

Moduli of Parabolic Stable Sheaves

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Introduction

Since a beautiful relationship between irreducible unitary representations of the fundamental group $\pi_1(X)$ of a compact Riemann surface X and stable vector bundles on X of degree zero was found out by M. S. Narasimhan and C. S. Seshadri in [16], their results have been extended to correspondences between representations of fundamental groups of various algebraic varieties and bundles with parabolic structure and with Higgs structure (and recently parabolic Higgs bundles). Stable parabolic bundles of parabolic degree zero correspond to irreducible unitary representations of the fundamental group of such a Riemann surface that a finite number of points are deleted from a compact Riemann surface. And stable Higgs bundles of degree zero correspond to irreducible (non-unitary) representations of the fundamental group of a compact Riemann surface. In the case of Higgs bundles, the results have been extended to higher dimensional cases. Moreover, in each case, the moduli of stable “bundles” had been constructed and proved that those moduli spaces are homeomorphic to those of the corresponding representations. In this paper, we shall give a definition of parabolic bundles in the higher dimensional cases and construct their moduli spaces.

In §1 we shall extend various notions on parabolic bundles on curves to that on non-singular projective varieties. We shall exploit C. T. Simpson’s description of parabolic bundles on curves for our case. It leads us to simpler descriptions of parabolic homomorphisms, parabolic Hilbert polynomials, parabolic exact sequences than our original forms or Seshadri’s. The moduli functor will be defined in this section. In §2 we shall deal with the boundedness of parabolic sheaves and the openness of the parabolic stability. C. T. Simpson’s techniques on the estimation of dimensions of $H^0(E(m))$ for semi-stable sheaves E enabled us to get a simpler proof of the “fundamental lemma” in [8]. §3 is devoted to constructing the moduli space. The method in this paper is a natural generalization

of that used by U. N. Bhosle [2].

Notation and Convention

Let $f : X \rightarrow S$ be a smooth, projective, geometrically integral morphism of locally noetherian schemes, E be a coherent \mathcal{O}_X -module and let $\mathcal{O}_X(1)$ be an f -very ample invertible sheaf. For a coherent \mathcal{O}_X -module E and a numerical polynomial H , we denote simply by $\text{Quot}(E, H)$ the Quot-scheme $\text{Quot}_{E/X/S}^H$. If s is a geometric point of S , then X_s means the geometric fiber of X over s and $E_s = E \otimes_{\mathcal{O}_S} k(s)$. For a coherent \mathcal{O}_{X_s} -module F , the degree of F with respect to $\mathcal{O}_X(1)$ is that of the first Chern class of F with respect to $\mathcal{O}_{X_s}(1) = \mathcal{O}_X(1) \otimes_{\mathcal{O}_{X_s}}$ and it is denoted by $\text{deg}_{\mathcal{O}_X(1)} F$ or simply $\text{deg } F$. Moreover the rank of F is denoted by $\text{rk}(F)$, $\mu(F) = \text{deg } F / \text{rk}(F)$ and $h^i(F) = \dim_{k(s)} H^i(X_s, F)$. For polynomials $f_1(n)$ and $f_2(n)$, $f_1(n) \prec f_2(n)$ (or, $f_1(n) \preceq f_2(n)$) means that $f_1(n) < f_2(n)$ (or, $f_1(n) \leq f_2(n)$, resp.) for all sufficiently large integers n . For a polynomial H and a number m , $H[m]$ denotes the polynomial such that $H[m](x) = H(m + x)$.

1 Elementary Notions on Parabolic Sheaves

Let X be a non-singular, projective variety over an algebraically closed field k , $D \subset X$ be an effective Cartier divisor and let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X .

Definition 1.1 Let E be a torsion free coherent \mathcal{O}_X -module. A quasi-parabolic structure (with respect to D) is giving a filtration

$$(1.1.1) \quad E = F_1(E) \supset F_2(E) \supset \cdots \supset F_l(E) \supset F_{l+1}(E) = E(-D),^1$$

where l is called the length of the filtration, $F_i(E)/F_{i+1}(E)$ is denoted by $G_i(E)$ and the Hilbert polynomial $\chi(G_i(E)(m))$ is called the i -th multiplicity polynomial of (E, F_*) .

