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Boundness of the degree of Fano manifolds with $b_2 = 1$

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1 Introduction

A normal projective algebraic variety X is called a **Q**-Fano variety if

1. X is **Q**-factorial.
2. X has at most canonical singularities.
3. $-K_X$ is ample.

Recently Mori completed the minimal model program in dimension 3 ([4]). And the minimal model conjecture is expected to be true in all dimension. Roughly speaking minimal model conjecture reduces birational geometry of algebraic varieties to the study of minimal varieties and **Q**-Fano varieties. Hence it is expected that Fano varieties play essential roles in the classification theory of algebraic varieties.

About Fano manifolds the following conjecture is well known (cf. [7, p. 599]).

Conjecture 1 *There exists a positive constant $C(n)$ which depends only on n such that for every Fano manifold M of dimension n*

$$c_1^n(M) \leq C(n).$$

holds.

The conjecture is known to be true classically in the case of surfaces (Del Pezzo surfaces). And Mori and Mukai proved that c_1^3 of Fano 3-folds are bounded by 64 ([6, Corollary 11]). In the case of toric Fano manifolds, Batyrev proved the conjecture affirmatively in all dimensions ([1]). But his estimate for $C(n)$ is fairly crude even in this very special case. The purpose of this paper is to prove the following partial answers.

Theorem 1 *There exists a constant $C(n)$ depending only on n such that for every Fano manifold M of dimension n and $b_2 = 1$*

$$c_1^n(M) \leq C(n).$$

holds.

Theorem 2 *Let M be a n-dimensional rationally connected Fano manifold. Then there exists a constant $C(n)$ depending only on n such that*

$$c_1^n(M) \leq C(n).$$

Remark 1 *Theorem 1,2 holds not only for Fano manifold but also* **Q**-*Fano variety with only quotient singularities. This generalization can be obtained only by replacing Fano manifolds to Fano orbifolds in the proof.*

The following conjecture is well known.

Conjecture 2 *Every Fano manifold is rationally connected.*

Hence by Theorem 2, the first conjecture is reduced to the second conjecture.

Our bound for c_1^n comes essentially from the Mori theory. Hence the bound seems to be quite natural. But we do not know that the bound is sharp or not. We should note that $c_1^n \leq c_1^n(\mathbf{P}^n)$ does not hold in general for a Fano manifold M of dimension greater than 3 ([1]).

The proof of Theorem 1,2 much depends on the method in [7].

2 Reduction to differential geometry

Let M be a Fano manifold of dimension n and let ω_0 be a Kähler form on M such that

$$[\omega_0] = 2\pi c_1(M),$$

where [ ] denotes the de Rham cohomology class.

Let us consider the equation:

$$(1-t)\omega_0 + t\omega = Ric_\omega, \tag{1}$$

where t is a parameter in $[0, 1]$.

To prove Theorem 1,2, we shall prove the following theorem.

Theorem 3 *Assume that M is rationally connected. Then the equation (1) has a smooth solution ω for*

$$t \in [0, \frac{1}{n+1})$$

For the first we shall show that Theorem 3 implies Theorem 2. In fact Theorem 3 implies that for every positive number ε, a rationally connected Fano manifold M admits a Kähler form ω such that

1. $[\omega] = 2\pi c_1(M)$,
2. $$\mathrm{Ric}_\omega \geq (\frac{1}{n+1} - \varepsilon)\omega.$$

By Myer's theorem the diameter of (M, ω) is bounded by a constant depending only on n. We note that

$$c_1^n(M) = (2\pi)^{-n} \int_M \omega^n.$$

Then by the volume comparison theorem ([2]), we see that $c_1^n(M)$ is bounded by a constant depending only on n.

3 Rationally connectedness

In this section we shall prepare the algebro-geometric tools for the proof of Theorem 1,2,3. First we recall the definition of rationally connectedness.

Definition 1 *Let X be a projective algebraic variety over an algebraically closed field. X is said to be rationally connected, if for general two closed points on X, there exists a chain of rational curves which contains the points.*

Using the Mori theory, Miyaoka proved the following theorem.

Theorem 4 *([3]). Let X be a Fano manifold over* $\mathbf{C}$ *with $b_2(X) = 1$. Then X is rationally connected.*

By using the deformation theory of curves, we obtain the following cororally.

Cororally 1 *Let X be a Fano manifold of dimension n over* $\mathbf{C}$ *with $b_2(X) = 1$. Then for every pair of points (p, q) on X, there exists a chain of curves $C = \sum C_i$ such that*

1. $p, q \in C$.

