \longrightarrow \bigwedge^{i-1}(R \otimes W) \longrightarrow \cdots \longrightarrow R \otimes W \xrightarrow{s} \mathcal{O}_{\tilde{G}} \to \mathcal{O}_Z \to 0$$

is a resolution of $\mathcal{O}_Z$. Geometrically, Z consists of the points $(\psi, L) \in X \times G = \tilde{G}$ such that $\psi : V \longrightarrow W^*$ factors through the quotient L of V. The rank of such a map ψ is at most $(t-1)$, so the projection map $\pi : \tilde{G} \longrightarrow X$ induces a map $\overline{\pi} : Z \longrightarrow Y$. If $\operatorname{rank} \psi = t-1$, then $(\psi, L) \in Z$ if and only if $L = \operatorname{Im} \psi$. So it is easy to see that $\overline{\pi}$ is a birational isomorphism. Since the codimension of Z in $\tilde{G}$ is $(m-t+1) \cdot n$, we have $\dim Z = \dim Y = mn - (m-t+1)(n-t+1)$. So we can recover (the special case of) the results on normality of Hochster and Eagon mentioned in section 0, as follows. Clearly, Y is a variety of dimension $mn-(m-t+1)(n-t+1)$. It is not so difficult to show that the singular locus of Y is defined by I_{t-1}. Since the codimension of $\operatorname{Spec} S/I_{t-1}$ in Y is $m+n-2t+3 \geq 2$ (unless $t=1$!), we have Y is normal.

We shall consider a double complex (a complex of $\mathcal{O}_{\tilde{G}}$-complexes)

$$B^{\cdot}: 0 \longrightarrow S_h(p\otimes W) \xrightarrow{d} \cdots \xrightarrow{d} S_1(p\otimes W) \longrightarrow \mathcal{O}_{\tilde{G}} \to 0$$

where $h=(m-t+1)\cdot n$, and the boundary $d: S(p\otimes W) = S(Q\otimes W)\otimes \bigwedge(V\otimes W) \longrightarrow S(p\otimes W)$ is $\mathrm{id}_{S(Q\otimes W)}\otimes \partial'$, where ∂' is the Koszul boundary derived from the cosection $u^{\#}: V\otimes W \longrightarrow \mathcal{O}_{\tilde{G}}$ obtained by u. By the theorem 1.2, $S_i(p\otimes W)$ is a cohomological resolution of $\bigwedge^i(R\otimes W)$. In fact, $\bigwedge^{\cdot}(R\otimes W) \longrightarrow B^{\cdot}$ is a resolution (of $\mathcal{O}_{\tilde{G}}$-complexes) (see for example, [11, chapter I]).

Proposition 2.1 *With the notation as above, each term of $B^{\cdot}$ is Γ-acyclic resolution of $\mathcal{O}_Z$. The complex $\pi_* B^{\cdot}$ is a finite free complex.*

The proof depends on the following theorem and the results on the characteristic-free representation theorey stated in section 1.

Theorem 2.2 (Kempf, [14]) *Let $\pi: \tilde{G} \longrightarrow X$ and Q be as above. For a partition λ and $i \geq 0$, we have*

$$R^i\pi_* L_\lambda Q \cong \begin{cases} L_\lambda V & (\text{if } \lambda_1 < t \text{ and } i=0) \\ 0 & (\text{otherwise}) \end{cases}.$$

In the first case, $\pi_* L_\lambda p : \pi_* L_\lambda \pi^* V = L_\lambda V \longrightarrow \pi_* L_\lambda Q = L_\lambda V$ is the identity map.

Since (cohomologically graded) all degree positive parts of $B^{\cdot}$ are zero, we have $R^i\pi_*\mathcal{O}_Z = 0$. On the other hand, since $\pi|_Z : Z \longrightarrow Y$ is proper birational and Y normal, we have $\pi_*\mathcal{O}_Z = \mathcal{O}_Y$. Now we can recover the results on Cohen-Macaulay property of Y due to Hochster and Eagon, using the results of Kempf [15]. Another consequence of the proposition is that $\pi_* B^{\cdot}$ is a free resolution of $\mathcal{O}_Y$.

