代数幾何学シンポジウム記録
1990年度 pp.208-226

A Method for The Construction of Pfaffian Systems with Finite Monodromy

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Introduction. Let Ω be a square-matrix-valued meromorphic 1-form on a complex manifold M such that $d\Omega + \Omega \wedge \Omega = 0$. For an unknown square-matrix-valued function F, the differential equation

$$dF = F\Omega$$

is called a Pfaffian system.

There is an important class of Pfaffian systems, called the class of Pfaffian systems of Fuchsian type (Deligne [2]). Some important differential equations such as Gauss' hypergeometric differential equations can be expressed as Pfaffian systems of Fuchsian type.

For a Pfaffian system, its monodromy group is defined. It is however a difficult problem in general to compute the monodromy group of a given Pfaffian system.

Hence it is also a difficult problem to determine if the monodromy group is finite or not. As for Gauss' hypergeometric differential equations, this problem was solved by Schwarz [9].

In this paper, we discuss the converse problem. That is, we give a general method for the construction of a wide class of Pfaffian systems, including those of Fuchsian type, with given finite monodromy groups.

1. Finite Galois coverings of complex manifolds.

Let M be a complex manifold. (The connectedness is always assumed.) We fix M once for all. A finite (branched) covering of M is , by definition, a finite proper holomorphic mapping

$$\pi : X \longrightarrow M$$

of an irreducible normal complex space X onto M.

Let $\pi : X \longrightarrow M$ and $\mu : Y \longrightarrow M$ be two finite coverings of M. A morphism (resp. an isomorphism) of π to μ is, by definition, a holomorphic (resp. biholomorphic) mapping φ of X onto Y such that $\mu \cdot \varphi = \pi$. When there is a morphism (resp. an isomorphism) of π to μ, we denote

$$\pi \geqslant \mu \quad \text{or} \quad \mu \leqslant \pi \qquad (\text{resp. } \pi \simeq \mu).$$

The set $\mathrm{Aut}(\pi)$ of all automorphisms of π forms a group under compositions and is called the automorphism group of π, which acts on every fiber of π.

A finite covering $\pi : X \longrightarrow M$ is called a finite Galois covering if $\mathrm{Aut}(\pi)$ acts transitively on every fiber of π. In this case, the quotient complex space $X/\mathrm{Aut}(\pi)$ (see Cartan [1]) is naturally biholomorphic to M.

Let $\pi : X \longrightarrow M$ be a finite covering. We put

$$R_\pi = \left\{ p \in X \;\middle|\; \pi \text{ is not biholomorphic around } p \right\}$$

and

$$B_\pi = \pi(R_\pi).$$

Then R_π and B_π are hypersurfaces (i.e., codimension 1 at every point) of X and M, respectively (see Fischer [3]), called the ramification locus and the branch locus of π, respectively.

Let B be a hypersurface of M. A finite covering $\pi: X \longrightarrow M$ is said to branch at most at B if B contains the branch locus B_π of π. In this case, the restriction π' of π to $X - \pi^{-1}(B)$

$$\pi': X - \pi^{-1}(B) \longrightarrow M - B \qquad \cdots\cdots (1)$$

is a usual (unbranched) covering. Its mapping degree is (independent of B and is) called the degree of π and is denoted by $\deg \pi$.

We have easily

Lemma 1.1. Let B be a hypersurface of M. Let π and μ be finite coverings of M which branch at most at B. Let π' and μ' be the restrictions of π and μ, respectively, as in (1). Then (i) $\pi \geqslant \mu$ if and only if $\pi' \geqslant \mu'$ and (ii) $\pi \simeq \mu$ if and only if $\pi' \simeq \mu'$.

Lemma 1.2. Under the same notations as in Lemma 1.1, (i) $\mathrm{Aut}(\pi)$ is naturally isomorphic to $\mathrm{Aut}(\pi')$ and (ii) π is a finite Galois covering if and only if π' is a finite Galois covering.

Corollary 1.3. $\#\mathrm{Aut}(\pi) \leqslant \deg \pi$, where $\#G$ means the order of the group G. The equality holds if and only if π is a finite Galois covering.