2. $(-K_X) \cdot C_i \leq n + 1$.

Proof. Let p, q be general points on X. Then by Theorem 4 there exists a chain of rational curve $C = \sum_{i=1}^{k} C_i$ such that $p, q \in C$. We may assume that

1. $p \in C_1$.
2. $q \in C_k$.
3. $C_i \cap C_{i+1} \neq \phi$ for $i = 1, 2, \ldots, k-1$.

Assume that there exists some i such that $(-K_X) \cdot C_i > n + 1$. Let us choose two points p_i,q_i on C_i such that $p_i \in C_i \cap C_{i-1}$ (if $i = 1$, we set $p_i = p$) and $q_i \in C_i \cap C_{i+1}$ (if $i = k$, we set $q_i = q$). Then by the deformation theory of morphisms and Riemann-Roch theorem, we can construct a nontrivial family of morphism from the normalization $\tilde{C}_i$ of C_i with the base points p_i, q_i which contains the noramlization $\tilde{C}_i \longrightarrow C_i$(cf. [5]). Then by the argument in [5], we can find a chain of rational curves $\sum_{j=1}^{m} C_{ij} (m \geq 2)$ which contains p_i and q_i. Continueing this process, we obtain a chain of rational curves thorough p, q with the desired properties. Q.E.D.

Cororally 2 implies that every Fano manifold of Picard number 1 contains abundant of curves of low degree (with respect to the anticanonical polarization). In other word, every two points in compact Kähler manifold with positive Ricci tensor and $b_2 = 1$ can be connected by a chain of minimal surfaces of small volume, if the Kähler form is in the anticanonical class.

4 Blow up of the solution

Let t_0 be the maximal existence time for the equation (1) in Section 2. To prove Theorem 3, we may assume that $t_0 < 1$ holds. We note that for every $t \in [0, t_0)$,

$$[\omega] = 2\pi c_1(M)$$

holds. We set

$$\omega = \omega_0 + \sqrt{-1}\partial\bar{\partial}u. \tag{2}$$

Let Ω be a smooth volume form on M such that

$$\omega_0 = \mathrm{Ric}\,\Omega = -\sqrt{-1}\partial\bar{\partial}\log\Omega. \tag{3}$$

Then the equation (1) is equivalent to the equation

$$\log\frac{(\omega_0 + \sqrt{-1}\partial\bar{\partial}u)^n}{\Omega} = -tu \ \text{ on } M \times [0, t_0). \tag{4}$$

We shall study how the solution ω of (1) blows up at $t = t_0$.

Lemma 1 *$\omega(t_0) = \lim_{t\to t_0}\omega$ exists as a positive d-closed (1,1)-current.*

Proof. We set $\omega(t) := \omega(\ ,t)$. Then since all $\omega(t)$ is cohomologous to ω_0 for all $t \in [0, t_0)$, we have

$$\int_M \omega(t) \wedge \omega_0^{n-1} = (2\pi)^n c_1^n(M).$$

We note that every $\omega(t)$ is positive for every $t \in [0, t_0)$. This implies the lemma. Q.E.D.

We set $u(t) := u(\ ,t)$. Let us set

$$v(t) = u(t) - \sup_M u(t)(t \in [0, t_0)). \tag{5}$$

Lemma 2 *$v(t_0) = \lim_{t\to t_0} v(t)$ exists as a upper semicontinuous function to $[-\infty, 0]$.*

Proof. The lemma follows from Lemma 1 and the $\partial\bar{\partial}$-Poincaré lemma. Q.E.D.

Since $\omega(t_0)$ is a d-closed positive (1,1)-current, $v(t_0)$ is locally a sum of plurisubharmonic function and a smooth function. Hence

$$S = \{p \in M \mid v(t_0)(p) = -\infty\}$$

is locally a pluripolar set. In particular S is of measure 0 with respect to ω_0.

The following proposition is crucial.

Proposition 1

$$S = \{p \in M \mid \exp(-t_0 v(t_0)) \textit{ is not locally integrable around } p\}.$$

To prove Proposition 1, we localize Siu's estimate in [7, pp.589-595] by a perturbation of $\omega(t_0)$. Let us prove Propositon 1 by contradiction. Assume that there exists a point $p \in S$ such that $\exp(-t_0 v(t_0))$ is integrable on an open neighbourhood U of p. We may assume that there exists $t_1 > t_0$ such that $\exp(-t_1 v(t_0))$ is integrable on U. Let B be a open geodesic ball with center p such that the closure $\bar{B}$ is contained in U. Let f be a smooth function on M such that

1. $\operatorname{Supp} f \subset U$.
2. $0 \leq f \leq 1$ on M.
3. $f \equiv 1$ on $\bar{B}$.