Lemma 2.3 *Let* $\mathbf{P} = \cdots \longrightarrow P_i \longrightarrow \cdots \longrightarrow P_0 = S \longrightarrow S/I_t \to 0$ *be the graded minimal free resolution of* S/I_t. *Then we have isomorphisms*

$$P_i \cong S \otimes \underline{\mathrm{Tor}}_i^S(K, S/I_t) \cong \bigoplus_{k-l=i} H^l(\tilde{G}, \wedge^k(R \otimes W))$$

as S*-modules. If the characteristic of the base field* K *is* 0, $\mathbf{P}$ *admits a unique* $\mathrm{GL}(V) \times \mathrm{GL}(W)$*-equivariant structure, and these isomorphisms are* $\mathrm{GL}(V) \times \mathrm{GL}(W)$*-equivariant.*

Proof. The first isomorphism is clear. By the preceeding remark, we have $\underline{\mathrm{Tor}}_i^S(K, S/I_t) \cong H_i(K \otimes_S \pi_* B^{\cdot})$. Since the boundary $d : S_i(p \otimes W) \longrightarrow S_{i-1}(p \otimes W)$ vanishes after tensored with K, the homology $H_i(K \otimes_S \pi_* B^{\cdot})$ is the direct sum $\bigoplus_{k-l=i} H^l(\pi'_* S_k(p \otimes W))$, where by $S_k(p \otimes W)$ we mean the complex on G, and $\pi' : G \longrightarrow \mathrm{Spec}\, K$ is the projection. Hence, we have

$$\begin{aligned} S \otimes H_i(K \otimes_S \pi_* B^{\cdot}) &\cong \bigoplus_{k-l=i} S \otimes H^l(\pi'_* S_k(p \otimes W)) \\ &\cong \bigoplus H^l(S \otimes \pi'_* S_k(p \otimes W)) \cong \bigoplus H^l(\pi_* S_k(p \otimes W)). \end{aligned}$$

So the first assertion follows from the fact $S_k(p \otimes W)$ is an acyclic resolution of $\wedge^i(R \otimes W)$. The last assertion is a consequence of the complete reducibility of the polynomial representations of $\mathrm{GL}(V) \times \mathrm{GL}(W)$. **Q.E.D.**

By Theorem 1.4, the exact sequence $0 \to \wedge^k(R \otimes W) \longrightarrow S_k(p \otimes W) \to 0$ admits a filtration whose associated object is

$$0 \to \bigoplus_{\lambda} K_\lambda R \otimes L_\lambda W \longrightarrow \bigoplus_{\lambda} L_\lambda p \otimes L_\lambda W \to 0,$$

where λ runs over all partitions of k (if the characteristic of K is zero, then $\wedge^k(R \otimes W)$ decomposes into the direct sum $\bigoplus_\lambda K_\lambda R \otimes L_\lambda W$)). Hence, there is a spectral sequence whose E_1-terms are of the form $H^l(\tilde{G}, K_\lambda R) \otimes L_\lambda W$, and converges to $H^l(\tilde{G}, \wedge^k(R \otimes W))$.

If the characteristic K is zero, then we have

$$H^l(\tilde{G}, \wedge^k(R \otimes W)) \cong \bigoplus_{\lambda} H^l(\tilde{G}, K_\lambda R) \otimes L_\lambda W.$$

The cohomology in the right hand side is calculated by the Bott's theorem. To state the Bott's theorem (in the special form for our need), we need some preparation from combinatorics. For a partition $\lambda = (\lambda_1, \ldots, \lambda_s)$, we define $\mathrm{ds}(\lambda) \stackrel{def}{=} \#\{i \mid \lambda_i \geq i\}$ and call it the *Durfee square number* of λ. For a positive integer t, we define $\lambda(t) \stackrel{def}{=} (\lambda_1+t-1, \ldots, \lambda_d+t-1, \lambda_{d+1}, \ldots, \lambda_s)$, where $d = \mathrm{ds}(\lambda)$. For example, if $\lambda = (5,4,2)$ and $t = 3$, then $\mathrm{ds}(\lambda) = 2$ and $\lambda(t) = (7,6,2)$.