The following theorem is deep.

Theorem 1.4 (Grauert-Remmert [4], see also Grothendieck-Raynaud [5]). Let B be a hypersurface of a complex manifold M. Let $\pi': X' \longrightarrow M - B$ be a finite unbranched covering. Then there exists a unique (up to isomorphisms) finite covering $\pi: X \longrightarrow M$ branching at most at B whose restriction to $X - \pi^{-1}(B)$ is isomorphic to π'.

Let o be a fixed point of $M - B$. We denote by $\pi_1(M - B, o)$ the fundamental group of $M - B$ with the reference point o.

By Theorem 1.4,

Theorem 1.5. Let B be a hypersurface of a complex manifold M. Then there exists a one-to-one correspondence $\pi \longmapsto H = H(\pi)$ between the set of all isomorphism classes of finite Galois coverings π of M branching at most at B and the set of all normal subgroups H of finite index of $\pi_1(M - B, o)$, which satisfies the following two conditions: (i) $\pi \geqslant \mu$ if and only if $H(\pi) \subset H(\mu)$ and (ii) $\mathrm{Aut}(\pi)$ is naturally isomorphic to $\pi_1(M - B, o)/H(\pi)$.

2. Pfaffian systems of meromorphic type with finite monodromy.

Let $\Omega = (\omega_{jk})$ be an $(m \times m)$-matrix-valued meromorphic 1-form on a complex manifold M which satisfies the integrability condition:

$$d\Omega + \Omega\wedge\Omega = 0 . \qquad \cdots\cdot(2)$$

Let B_{jk} be the polar set of ω_{jk}. We put

$$B = \bigcup_{j,k} B_{jk}$$

and call it the <u>polar set of</u> Ω. B is a hypersurface of M. We put $M' = M - B$ and denote by $\widetilde{M}'$ the universal covering space of M'.

A theorem of Frobenius asserts that the Pfaffian system

$$dF = F\Omega \qquad \cdots\cdot(3)$$

has a solution F such that (i) F is an $(m\times m)$-matrix-valued holomorphic function on $\widetilde{M}'$, (ii) the determinant $\det F$ of F is nowhere vanishing and (iii) any other solution of (3) can be written as AF, where A is a constant $(m\times m)$-matrix.

We call such an F a <u>fundamental solution of</u> (3).

Let $o\in M' = M - B$ be a fixed point. The fundamental group $\pi_1(M', o)$ naturally acts on $\widetilde{M}'$. Let F be a fundamental solution of (3). For each element $\gamma\in\pi_1(M', o)$, we put $\gamma^* F = F\cdot\gamma$. Then, by (3),

$$d(\gamma^* F) = \gamma^* dF = \gamma^*(F\Omega) = (\gamma^* F)\Omega.$$

Hence we may write

$$\gamma^* F = R(\gamma)F \qquad \text{for } \gamma\in\pi_1(M', o), \qquad \cdots\cdot(4)$$

where

$$R : \gamma\in\pi_1(M', o) \longmapsto R(\gamma)\in GL(m,\mathbb{C})$$

is a homomorphism, called the <u>monodromy representation of the Pfaffian system</u> (3). Its image $G = R(\pi_1(M', o))$ is called the <u>monodromy group of</u> (3).

Assume now that the monodromy group G is a finite subgroup of $GL(m,\mathbb{C})$. In this case, (3) is called a Pfaffian system <u>with finite monodromy</u>. The kernel $Ker(R)$ of R is a normal subgroup of finite index of $\pi_1(M', o)$. By Theorem 1.5, there corresponds a unque (up to isomorphisms) finite Galois covering

$$\pi : X \longrightarrow M \qquad \cdots\cdots (5)$$

such that $Aut(\pi) \simeq G$ naturally. A fundamental solution F can be regarded as a holomorphic function on $X' = \pi^{-1}(M') = X - \pi^{-1}(B)$. In general, F has essential singularity along $\pi^{-1}(B)$.

The Pfaffian system (3) is said to be <u>of meromorphic type with finite monodromy</u> if (i) its monodromy group G is a finite subgroup of $GL(m,\mathbb{C})$ and (ii) F can be extended to a meromorphic function on the normal complex space X in (5).