Definition 1.2 A quasi-parabolic sheaf (E, F_*) is said to be a parabolic sheaf if it has a system of weights $\alpha_1, \alpha_2, \dots, \alpha_l$ such that $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_l < 1$ where l is the length of the filtration. The weight α_i is called the weight of the filter $F_i(E)$ and $\chi(G_i(m))$ is called the multiplicity polynomial of the weight α_i .

Let (E, F_*, α_*) be a parabolic sheaf. For all real numbers α , we put

$$E_\alpha = F_l(E)(-\lceil \alpha \rceil D)$$

¹ $E(-D)$ is the Image $(E \otimes \mathcal{O}_X(-D) \rightarrow E)$

if $\alpha_{i-1} < \alpha - [\alpha] \leq \alpha_i$ where $\alpha_0 = \alpha_l - 1$ and $\alpha_{l+1} = 1$. In particular, we have $E_{\alpha_i} = F_i(E)$ for $i = 1, 2, \dots, l+1$. Then we have the following filtration

$$(1.3.1) \quad j_* j^* E = \bigcup_{m \in \mathbf{Z}} E(-mD) = \bigcup_{\alpha \in \mathbf{R}} E_\alpha \supset \cdots \supseteq E_\alpha \supseteq E_\beta \supseteq \cdots,$$

where j is the inclusion of $X - D$ to X . This filtration satisfies the following conditions:

1. If $\alpha \leq \beta$, then $E_\alpha \supseteq E_\beta$.
2. If ε is a sufficiently small number, then $E_{\alpha-\varepsilon} = E_\alpha$.
3. $E_{\alpha+1} = E_\alpha(-D)$ for all real numbers α .
4. $E_0 = E$.
5. The length of the filtration for $0 \leq \alpha \leq 1$ is finite.

Conversely, if $\bigcup_{m \in \mathbf{Z}} E(-mD)$ has a filtration $\bigcup_{\alpha \in \mathbf{R}} E_\alpha$ which satisfies the above conditions, then E has a unique parabolic structure giving the filtration. So we often denote a parabolic sheaf (E, F_*, α_*) by simply E_* .

Definition 1.4 Let E_* and F_* be two parabolic sheaves. An \mathcal{O}_X -homomorphism $f : E \rightarrow F$ is called a homomorphism of parabolic sheaves when the following condition is satisfied:

$$f(E_\alpha) \subseteq F_\alpha \quad \text{for all } 0 \leq \alpha < 1.$$

We denote by $\text{Hom}^{\text{par}}(E_*, F_*)$ the k -module consisting of all parabolic homomorphisms of E_* to F_* and by $\mathcal{H}om^{\text{par}}(E_*, F_*)$ the sheaf of all parabolic homomorphisms.

Definition 1.5 A parabolic sheaf F_* is a parabolic subsheaf of a parabolic sheaf E_* if the following conditions are satisfied:

- 1) F is a subsheaf of E and E/F is torsion free.
- 2) $F_\alpha \subseteq E_\alpha$ for all α .
- 3) If $F_\alpha \subseteq E_\beta$ for some $\beta > \alpha$, then $F_\alpha = F_\beta$.

Definition 1.6 A homomorphism $f : E_* \rightarrow F_*$ of parabolic sheaves makes F_* a quotient parabolic sheaf of E_* if the following conditions are satisfied:

- 1) f is surjective.
- 2) If $f(E_\alpha) \subseteq F_\beta$ for some $\beta > \alpha$, then $F_\alpha = F_\beta$.

Let E_* be a parabolic sheaf and F be a subsheaf of E such that E/F is torsion free. If we put $F_\alpha = E_\alpha \cap F$, then we get a parabolic subsheaf F_* of E_* .² This parabolic structure of F is the maximal one among all parabolic

²Since E/F is torsion free, $E(-D) \cap F = F(-D)$.

subsheaves of E_* with the underlying subsheaf F . We call this structure of F the *induced sub-structure*. Let f be a surjective homomorphism of E to a torsion free sheaf G . Put $G_\alpha = f(E_\alpha)$. Then we get a parabolic quotient sheaf G_* of E_* . This parabolic structure of G is the minimal one among all parabolic quotient sheaves of E_* with the underlying quotient sheaf $f : E \rightarrow G$. We call this structure of G the *induced quotient structure*.