Theorem 2.4 (Bott) *If the characteristic of the base field K is zero, then we have*

$$H^l(\tilde{G}, K_\lambda R) = \begin{cases} L_{\tilde{\mu}(t)}V & (\text{if } l = \mathrm{ds}(\lambda)\cdot(t-1),\ \lambda = \mu(t) \text{ for some } \mu) \\ 0 & (\text{otherwise}) \end{cases}$$

Using this theorem, we have:

Theorem 2.5 (Lascoux, [20]) *If the characteristic of the base field K is zero, then the i^{th} term of the minimal free resolution of S/I_t is isomorphic to $\bigoplus_\mu L_{\tilde{\mu}(t)}V \otimes L_{\mu(t)}W$, where the sum is taken over all partitions of i.*

If μ is a partition of $(m-t+1)(n-t+1)$, then both $L_{\tilde{\mu}(t)}V$ and $L_{\mu(t)}W$ are non-zero if and only if $\mu = (n-t+1, \ldots, n-t+1)$ ($(m-t+1)$-times $n-t+1$) which we denote $(n-t+1^{m-t+1})$. For this μ, we have $d \stackrel{def}{=} \mathrm{ds}(\mu) = \min(m,n)$, $\tilde{\mu}(t) = (m^d, (m-t+1)^{n-d})$, and $\mu(t) = (n^d, (n-t+1)^{m-d})$. Hence, we have S/I_t is of type one if and only if $t = 1$ or $m = n$. This result is first proved by T. Svanes [23]. Note that this result remains true for ground field of positive characteristic, since Gorenstein property of graded Cohen-Macaulay domain depends only on its Poincaré series (see [22]), and it does not depend on characteristic for S/I_t.

3 E-N and A-B-W resolution

In this section, we shall try to make a brief sketch of three important characteristic-free minimal free resolutions of S/I_t for special values of t, m, and n. Though R is an arbitrary noetherian commutative ring, we shall use the notation in section 2. By abuse of notation, we denote $S \otimes V$ and $S \otimes W$ simply by V and W, respectively. All tensor products are over S unless specified otherwise. Without loss of generality, we may assume that $m \leq n$. The first one is, the Koszul complex for $t = 1$. Namely,

$Su^{\#} \longrightarrow S/I_1 = K \to 0$ is the minimal free resolution of S/I_1, where $u^{\#} : V \otimes W \longrightarrow S$ is a natural map derived from the generic map u.

For $i \geq 0$, there is a GL(W)-isomorphism $\omega_i : \bigwedge^i W^* \otimes \bigwedge^n W \longrightarrow \bigwedge^{n-i} W$ given by

$$\omega_i(y_{\sigma 1} \wedge \cdots \wedge y_{\sigma i} \otimes x_1 \wedge \cdots \wedge x_n) = (-1)^\sigma x_{\sigma(i+1)} \wedge \cdots \wedge x_{\sigma n}$$

for $\sigma \in \mathfrak{S}_n$, where $x_1, \ldots, x_n$ is a basis of W, $y_1, \ldots, y_n$ is the dual basis. Since (minor) determinants are alternative about rows and columns, there is a well-defined map $\phi_i : \bigwedge^i V \otimes \bigwedge^i W \longrightarrow S$ given by

$$\phi_i(v_1 \wedge \cdots \wedge v_i \otimes w_1 \wedge \cdots w_i) = det(v_\alpha \otimes w_\beta).$$

It is easy to see that the image of ϕ_t is I_t.

The second one is for the case $t = \min(m, n) = m$, the Eagon-Northcott complex. We shall describe this complex in the version re-constructed by Buchsbaum-Eisenbud [7]. See also [8] and [5]. The key lemma is:

Lemma 3.1 *Let A be a noetherian ring, M be a finitely generated A-module, and*

$$\mathbf{E} : 0 \longrightarrow E_k \longrightarrow \cdots \longrightarrow E_1 \longrightarrow E_0$$

be a finite free complex of A-modules. If for every prime ideal $\mathfrak{p} \subset A$ with depth$(\mathfrak{p}, M) < k$ *the localized complex $(M \otimes_A \mathbf{E})_\mathfrak{p}$ is exact, then $M \otimes_A \mathbf{E}$ is exact.*

For the proof of this lemma, we refer the reader to [6]

Theorem 3.2 (Eagon-Northcott complex) *If $t = m$, then the complex*

$$\mathbf{E} : \bigwedge^m V \otimes \bigwedge^{n-m} u \otimes \bigwedge^n W \xrightarrow{f} S \longrightarrow S/I_m \to 0$$

is the minimal free resolution of S/I_m, where $u : V \longrightarrow W^*$ is the generic map, and

$$f : E_1 = \bigwedge^m V \otimes \left[\bigwedge^{n-m} u\right]_0 \otimes \bigwedge^n W = \bigwedge^m V \otimes \bigwedge^{n-m} W^* \otimes \bigwedge^n W \longrightarrow E_0 = S = S(V \otimes W)$$

is the composite map $\phi_m \circ \mathrm{id} \otimes \omega_{n-m}$.