<u>Proposition 2.1.</u> Every Pfaffian system of Fuchsian type with finite monodromy is of meromorphic type with finite monodromy.

<u>Proof.</u> If the Pfaffian system (3) is of Fuchsian type, then (see Deligne [2]), around a generic point q of B, Ω can be written as

$$\Omega = A_1(w)dw_1/w_1 + A_2(w)dw_2 + \cdots + A_n(w)dw_n,$$

where (i) $(w_1, w_2, \cdots, w_n)$ is a local coordinate system in M

around q such that $q = (0,0,\cdots,0)$ and $B = \{w_1 = 0\}$ (around q) and (ii) $A_j(w) = A_j(w_1, w_2, \cdots, w_n)$ are $(m \times m)$-matrix-valued holomorphic functions.

By Yoshida-Takano [11], a fundamental solution F of (3) can be written as

$$F(w) = (\exp C \log w_1)(\exp N \log w_1) G(w)$$

around q, where C is a constant $(m \times m)$-matrix, N is a diagonal $(m \times m)$-matrix with integral coefficients and $G(w)$ is an $(m \times m)$-matrix-valued holomorphic function with $\det G(w)$ nowhere vanishing.

If (3) is of finite monodromy, then C must be diagonalizable and every eigenvalue of C must be a rational number. Thus $F(w)$ can be meromorphically extended to an open neighborhood of $\pi^{-1}(q)$ in X.

By Levi's extension theorem (see Fischer [3]), F can be meromorphically extended to X.

q.e.d.

3. Construction. Put $s = m^2$ and let $Y = \mathbb{P}^s$ be the s-dimensional complex projective space. Y is the disjoint union of $\mathbb{C}^s$ and H_∞, where H_∞ is the hyperplane at infinity. We identify $\mathbb{C}^s$ with the set of all complex $(m \times m)$-matrices.

$GL(m,\mathbb{C})$ acts on $\mathbb{C}^s$ as the product of matrices:

$$(A,\ y) \in GL(m,\mathbb{C}) \times \mathbb{C}^s \longmapsto Ay \in \mathbb{C}^s.$$

This action can be naturally extended to that on Y by defining

$$(A,\ (0:y)) \in GL(m,\mathbb{C}) \times H_\infty \longmapsto (0:Ay) \in H_\infty .$$

Put

$$\Delta = \left\{ y \in \mathbb{C}^s \;\middle|\; \det y = 0 \right\} \bigcup H_\infty . \qquad \cdots\cdot (6)$$

Then Δ is a hypersurface of Y which is invariant under the action of $GL(m,\mathbb{C})$.

The $(m \times m)$-matrix-valued meromorphic 1-form

$$\Xi = y^{-1}dy \qquad \cdots\cdot (7)$$

on Y is clearly $GL(m,\mathbb{C})$-invariant.

Now, suppose that the Pfaffian system (3) on a complex manifold M is given and is of meromorphic type with finite monodromy. Then the fundamental solution F of (3) can be regarded as a meromorphic mapping

$$F:\ X \longrightarrow Y = \mathbb{P}^s,$$

where $\pi : X \longrightarrow M$ is the finite Galois covering in (5).
(4) means that the meromorphic mapping F is equivariant under the actions of $\mathrm{Aut}(\pi)$ on X and of the monodromy group G on Y. Hence a meromorphic mapping

$$f:\ M \longrightarrow N = Y/G \qquad \cdots\cdot (8)$$

is indeced from F and makes the diagram

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[d, "\pi"'] & Y \arrow[d, "\mu"] \\ M \arrow[r, "f"] & N = Y/G \end{tikzcd}$$

commutative. (μ is the natural projection.)

In general,

Definition 3.1. Let G be a finite subgroup of $GL(m,\mathbb{C})$. A meromorphic mapping $g: M \longrightarrow N = Y/G$ is said to be G-primitive if (i) there is a hypersurface B of M such that g is holomorphic on $M - B$ and $g(M-B) \cap \mu(\Delta) = \phi$, where Δ is defined in (6) and (ii) there is no decomposition of g as follows: $g = h \cdot \nu$. Here, $h: M \longrightarrow Y/H$ is a meromorphic mapping and $\nu : Y/H \longrightarrow N = Y/G$ is the natural projection, where H is a proper subgroup of G.