Definition 1.7 Let $f : E_* \rightarrow F_*$ and $g : F_* \rightarrow G_*$ be two parabolic homomorphisms. The sequence

$$(1.7.1) \quad 0 \longrightarrow E_* \xrightarrow{f} F_* \xrightarrow{g} G_* \longrightarrow 0$$

is said to be exact if for all real numbers α the sequence

$$(1.7.2) \quad 0 \longrightarrow E_\alpha \xrightarrow{f_\alpha} F_\alpha \xrightarrow{g_\alpha} G_\alpha \longrightarrow 0$$

is exact.

Note that if the sequence (1.7.1) is exact, then E_* and G_* have the induced structures. Conversely, the sequence (1.7.1) is exact when E_* and G_* have the induced structures and the sequence

$$0 \longrightarrow E \xrightarrow{f} F \xrightarrow{g} G \longrightarrow 0$$

is exact.

Every torsion free sheaf E has a natural parabolic structure:

$$E = F_1(E) \subset F_2(E) = E(-D), \alpha_1 = 0.$$

This structure is called the *special (parabolic) structure*.

For a locally free sheaf \mathcal{M} on X , $E_* \otimes \mathcal{M}$ has a natural parabolic structure such that

$$(E_* \otimes \mathcal{M})_\alpha = E_\alpha \otimes \mathcal{M}.$$

Hence, we regard $E_* \otimes \mathcal{M}$ as a parabolic sheaf with this structure.

Definition 1.8 1) The parabolic Euler characteristic of E_* is

$$\chi(E(-D)) + \sum_{i=1}^l \alpha_i \chi(G_i),$$

where l is the length of the filtration. It is denoted by $\text{par-}\chi(E_*)$. The polynomial $\text{par-}\chi(E_*(m))$ is called the parabolic Hilbert polynomial of E_* and the polynomial $\text{par-}\chi(E_*(m))/\text{rk}(F)$ is denoted by $\text{par-}P_{E_*}(m)$.

2) The parabolic degree of E_* is

$$\deg E + (n-1)! \times \left\{ \begin{array}{l} \text{the coefficient of } m^{n-1} \text{ of the polynomial} \\ \sum_{i=1}^r \alpha_i \chi(G_i(m)) \end{array} \right\},$$

where n is the dimension of X , r is the rank of E and $\mathcal{O}_X^{\oplus r}$ has the special structure. The parabolic degree of E_* is denoted by $\text{par-deg } E_*$ and we denote $\text{par-deg } E_*/\text{rk}(E)$ by $\text{par-}\mu(E_*)$. The second term of the right side is denoted by $\text{wt}(E_*)$.

It is easy to see that

$$(1.9.1) \quad \text{par-}\chi(E_*(m)) = \int_0^1 \chi(E_\alpha(m)) d\alpha$$

and

$$(1.9.2) \quad \begin{aligned} \text{par-deg}(E_*) &= (n-1)! \times \left\{ \begin{array}{l} \text{the coefficient of } m^{n-1} \text{ of the polynomial} \\ (\text{par-}\chi(E_*(m)) - \text{par-}\chi(\mathcal{O}_X^{\oplus r}(m))) \end{array} \right\} \\ &= \int_0^1 \deg E_\alpha d\alpha + r \cdot \deg D, \end{aligned}$$

where r is the rank of E . Hence, if (1.7.1) is exact, then we have

$$\text{par-}\chi(E_*(m)) + \text{par-}\chi(G_*(m)) = \text{par-}\chi(F_*(m)),$$

$$\text{par-deg}(E_*) + \text{par-deg}(G_*) = \text{par-deg}(F_*)$$

and

$$\text{wt}(E_*) + \text{wt}(G_*) = \text{wt}(F_*).$$

Definition 1.10 1) E_* is said to be parabolic stable (or, parabolic semi-stable) if for every parabolic subsheaf F_* of E_* with $0 \neq F \neq E$, we have

$$\text{par-}P_{F_*}(m) \prec \text{par-}P_{E_*}(m) \quad (\text{or, } \leq, \text{ resp.}).$$