Proof. First note that

$$E_1 \xrightarrow{f} E_0 \longrightarrow S/I_m \to 0$$

is exact by the preceeding remark. We fix the bases $x_1, \ldots, x_n$ and $\xi_1, \ldots, x_m$ of W and V respectively, and denote the dual bases $y_1, \ldots, y_n$ of W^*. The composite map $E_2 \xrightarrow{d_2} E_1 \xrightarrow{f} E_0$ is zero. In fact, for $1 \leq i \leq m$ and $\sigma \in \mathfrak{S}_n$, we have

$$\begin{aligned} & f \circ d_2(\xi_1 \wedge \cdots \wedge \xi_m \otimes \xi_i \otimes y_{\sigma 1} \wedge \cdots \wedge y_{\sigma(n-m-1)} \otimes x_1 \otimes \cdots x_n) \\ = & \textstyle\sum_{j=1}^{n} (\xi_i \otimes x_j) f(\xi_1 \wedge \cdots \wedge \xi_m \otimes y_j \wedge y_{\sigma 1} \wedge \cdots \wedge y_{\sigma(n-m-1)} \otimes x_1 \otimes \cdots x_n) \\ = & \textstyle\sum_{k=1}^{m+1} (-1)^{k+1} (\xi_i \otimes x_{\sigma(n-m-1+k)}) \cdot det(\xi_\alpha \otimes x_{\sigma(n-m-1+\beta)})_{\beta \neq k} = 0. \end{aligned}$$

Hence, $\mathbf{E}$ is certainly a complex. The minimality is clear by definition. So it suffices to show the acyclicity of $\bigwedge^{n-m} u$. By the lemma above, we may localize at a prime $\mathfrak{p}$ with $\operatorname{depth}(\mathfrak{p}, S) \leq n-m-1$. Since $\dim S/I_t = mn - (n-m+1)$ and S/I_t is Cohen-Macaulay, we have $\operatorname{depth}(I_t, S) = n-m+1$. Hence, I_t is not contained in $\mathfrak{p}$, and $u_\mathfrak{p}$ is a split injection. By Theorem 1.2, $\left[\bigwedge^{n-m} u\right]_\mathfrak{p} \cong \bigwedge^{n-m} u_\mathfrak{p}$ is acyclic as desired. **Q.E.D.**

Remark 3.3 *The complex $\bigwedge^{n-m} u$ is of the form*

$$\begin{aligned} 0 \to D_{n-m}V \longrightarrow D_{n-m-1} \otimes W^* \longrightarrow \cdots \longrightarrow D_i V \otimes \textstyle\bigwedge^{n-m-i} W^* \longrightarrow \\ D_{i-1} V \otimes \textstyle\bigwedge^{n-m-i+1} W^* \cdots \longrightarrow V \otimes \textstyle\bigwedge^{n-m-1} W^* \longrightarrow \textstyle\bigwedge^{n-m} W^* \to 0 \end{aligned}$$

where $D_i V \stackrel{def}{=} (S_i V^*)^* = K_{(i)} V$ which is called the i^{th} divided power of V. Taking the dual of the multiplication map $T_i V^* \longrightarrow S_i V^*$, we obtain a inclusion $D_i V \longrightarrow T_i V$. In fact, $D_i V$ is the invariance $(T_i V)^{\mathfrak{S} i}$ under the natural action of $\mathfrak{S} i$ on $T_i V$. The construction above is extended to the case base scheme is non-affine.

The third one is for the case $m = t+1$, the Akin-Buchsbaum-Weyman complex [2]. The construction of this complex is too complicated to describe in this short survay. Note that the following argument without any proof is a consequence of this complicated construction, but not (so far) proved directly.