Remark 3.2. In Namba [8] , a meromorphic mapping $g: M \longrightarrow N = Y/G$ was said to be G-indecomposable if g satisfies (i)' $g(M) \not\subset \mu(\text{Fix } G)$ and (ii) in Definition 3.1. Here, $\text{Fix } G = \bigcup \text{Fix } A$, where $\text{Fix } A$ is the fixed point set of A and the union runs over all elements A of G with $A \neq 1$. It is clear that a G-primitive meromorphic mapping is G-indecomposable.

We can easily show (see Namba [8]) that the meromorphic mapping $f: M \longrightarrow N = Y/G$ in (8) is G-primitive.

The $(m \times m)$-matrix-valued 1-form Ξ on Y in (7) is $GL(m,\mathbb{C})$-invariant. So, it is G-invariant. Hence it can be regarded as an $(m \times m)$-matrix-valued rational 1-form on $L/\mathbb{C}$, (see Iitaka [6]), where $L = \mathbb{C}(N)$ is the field of meromorphic functions on $N = Y/G$.

In fact, if we take algebraically independent elements

$$u_1, u_2, \cdots, u_s \qquad (s = m^2)$$

in L, then $L/\mathbb{C}(u_1, u_2, \cdots, u_s)$ is a finite extension. Put $K = \mathbb{C}(Y)$. Then K/L is also a finite extension. By Iitaka [6] ,

Ξ can be written as

$$\Xi = \Xi_1 du_1 + \Xi_2 du_2 + \cdots + \Xi_s du_s, \qquad \cdots\cdot(9)$$

where Ξ_j are $(m \times m)$-matrix-valued meromorphic functions on Y. For each element A of G, we have

$$\Xi = A^*\Xi = A^*\Xi_1 du_1 + A^*\Xi_2 du_2 + \cdots + A^*\Xi_s du_s.$$

Since $du_1, du_2, \cdots, du_s$ are linearly independent over K, we have $A^*\Xi_j = \Xi_j$ $(1 \leqslant j \leqslant s)$. Hence Ξ_j are $(m \times m)$-matrix-valued meromorphic functions on $N = Y/G$. Thus Ξ can be regarded as an $(m \times m)$-matrix-valued rational 1-form on $L/\mathbb{C}$.

<u>Example 3.3.</u> Put $m = 2$ and $G = \{1, A\}$, where

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We put

$$y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}$$

and

$$p = -y_{11}-y_{21}, \qquad q = y_{11}y_{21},$$

$$r = -y_{12}-y_{22}, \qquad s = y_{12}y_{22},$$

$$t = y_{11}y_{12}+y_{21}y_{22}, \qquad u = y_{11}y_{22}+y_{12}y_{21}.$$

Then p, q, r, s, t and u generate the ring $\mathbb{C}[y_{11}, y_{21}, y_{12}, y_{22}]^G$ of G-invariants and have the following two relations:

$$t + u = pr,$$

$$tu = p^2 s + r^2 q - 4qs.$$

In this case, $\Xi = y^{-1}dy$ is written as

$$\Xi = \begin{pmatrix} \dfrac{-udp+rdq}{-pu+2qr} & \dfrac{2sdr-rds}{-ru+2ps} \\ \\ \dfrac{2qdp-pdq}{-pu+2qr} & \dfrac{-udr+pds}{-ru+2ps} \end{pmatrix}$$

which is a (2×2)-matrix-valued rational 1-form on $L/\mathbb{C}$, where L is the quotient field of $\mathbb{C}[y_{11}, y_{21}, y_{12}, y_{22}]^G$.

Now, let
$$\begin{tikzcd} M_o \arrow[r, "f_o"] \arrow[d, "\rho"'] & N \arrow[d, "id"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$
be the resolution of indeterminacy of the meromorphic mapping f in (8), where id is the identity mapping and ρ is a proper modification (see Ueno [10]).