2) E_* is said to be parabolic μ -stable (or, parabolic μ -semi-stable) if for every parabolic subsheaf F_* of E_* with $0 \neq F \neq E$, we have

$$\text{par-}\mu(F_*) < \text{par-}\mu(E_*) \quad (\text{or, } \leq, \text{ resp.}).$$

3) Let e be an integer. E_* is said to be of type e if for every parabolic subsheaf F_* of E_* with $0 \neq F \neq E$, we have

$$\text{par-}\mu(F_*) \leq \text{par-}\mu(E_*) + e.$$

Remark 1.11 1. Let $f : E_* \rightarrow F_*$ be an injective parabolic homomorphism of parabolic sheaves with the same rank. Since $\int_0^1 \chi(F_\alpha/E_\alpha(m))d\alpha \geq 0$, $\text{par-}P_{E_*}(m) \leq \text{par-}P_{F_*}(m)$ and if the equality holds, then f is an isomorphism of parabolic sheaves. Moreover, we have $\text{par-}\mu(E_*) \leq \text{par-}\mu(F_*)$. Hence, to verify the (μ_-) stability (or (μ_-) semi-stability) of E_* , it is enough to check the inequality for only parabolic subsheaves F_* such that E/F is torsion free and F_* has the induced structure.

2. For a parabolic sheaf E_* with the special structure, the parabolic Hilbert polynomial is not $\chi(E(m))$ but $\chi(E(-D)(m))$. So in general, the stability of a sheaf E does not mean the parabolic stability of a parabolic sheaf E_* with the special structure.

Every parabolic semi-stable sheaf has a Jordan-Hölder filtration. More precisely, every parabolic semi-stable sheaf E_* has a filtration

$$E = E^0 \supset E^1 \supset \dots \supset E^m = 0$$

such that for all i , $(E^i/E^{i+1})_*$ with the induced structure is parabolic stable and

$$\text{par-}P_{E_*}(m) = \text{par-}P_{(E^i/E^{i+1})_*}(m) \quad \text{for all } m.$$

We denote by $\text{gr}(E_*)$ the direct sum of parabolic sheaves $\bigoplus_0^{m-1} (E^i/E^{i+1})_*$. It exists unique up to isomorphisms. The proof of this fact is given by the similar arguments as the case of semi-stable sheaves. (cf. [9]) Two parabolic sheaves E_* and F_* are said to be equivalent if $\text{gr}(E_*)$ is isomorphic to $\text{gr}(F_*)$.

Remark 1.12 We have two induced structures on $(E^i/E^{i+1})_*$, i.e. the induced sub-structure from the induced quotient structure of E/E^{i+1} and the induced quotient structure from the induced sub-structure of E^i . But it is easy to see that the two structures coincide.

Moreover every parabolic sheaf E_* has the Harder-Narasimhan filtration, i.e. there exists the unique filtration

$$E = E^0 \supset E^1 \supset \dots \supset E^m = 0$$

such that all $(E^i/E^{i+1})_*$ are parabolic μ -semi-stable and for all i

$$\text{par-}\mu((E^i/E^{i+1})_*) < \text{par-}\mu((E^{i+1}/E^{i+2})_*).$$

We can show this by a similar proof to that in the case of semi-stable sheaves. (cf. [21])

Let $f : X \rightarrow S$ be a smooth, projective, geometrically integral morphism of noetherian schemes, $D \subset X$ be a relative effective Cartier divisor with respect to f and let $\mathcal{O}_X(1)$ be an f -very ample invertible sheaf.

Definition 1.13 A coherent \mathcal{O}_X -module E is said to be f -torsion free if it is flat over S and for every geometric fiber X_s of f , $E \otimes \mathcal{O}_{X_s}$ is a torsion free \mathcal{O}_{X_s} -module.

Note that if E is f -torsion free, then the canonical homomorphism

$$E \otimes \mathcal{O}_X(-D) \rightarrow E$$

is injective. In fact, if it is not injective, then D contains an associated point x of E and the point x is the associated point $E \otimes \mathcal{O}_{f^{-1}(f(x))}$ by the flatness of E over S . The torsion freeness means that x is the generic point of the fiber $f^{-1}(f(x))$. It contradicts the flatness of D .