Consider the double complex $\pi_* B^{\cdot}_{\cdot}$ appeared in section 2 again. First taking the homology of $\pi_* S_k(p \otimes W)$, we have a spectral sequence whose

E_1-terms are of the form $H^l(\tilde{G}, \bigwedge^k(R \otimes W))$ which converges to S/I_t. As a result, all E^1-terms are S-free in this case. So we obtain finite free S-complexes

$$\mathbf{X} : \cdots \longrightarrow X_i = H^{t-1}(\tilde{G}, \textstyle\bigwedge^{i+t-1}(R \otimes W)) \xrightarrow{d_1} X_{i-1} = H^{t-1}(\tilde{G}, \textstyle\bigwedge^{i+t-2}(R \otimes W)) \longrightarrow \cdots$$

and

$$\mathbf{Y} : \cdots \longrightarrow Y_i = H^{2(t-1)}(\tilde{G}, \textstyle\bigwedge^{i+2(t-1)}(R \otimes W)) \xrightarrow{d_1} Y_{i-1} = H^{2(t-1)}(\tilde{G}, \textstyle\bigwedge^{i-1+2(t-1)}(R \otimes W)) \longrightarrow \cdots$$

whose homologies are E_2-terms of this spectral sequence. The non-vanishing E_1-terms are X_i's for $i \geq 1$, Y_i's for $i \geq 4$, and $S = H^0(\tilde{G}, \bigwedge^0(R \otimes W))$. The non-vanishing E_2-terms are $H_4(\mathbf{Y}) \cong I_{t+1}^2 \cong H_3(\mathbf{X})$, $H_1(\mathbf{X}) \cong I_t$, and S. Using the comparison theorem, we obtain a chain map $f : \mathbf{Y} \longrightarrow \mathbf{X}$ whose mapping cone $C(f)$ is a resolution of I_t.

Similarly to Eagon-Northcott complex, the minimal free resolution of I_{t+1}^2 is isomorphic to $L_{(m-n,m-n)}u$. Hence, $\mathbf{Y} \cong L_{(m-n,m-n)}u$. The complex $\mathbf{X}$ is constructed concretely, but f is not given explicitely in [2].

Remark 3.4 *Let X' be an algebraic K-scheme, V' and W' be vector bundles of rank m and n respectively on X', and $\varphi : V' \longrightarrow W'^*$ be a map of bundles. We denote the zero of $\bigwedge^t \varphi$ by Y'. The map φ determines a section $s(\varphi) : X' \longrightarrow \tilde{X} \stackrel{def}{=} \mathrm{Hom}(V', W'^*)$. Let $u' : V' \longrightarrow W'^*$ be the generic map on $\tilde{X}$. The section $s(\varphi)$ is determined by $s(\varphi)^*(u') = \varphi$. We denote the zero of $\bigwedge^t \varphi$ by $\tilde{Y}$. Let $\mathbf{P}$ be a resolution of $\mathcal{O}_{\tilde{Y}}$ of finite complex of locally free sheaves on $\tilde{X}$ (the construction of $\pi_* B^{\cdot}$ generalizes to this case and gives an example. If $t = \min(m,n)$, then the Eagon-Northcott complex is also an example). It holds that $s(\varphi)^*\mathbf{P}$ is a resolution of $\mathcal{O}_{Y'}$ if and only if $\mathrm{depth}(Y', X') = (m-t+1)(n-t+1)$ (the question is local, so we may assume that all schemes are affine, and V' and W' are free modules. In this case, $\mathrm{pd}_S S/I_t = (m-t+1)(n-t+1)$ $(S = \Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}}))$. Now use Lemma 3.1 for $M = s(\varphi)_* \mathcal{O}_{X'}$).*

4 Calculations of the Betti numbers

We have seen that there exists a minimal free resolution if $t = 1$ or $t \geq \min(m,n)-1$. We shall consider the remaining cases: $2 \geq t \geq \min(m,n)-2$. The question is, whether there is any minimal free rosolutions of S/I_t over any commutative ring R or not. By the general theory of resolutions, the answer is yes when R is a field, since S/I_t is graded. To consider the case R general, it suffices to consider the case $R = \mathbb{Z}$, the ring of integers, as mentioned in section 0. For this problem, the Betti numbers $\beta_i^p \stackrel{def}{=} \dim_K \mathrm{Tor}_i^S(K, S/I_t)$ plays important role, where p is a prime number or zero, $i \geq 0$, and $R = K$ is a base field of characteristic p. Since β_i^p is the rank of the i^{th} component of the minimal free resolution of S/I_t over K, β_i^p must not depends on p for any i if there exists a minimal free resolution $\mathbf{F}$ over $\mathbb{Z}$. The converse is true.

We define $\beta_{i,j}^p \stackrel{def}{=} \dim_K \left[\underline{\mathrm{Tor}}_i^S(K, S/I_t)\right]_j$ for $i, j \geq 0$, where $[]_j$ means the degree j component.