We operate f_o^* on Ξ. Then, by (9), we have

$$f_o^*\Xi = (f_o^*\Xi_1)d(f_o^*u_1) + (f_o^*\Xi_2)d(f_o^*u_2) + \cdots + (f_o^*\Xi_s)d(f_o^*u_s).$$

This is well defined by the condition (i) in Definition 3.1 and is an $(m \times m)$-matrix-valued meromorphic 1-form on M_o. Moreover, we can easily see that the original meromorphic 1-form Ω on M is recovered by the relation

$$f_o^*\Xi = \rho^*\Omega. \qquad \cdots\cdots(10)$$

(Note that $\rho : M_o - \rho^{-1}(S) \longrightarrow M - S$ is biholomorphic, where S is the points of indeterminacy of f.)

Conversely, if $f: M \longrightarrow N = Y/G$ is a G-primitive meromorphic mapping for a given finite subgroup G of $GL(m,\mathbb{C})$, then we can define an $(m\times m)$-matrix-valued meromorphic 1-form Ω on M by (10). Ω satisfies the integrability condition (2), for Ξ satisfies it. Let $M_o \times_N Y$ be the fiber product of M_o and Y over N. Then $M_o \times_N Y$ is irreducible (see Namba [8]). Let

$$\alpha : X_o \longrightarrow M_o \times_N Y$$

be its normalization and put $\pi_o = \pi' \cdot \alpha$, where

$$\pi' : M_o \times_N Y \longrightarrow M_o$$

is the natural projection. Then

$$\pi_o : X_o \longrightarrow M_o$$

is a finite Galois covering with $\mathrm{Aut}(\pi_o) \simeq G$ naturally. By Theorem 1.4, π_o induces a finite Galois covering

$$\pi : X \longrightarrow M$$

with $\mathrm{Aut}(\pi) \simeq G$ naturally. Moreover, there is a commutative diagram:

$$\begin{array}{ccc} X_o & \xrightarrow{\eta} & X \\ {\scriptstyle \pi_o}\downarrow & & \downarrow{\scriptstyle \pi} \\ M_o & \xrightarrow[\rho]{} & M \end{array}$$

where η is a proper modification.

Now, look at the following commutative diagram:

$$\begin{array}{ccccc} X_o & \xrightarrow{\alpha} & M_o \times_N Y & \xrightarrow{\beta} & Y \\ {\scriptstyle \pi_o}\downarrow & & {\scriptstyle \pi'}\downarrow & & {\scriptstyle \mu}\downarrow \\ M_o & \xrightarrow[\mathrm{id}]{} & M_o & \xrightarrow[f]{} & N = Y/G \end{array}$$

Here, id is the identity mapping and β is the natural projection. The holomorphic mapping

$$F_o = \beta\cdot\alpha : X_o \longrightarrow Y$$

infuces a meromorphic mapping

$$F : X \longrightarrow Y = \mathbb{P}^s \qquad (s = m^2)$$

such that $F\cdot\eta = F_o$.

It is now easy to see that F gives a fundamental solution of the Pfaffian system

$$dF = F\Omega ,$$

which is of meromorphic type with the monodromy group G. Thus we conclude

Theorem 3.4. Let M be a complex manifold and G be a finite subgroup of $GL(m,\mathbb{C})$. Then every Pfaffian system on M of meromorphic type with the monodromy group G can be obtained by Ω in (10) for a G-primitive meromorphic mapping $f: M \longrightarrow N = \mathbb{P}^s/G \quad (s = m^2)$.

Remark 3.5. (i) An idea similar to our method can be found in Klein [7]. (ii) Our method can also be applied to Pfaffian systems with discrete monodromy groups. (iii) It is not easy in general to check the condition (ii) in Definition 3.1 for a given meromorphic mapping $f: M \longrightarrow Y/G$. If a meromorphic mapping $f: M \longrightarrow Y/G$ satisfies only the condition (i) in Definition 3.1, then, by the same method, we still have a Pfaffian system on M of meromorphic type with finite monodromy, whose monodromy group is however a subgroup of G. (iv) If

$f_o: M_o \longrightarrow Y/G$ is surjective with connected fibers, then we can show that $f: M \longrightarrow Y/G$ is G-primitive (see Namba [8]).