Now we shall define the moduli functor for the family of parabolic semi-stable sheaves. Let (Sch/S) be the category of locally noetherian schemes over S . Let H and H_1, H_2, \dots, H_l be polynomials and fix a set of real numbers $\alpha_* = \{\alpha_1, \alpha_2, \dots, \alpha_l\}$ such that $0 \leq \alpha_1 < \alpha_2 < \dots < \alpha_l < 1$.

Let T be an object of (Sch/S) , E be a coherent \mathcal{O}_{X_T} -module, F_* be a filtration of E as in Definition 1.1 and α_* be a system of weights as in Definition 1.2. (E, F_*, α_*) is called a *flat family of parabolic sheaves* on X_T/T if E is f_T -torsion free and all $E/F_i(E)$ are flat over T (hence, all F_i are flat over T). Note that a flat family of parabolic sheaves has a filtration as in (1.3.1), hence we denote it simply by E_* .

Definition 1.14 For an object T of (Sch/S) , set

$$\text{par-}\overline{\Sigma}_{D/X/S}^{H_*, \alpha_*}(T) = \left\{ E_* \left| \begin{array}{l} E_* \text{ is a flat family of parabolic sheaves} \\ \text{on } X_T/T \text{ with the property (1.14.1)} \end{array} \right. \right\} / \sim$$

where \sim is the equivalence relation defined by (1.14.2).

(1.14.1) For every geometric point t of T , $(E_t, F_{*,t}, \alpha_*)$ is parabolic semi-stable, $\chi(E_t(m)) = H(m)$ and $\chi((E_t/F_{i+1}(E_t))_t(m)) = H_i(m)$, where $F_{*,t}$ is the filtration :

$$E_t = F_1(E)_t \supset F_2(E)_t \supset \dots \supset F_{l+1}(E)_t = E_t(-D).$$

(1.14.2) $E_* \sim E'_*$ if and only if there exist filtrations $E = E^0 \supset E^1 \supset \dots \supset E^m = 0$ and $E' = E'^0 \supset E'^1 \supset \dots \supset E'^m = 0$ such that for every geometric point t of T , their restrictions to X_t provide us with Jordan-Hölder filtrations of $E_{t,*}$ and $E'_{t,*}$, respectively, $\text{gr}(E_*) = \bigoplus_{i=0}^m E^i/E^{i+1}$ is T -flat and that $\text{gr}(E_*) \cong \text{gr}(E'_*) \otimes L$ for some invertible sheaf L on T .

For a morphism $g : T' \rightarrow T$ in (Sch/S) , g^* defines a map of $\text{par-}\overline{\Sigma}_{D/X/S}^{H_*, \alpha_*}(T)$ to $\text{par-}\overline{\Sigma}_{D/X/S}^{H_*, \alpha_*}(T')$. Then $\text{par-}\overline{\Sigma}_{D/X/S}^{H_*, \alpha_*}$ is a contravariant functor of (Sch/S) to $(Sets)$. We denote by $\text{par-}\Sigma_{D/X/S}^{H_*, \alpha_*}$ the sub-functor of $\text{par-}\overline{\Sigma}_{D/X/S}^{H_*, \alpha_*}$ consisting of all flat families of parabolic stable sheaves.

2 Boundedness and Openness

Let $f : X \rightarrow S$ be a smooth, projective, geometrically integral morphism of noetherian schemes, $D \subset X$ be a relative effective Cartier divisor with respect to f and let $\mathcal{O}_X(1)$ be an f -very ample invertible sheaf. Assume S is connected. We denote by h the degree of $\mathcal{O}_X(1)$ on fibers. Fix polynomials H and H_1, H_2, \dots, H_l and a set of real numbers $\alpha_* = \{\alpha_1, \alpha_2, \dots, \alpha_l\}$ such that $0 \leq \alpha_1 < \alpha_2 < \dots < \alpha_l < 1$.

Let $\mathcal{F}(H, H_*, \alpha_*)$ be the family of classes of parabolic sheaves on the fibers of X over S such that E_* is contained in $\mathcal{F}(H, H_*, \alpha_*)$ if and only if E_* is a parabolic semi-stable sheaf on some geometric fiber of X over S , $\chi(E(m)) = H(m)$, $\chi((E/F_{i+1}(E))(m)) = H_i(m)$ and the system of weights is α_* . For every member E_* of $\mathcal{F}(H, H_*, \alpha_*)$, the parabolic Hilbert polynomial has the following description;

$$(2.0.1) \quad \text{par-}\chi(E_*(m)) = H(m) - \sum_{i=1}^l \varepsilon_i H_i(m),$$

where $\varepsilon_i = \alpha_{i+1} - \alpha_i$ (set $\alpha_{l+1} = 1$).