Proposition 4.1 (Roberts [21]) *The followings are equivalent.*

1 *There exists a graded minimal free resolution of S/I_t over any commutative ring R.*

2 *The numbers $\beta_{i,j}^p$ are independent of the characteristic p for any non-negative integers i and j.*

2′ *The Betti numbers of S/I_t is independent of the base field.*

3 *If the base ring R is $\mathbb{Z}$, then for any $i \geq 0$, $\mathrm{Tor}_i^S(S/I_1, S/I_t)$ is $\mathbb{Z}$-free.*

Clearly, $\beta_0^p = 1$ is independent of p. The first Betti number β_1^p is the number of minimal generators of I_t. It is the number of t-minors of the generic matrix (x_{ij}) and independent of p, since distinct t-minors doesn't have any common monomial with non-zero coefficient and are linearly independent over K. The second Betti number β_1^p is the number of minimal generators of the relation module of I_1.

Let M be the relation module of S/I_t, namely, the kernel of the natural map $\phi_t : \bigwedge^t V \otimes_S \bigwedge^t W \longrightarrow I_t$. We fix the bases $\xi_1 \ldots \xi_m$ of V and $x_1, \ldots, x_n$ of W. For sequence of integers $I = (i(1), \ldots, i(t+1))$ with $1 \leq i(1), \ldots, i(t+$

$1) \leq m$ and $J = (j(1), \ldots, j(t+1))$ with $1 \leq j(1), \ldots, j(t+1) \leq n$, we define:

$$\begin{aligned}\mathrm{rel}_1(I,J) &\stackrel{def}{=} \sum_{l=1}^{t+1}(-1)^{l+1}\xi_{i(1)}\otimes x_{j(l)}\cdot \xi_{i(2)}\wedge\cdots\wedge\xi_{i(t+1)}\otimes x_{j(1)}\wedge\cdots\overset{l}{\check{}}\cdots\wedge x_{j(t+1)} \\ &\quad -\sum_{l=1}^{t+1}(-1)^{l+t+1}\xi_{i(t+1)}\otimes x_{j(l)}\cdot \xi_{i(1)}\wedge\cdots\wedge\xi_{i(t)}\otimes x_{j(1)}\wedge\cdots\overset{l}{\check{}}\cdots\wedge x_{j(t+1)}\end{aligned}$$

and

$$\begin{aligned}\mathrm{rel}_2(I,J) &\stackrel{def}{=} \sum_{l=1}^{t+1}(-1)^{l+1}\xi_{i(1)}\otimes x_{j(l)}\cdot \xi_{i(2)}\wedge\cdots\wedge\xi_{i(t+1)}\otimes x_{j(1)}\wedge\cdots\overset{l}{\check{}}\cdots\wedge x_{j(t+1)} \\ &\quad -(-1)^t\sum_{l=1}^{t+1}(-1)^{l+t+1}\xi_{i(l)}\otimes x_{j(t+1)}\cdot \xi_{i(1)}\wedge\cdots\overset{l}{\check{}}\cdots\wedge\xi_{i(t+1)}\otimes x_{j(1)}\wedge\cdots\wedge x_{j(t)}.\end{aligned}$$

Since $\phi_t(\mathrm{rel}_1(I,J))$ (resp. $\phi_t(\mathrm{rel}_2(I,J))$) is the difference of the expantion of the same determinant $det(\xi_{i(\alpha)}\otimes x_{j(\beta)})$ with respect to the first row and the last row (resp. the first row and the last column), it is equal to zero. Hence, $\mathrm{rel}_1(I,J)$ and $\mathrm{rel}_2(I,J)$ are relations of I_t (i.e., elements of M).

Theorem 4.2 *The set*

$$\begin{aligned}B \stackrel{def}{=} &\{\mathrm{rel}_1(I,J) \mid i(1)<\cdots<i(t),\ i(1)\leq i(t+1),\ j(1)<\cdots<j(t+1)\} \\ &\cup\{\mathrm{rel}_2(I,J) \mid i(1)<\cdots<i(t+1),\ j(1)<\cdots<j(t),\ j(1)\leq j(t+1)\}\end{aligned}$$

minimally generates the relation module M of I_t. In particular, we have $\beta^p_{2,j} = 0$ for $j \neq t+1$ and $\beta^p_{2,t+1} = \#B$. So $\beta^p_2 = \#B$ does not depend on the characteristic p.

For the precise, we refer the reader to [17]. See also [9, Remark 2.2].