<u>Example 3.6.</u> We take the same group $G = \{1, A\}$ as in Example 3.3 and use the same notations. A meromorphic mapping $f: M \longrightarrow N = \mathbb{P}^4/G$ can be written as

$$f = (f_1, f_2, f_3, f_4, f_5, f_6) = (p, q, r, s, t, u),$$

where f_j $(1 \leqslant j \leqslant 6)$ are meromorphic functions on M such that

$$f_5 + f_6 = f_1 f_3 \qquad \text{and}$$

$$f_5 f_6 = f_1^2 f_4 + f_3^2 f_2 - 4 f_2 f_4.$$

In this case, f is G-primitive if and only if (i) $f_6^2 - 4f_2f_4 \neq 0$ and (ii) one of the following quadratic equations does not have a solution in $\mathbb{C}(M)$, the field of meromorphic functions on M:

$$x^2 + f_1 x + f_2 = 0,$$

$$x^2 + f_3 x + f_4 = 0.$$

Suppose that f is G-primitive. Then the Pfaffian system $dF = F\Omega$ is of meromorphic type with the monodromy group $G = \{1, A\}$, where

$$\Omega = \begin{pmatrix} \dfrac{-f_6 df_1 + f_3 df_2}{-f_1 f_6 + 2f_2 f_3} & \dfrac{2f_4 df_3 - f_3 df_4}{-f_3 f_6 + 2f_1 f_4} \\ \\ \dfrac{2f_2 df_1 - f_1 df_2}{-f_1 f_6 + 2f_2 f_3} & \dfrac{-f_6 df_3 + f_1 df_4}{-f_3 f_6 + 2f_1 f_4} \end{pmatrix}.$$

Conversely, by Theorem 3.4, every Pfaffian system on M

of meromorphic type with the monodromy group $G = \{1, A\}$ can be obtained in this way.

Example 3.7. Put $G = \{1, A, B, AB\}$, where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We put

$$y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}$$

and

$$u_1 = y_{11}^2,\ u_2 = y_{11}y_{12},\ u_3 = y_{12}^2,$$

$$v_1 = y_{21}^2,\ v_2 = y_{21}y_{22},\ v_3 = y_{22}^2.$$

Then u_1, u_2, u_3, v_1, v_2 and v_3 generate the ring $\mathbb{C}[y_{11}, y_{12}, y_{21}, y_{22}]^G$ of G-invariants and have the following two relations:

$$u_1u_3 = u_2^2, \quad v_1v_3 = v_2^2.$$

Hence a meromorphic mapping $f: M \longrightarrow N = \mathbb{P}^4/G$ can be written as

$$f = (f_1,f_2,f_3,g_1,g_2,g_3) = (u_1,u_2,u_3,v_1,v_2,v_3),$$

where f_j and g_j $(1 \leqslant j \leqslant 3)$ are meromorphic functions on M such that

$$f_1f_3 = f_2^2 \quad \text{and} \quad g_1g_3 = g_2^2.$$

In this case, f is G-primitive if and only if (i) $f_1g_3 \neq f_3g_1$ and (ii) none of the following two equations has a solution in $\mathbb{C}(M)$:

$$x^2 - f_1 = 0, \quad x^2 - g_1 = 0.$$

Suppose that f is G-primitive. Then the Pfaffian system $dF = F\Omega$ is of meromorphic type with the monodromy group $G = \{1, A, B, AB\}$, where

$$\Omega = \frac{1}{f_1g_3 - f_3g_1}\begin{pmatrix} \frac{(f_1g_3+f_2g_2)df_1}{2f_1} - \frac{(f_2g_2+f_3g_1)dg_1}{2g_1} & \frac{(f_2g_3+f_3g_2)df_3}{2f_3} - \frac{(f_2g_3+f_3g_2)dg_3}{2g_3} \\ \frac{(f_1g_2+f_2g_1)dg_1}{2g_1} - \frac{(f_1g_2+f_2g_1)df_1}{2f_1} & \frac{(f_1g_3+f_2g_2)dg_3}{2g_3} - \frac{(f_2g_2+f_3g_1)df_3}{2f_3} \end{pmatrix}.$$

Conversely, every Pfaffian system on M of meromorphic type with the monodromy group $G = \{1, A, B, AB\}$ can be obtained in this way.