For $t \in [0, t_0)$ and $\varepsilon \in [0, t]$, let us consider the following equation

$$\begin{aligned} \log \frac{(\omega_0 + \sqrt{-1}\partial\bar{\partial}u(t)_\varepsilon)^n}{\Omega} &= -t_0\{(1-f)u(t_0-\varepsilon) + f \cdot u(t_0)\} \\ &\quad - \log \frac{\int_M \exp(-t_0\{(1-f)u(t_0-\varepsilon) + f \cdot u(t_0)\})\omega_0^n}{\int_M \omega_0^n} \end{aligned} \tag{6}$$

If we replace t_0 to $\varepsilon < t < t_0$, we have a smooth solution of (6) by the solution of Calabi's conjecture ([9]). Hence

$$\omega(t)_\varepsilon = \omega_0 + \sqrt{-1}\partial\bar{\partial}u(t)_\varepsilon$$

exists as a d-closed positive $(1,1)$-current on M for $0 \leq \varepsilon < t_0$. We note that

$$\mathrm{Ric}(\omega(t)) > t\omega(t). \tag{7}$$

and

$$\exp(-t_1 f \cdot u(t_0)) \tag{8}$$

is integrable on (M, ω_0). Now we note the following proposition due to Siu.

Proposition 2 *([7, p. 589, Proposition 2.1 and p.592, Proposition 3.1]) Given any small positive number ε there exists a positive constant C_ε^* such that for $0 \leq t < t_0$,*

$$\sup_M(-u(t)) \leq (n+\varepsilon)\sup_M(u(t)) + C_\varepsilon^*$$

and

$$\sup_M(u(t)) \leq (n+\varepsilon)\sup_M(-u(t)) + C_\varepsilon^*$$

hold on M.

Then by the Bochner-Kodaira formula letting t tend to t_0, for a sufficuently small ε we obtain that

$$\mathrm{osc}((1-f)u(t_0-\varepsilon) + fu(t_0)) < \infty$$

by Proposition 2 and the same argument as in [7, pp. 592-595 and p. 596 Proposition 4.3] (In the proof, we do not need the lower bound of the Ricci curvature of $\omega(t)_\varepsilon$ but the inequality (7). Because we only need the following inequality,

$$\int_M \sqrt{-1}\partial h \wedge \bar{\partial} h \wedge \omega^{n-1} \geq \lambda_1 \int_M |\, h\,|^2\, \omega^n,$$

where ω is a Kähler form on M and λ_1 is the first eigenvalue of the Laplacian of (M, ω) and f is a smooth function on M such that

$$\int_M h\omega^n = 0).$$

This is the contrtadiction because

$$\mathrm{Supp} f \cap S \neq \phi$$

and S is of measure 0. Q.E.D.

By Proposition 1 and the result in [8], we obtain the following corollary. For the definition of Lelong number and its basic properties, see [8].

Cororally 2 *Let $n(\omega(t_0), p)$ be the Lelong number of the d-closed positive (1,1)-current $\omega(t_0)$ at $p \in M$. Then we have:*

1. *$n(\omega(t_0), p) \geq 1/t_0$ for $p \in S$.*
2. *S is a proper subvariety of M.*

Now we prove Theorem 1,2 and 3. Let p be a point on S and let q be a point on $M - S$. Then by Cororally 1, there exists a chain of rational curves $C = \sum C_i$ which satisfies the properties in Cororally 1. Then there exists an irreducible component C_j of C which intersects with S with finitely many points. We note that the restriction $\omega(t_0) \mid C_j$ is well defined. Let us choose a point p_0 on $C_j \cap S$. Then by the structure theorem of d-closed positive $(1,1)$-current in [8], we have

$$n(\omega(t_0) \mid C_j, p_0) \geq n(\omega(t_0), p_0) \tag{9}$$

Since $(-K_M) \cdot C_j \leq n + 1$, we have

$$n(\omega(t_0) \mid C_j, p_0) \leq n + 1. \tag{10}$$

Then (9),(10) and Cororally 2, we have

$$t_0 \geq \frac{1}{n+1}.$$

This completes the proof of Theorem 1,2 and 3.

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