By the proof of Lemma 2.3, $\beta^p_{i,j}$ is calculated by $H^{j-i}(G, \wedge^j(R\otimes W)) = H_i(H^0(G, S_j(p\otimes W)))$. In fact, the complex $H^0(G, S_j(p\otimes W))$ is isomorphic to the degree j component of

$$S/I_t \otimes_S S(\mathrm{id}_{V\otimes_R W}) \cong S/I_t \otimes_S Su^{\#}.$$

So this information gives us nothing new. But the complex $S_j(p\otimes W)$ admits a filtration of Γ-acyclic complexes whose associated graded object is $\bigoplus_\lambda L_\lambda p \otimes L_\lambda W$, the sum is taken over partitions of j. Hence, we obtain some filtration $\{M^{t,\lambda}\}$ of $S/I_t \otimes_S S_j(\mathrm{id}_{V\otimes W})$ whose associated graded object is $\bigoplus_\lambda H^0(G, L_\lambda p) \otimes L_\lambda W$. This filtration is constructed explicitely in [10, section 2], using the characteristic-free representation theorey. We shall discuss more about the complex $H^0(G, L_\lambda p)$. Note that the homologies of $H^0(G, L_\lambda p)$ is cohomologies of $K_\lambda R$. So this is a characteristic p problem of the Bott's theorem. The Bott's theorem does not hold, as below. To describe this complex without grassmanian, we define *t-Schur complexes.*

Let X be a scheme, and $\varphi : F \longrightarrow E$ be a map of bundles on X. For $k \geq 0$ and $t \geq 0$, we denote the truncated complex

$$0 \to D_{k-t}F \otimes \bigwedge^t E \longrightarrow D_{k-t-1}F \otimes \bigwedge^{t+1} E \longrightarrow \cdots \longrightarrow \bigwedge^k E \to 0$$

of $\bigwedge^k \varphi$ by $\bigwedge^{k,t} \varphi$. For a partition λ, we define $L_{t,\lambda}\varphi$ to be the image of $\bigwedge^{\lambda_1,t} \varphi \otimes \bigwedge^{\lambda_2} \varphi \otimes \cdots$ by the Schur map $d_\lambda : \bigwedge_\lambda \varphi \longrightarrow L_\lambda \varphi$ and call it the t-Schur complex of φ with respect to the partition λ. This complex is a complex of vector bundles, and a subcomplex of $L_\lambda \varphi$. Using the Kempf vanishing theorem (theorem 2.2), it is not so difficult to show that $H^0(G, L_\lambda p)$ is isomorphic to $\overline{L}_{t,\lambda}\mathrm{id}_V \stackrel{def}{=} L_\lambda \mathrm{id}_V / L_{t,\lambda}\mathrm{id}_V$.

Passing through some combinatorial argument, we obtained the following result.

Proposition 4.3 *Let K be a field, $t \geq 1$, $i, j \geq 0$, V be a vector space over K with $\dim_K V - t \leq 2$, and λ be a partition of j. If $\dim_K H_i(\overline{L}_{t,\lambda}\mathrm{id}_V)$ is different from the value for $K = \mathbb{Q}$, then $\mathrm{char}(K) = 2$, $t \geq 2$, $\dim V - t = 2$, and $\lambda = (\lambda_1, 2)$. In this case, it holds $i = j - t$ or $i = j - t + 1$.*

So the E^1-terms of the spectral sequence associated with the filtration $\{M^{t,\lambda}\}$ of $S/I_t \otimes_S S\mathrm{id}_V$ depend on characteristic. But this dependence is resolved in the E^2-terms. Hence, we have:

Theorem 4.4 *If $t = m - 2$, then the Betti numbers β_i^p of S/I_t is independent of the characteristic. Hence, there exists a minimal free resolution of S/I_t over any commutative ring R.*

For the details, we refer the reader to [10, section 3]. This theorem is first proved for the case $n = m$, using the Gorenstein property [11]. We don't know any explicit form of the minimal free resolution, even in the case

$n = m$. On the other hand, the homologies of $\overline{L}_{t,\lambda}\mathrm{id}_V$ varies in quite caotic way, depending on the characteristic. This really effects on the syzygies of S/I_t, when $t \leq m-3$.