4. <u>Finite projective monodromy groups.</u> Next, let $PGL(m,\mathbb{C})$ be the projective linear group and

$$1 \longrightarrow \mathbb{C}^* \longrightarrow GL(m,\mathbb{C}) \xrightarrow{\lambda} PGL(m,\mathbb{C}) \longrightarrow 1$$

the the natural exact sequence, where $\mathbb{C}^* = \mathbb{C} - \{0\}$.

For a Pfaffian system (3), let R be its monodromy representation. The homomorphism

$$\hat{R} = \lambda\cdot R:\ \pi_1(M - B,\ o) \longrightarrow PGL(m,\mathbb{C})$$

is called the <u>projective monodromy representation of</u> (3). Its image $\hat{G}$ is called the <u>projective monodromy group</u>. The Pfaffian system (3) is said to be <u>with finite projective monodromy</u> if $\hat{G}$ is a finite subgroup of $PGL(m,\mathbf{C})$. In this case, we have a finite Galois covering

$$\pi : X \longrightarrow M$$

which corresponds to $\ker \hat{R}$. Put

$$\hat{Y} = \mathbb{P}^{s-1} = \mathbf{C}^s/\mathbf{C}^* \quad (s = m^2).$$

Then a fundamental solution F of (3) induces a holomorphic mapping

$$\hat{F}: X - \pi^{-1}(B) \longrightarrow \hat{Y}. \qquad \cdots\cdots(11)$$

(B is the polar set of Ω.)

The Pfaffian system (3) is said to be <u>of meromorphic type with finite projective monodromy</u> if (i) it has a finite projective monodromy group and (ii) $\hat{F}$ in (11) can be extended to a meromorphic mapping $\hat{F}: X \longrightarrow \hat{Y}$.

A similar argument to the proof of Proposition 2.1 shows

<u>Proposition 4.1.</u> Every Pfaffian system of Fuchsian type with finite projective monodromy is of meromorphic type with finite projective monodromy.

A meromorphic mapping

$$g: M \longrightarrow \hat{N} = \hat{Y}/\hat{G}$$

for a finite subgroup $\hat{G}$ of $PGL(m,\mathbf{C})$ is said to be <u>$\hat{G}$-primitive</u> if (i) there is a hypersurface B of M such that g is holomorphic on $M - B$ and $g(M-B) \cap \hat{\mu}(\hat{\Delta}) = \emptyset$, where

$$\hat{\Delta} = \left\{ \hat{y} \in \hat{Y} \;\middle|\; \det y = 0 \right\}$$

and $\hat{\mu}: \hat{Y} \longrightarrow \hat{N}$ is the natural projection ($\hat{y}$ is the image of y under the natural projection $\mathbb{C}^s \longrightarrow \hat{Y} = \mathbb{P}^{s-1}$) and (ii) a similar condition to (ii) in Definition 3.1.

The $(m \times m)$-matrix-valued meromorphic 1-form $\Xi = y^{-1}dy$ on $\mathbb{C}^s$ $(s = m^2)$ is $GL(m,\mathbb{C})$-invariant. In particular, it is $\mathbb{C}^*$-invariant. Hence it can be regarded as an $(m \times m)$-matrix-valued meromorphic 1-form on $\hat{Y} = \mathbb{P}^{s-1}$ and is $PGL(m,\mathbb{C})$-invariant.

A similar argument to the proof of Theorem 3.4 shows

<u>Theorem 4.2.</u> Let M be a complex manifold and $\hat{G}$ be a finite subgroup of $PGL(m,\mathbb{C})$. Then every Pfaffian system on M of meromorphic type with the projective monodromy group $\hat{G}$ can be obtained by Ω in (10) for a $\hat{G}$-primitive meromorphic mapping $f: M \longrightarrow \hat{N} = \mathbb{P}^{s-1}/\hat{G}$ $(s = m^2)$.

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