Theorem 4.5 *If $2 \leq t \leq m-3$, then the third Betti number $\beta_3^{\mathbf{F}_p}$ of S/I_t depends on characteristic. More precisely, $\beta_3^{\mathbf{F}_p} > \beta_3^{\mathbf{Q}}$ if (and only if) $p = 3$. So there is no minimal free resolution of S/I_t over $\mathbb{Z}$ in this case.*

For the precise, we refer the reader to [9].

5 Pfaffian ideals and minors of symmetric matrices

The problem of syzygies of Pfaffian ideals and the ideals generated by minors of symmetric matrices is closely related to the problem of syzygies of determinantal ideals.

Let $A = (a_{ij})$ be a alternative $2m \times 2m$ matrix over a commutative ring R. Namely, $a_{ij} = -a_{ji}$ for $1 \leq i, j \leq 2m$ and $a_{ii} = 0$ for $1 \leq i \leq 2m$ (the second condition is indispensable if 2 is a zero-divisor in R!). We define the *Pfaffian* of A, denoted by $pf(A)$, by

$$\frac{1}{m!\cdot 2^m}\left[\sum_{\sigma\in\mathfrak{S}_n}(-1)^\sigma a_{\sigma(1)\sigma(2)}\cdots a_{\sigma(2m-1)\sigma(2m)}\right] \in R.$$

Note that in the sum in the bracket, the same terms appear $(m!\cdot 2^m)$ times, and this definition is effective (but awkward) for any R. It holds that $det(A) = pf(A)^2$. On the other hand, $det(A) = 0$ for any alternative $n \times n$ matrix with odd n.

Now consider a noetherian commutative ring R and the polynomial ring $S = R[x_{ij}]_{1\leq i<j\leq n}$ for $n \geq 1$. For $1 \leq t \leq [n/2]$, we define the Pfaffian ideal Pf_{2t} by the ideal of S generated by $pf(x_{i(\alpha)i(\beta)})_{1\leq\alpha\leq 2t}$ for all indices $I = (i_1 < \cdots < i_{2t})$, where $x_{ji} \stackrel{def}{=} -x_{ij}$ for $i < j$ and $x_{ii} \stackrel{def}{=} 0$. Note that $Y = \operatorname{Spec} S/Pf_{2t}$ is the zero of $\bigwedge^{2t} u^{\#} : \bigwedge^{2t} V \longrightarrow \bigwedge^{2t} V^*$, where $V = S \otimes V_0$ is a rank n free module over S, $u^{\#} : V \longrightarrow V^*$ is the map associated with the generic form $u : \bigwedge^2 V \longrightarrow S$ on $X = \operatorname{Spec} S = \operatorname{Spec} S(\bigwedge^2 V_0) = \operatorname{Hom}(\bigwedge^2 V_0, R)$. It is known that S/Pf_{2t} is R-free, and is Gorenstein normal domain if so is R. The result on characteristic zero case is obtained by

Józefiak-Pragacz-Weyman [13]. They also constructed the minimal free resolution of the ideals generated by the minors of generic symmetric matrices (ibid.).

Theorem 5.1 ([19]) *Let $K = R$ be a field. We set $p = \mathrm{char}(K)$ if $\mathrm{char}(K) > 0$ and $p = \infty$ if $\mathrm{char}(K) = 0$. With the notation as above, the relation module of Pf_{2t} is generated by the linear relations when $2p > n-2t$. In particular, the second Betti number of S/Pf_{2t} is independent of p when $n - 2t \leq 3$.*

On the other hand, the answer for the question of the existence of the minimal free resolutions of S/Pf_{2t} is negative in general.

Theorem 5.2 ([18]) *With the notation as above, the relation module of Pf_{2t} is not generated by the linear relations when $p = 2$, $t = 2$, and $n = 8$. There is no minimal free resolutions of S/Pf_{2t} over $\mathbb{Z}$ in this case.*

For more about this problem, see [19], [18], and references there in.

Let $n \geq t \geq 1$. Consider the polynomial ring $S = R[x_{ij}]_{1 \leq i \leq j \leq n}$. We denote by J_t the ideals of S generated by t-minors of the generic symmetric matrix (x_{ij}), where $x_{ji} \overset{def}{=} x_{ij}$ for $i < j$.

Theorem 5.3 ([16]) *With the notation as above, the relation module of J_t is generated by the linear relations. In particular, the second Betti number of S/J_t is independent of the characteristic.*

The problem of the existence of minimal free resolutions of S/J_t over $\mathbb{Z}$ is still open. For more about this problem, see [16] and the references there in.

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