If E_* is of type e , then E is of type $e + \deg D$. In fact, we have

$$\text{wt}(E_*) \leq (n-1)! \times \left(\begin{array}{c} \text{the coefficient of } m^{n-1} \text{ of} \\ \text{the polynomial } \chi(E|_D(m)) \end{array} \right) = \text{rk}(E) \deg D.$$

Hence, $\mu(F) \leq \text{par-}\mu(F_*) \leq \text{par-}\mu(E_*) + e \leq \mu(E) + \deg D + e$.

By the boundedness result on the family of coherent sheaves (cf. [10]), we have

Proposition 2.1 *The family $\mathcal{F}(H, H_*, \alpha)$ is bounded if one of the following conditions is satisfied :*

- 1) S is a noetherian scheme over a field of characteristic zero.
- 2) The rank is not greater than 3.
- 3) The dimension of X over S is not greater than 2.

Definition 2.2 Let e be a non-negative integer and let E_* be a parabolic sheaf on some geometric fiber X_s .

1) E is said to be of c-type e if for general non-singular curves $C = D_1 \cdot D_2 \cdot \dots \cdot D_{n-1}$, $D_i \in |\mathcal{O}_{X_s}(1)|$, every subsheaf $E' (\neq 0)$ of $E \otimes \mathcal{O}_C$ has a degree $\leq \text{rk}(E')(\mu(E) + e)$.

2) E_* is said to be parabolic e -stable (or, parabolic e -semi-stable) if E_* is parabolic stable (or, parabolic semi-stable, resp.) and E is of c-type e .

3) E_* is said to be strictly parabolic e -semi-stable if it is e -semi-stable and if for every parabolic quotient sheaf F_* of E_* with $\text{par-}P_{E_*} = \text{par-}P_{F_*}$, F_* is parabolic e -semi-stable.

Let $\mathcal{F}^e(H, H_*, \alpha_*)$ be the sub-family of \mathcal{F} such that E_* is contained in $\mathcal{F}^e(H, H_*, \alpha_*)$ if and only if E_* is parabolic e -semi-stable.

By Lemma 3.3 in [8], we have

Proposition 2.3 *The family $\mathcal{F}^e(H, H_*, \alpha_*)$ is bounded.*

Let $\mu\text{-}\mathcal{F}(e, r)$ be the family of coherent sheaves on the fibers of X over S such that E is contained in $\mu\text{-}\mathcal{F}(e, r)$ if and only if E is μ -semi-stable, E is of c -type e and $\text{rk}(E) \leq r$. The following proposition is Corollary 2.5 in [22] for our case. (See for the proof, C. T. Simpson [22].)

Proposition 2.4 *For each $\mu\text{-}\mathcal{F}(e, r)$, there exists an integer B such that for all members E of $\mu\text{-}\mathcal{F}(e, r)$, we have*

$$h^0(E(m)) \leq h \cdot \text{rk}(E)(m + B + \mu(E)/h)^n/n! \quad \text{for all } m \geq -B - \mu(E)/h,$$

and $h^0(E(m)) = 0$ for all $m \leq -B - \mu(E)/h$.

In the case of stable sheaves, the following was proved by M. Maruyama [8] using the fundamental lemma 2.2 in [8]. C. T. Simpson got a simpler proof in the case of characteristic zero using the Harder-Narasimhan filtration ([22]). In the parabolic case, we can also prove the following using the fundamental lemma 2.2 in [8]. Lemma 2.6 is equivalent to it. We give a simple proof using the technique of C.T.Simpson in [22].

Proposition 2.5 *There exists an integer N such that*

1) *if $E_* \in \mathcal{F}^e(H, H_*, \alpha_*)$ is parabolic stable, then for all $m \geq N$ and all parabolic subsheaves F_* of E_* with $0 \neq F \neq E$,*

$$\int_0^1 h^0(F_\alpha(m))d\alpha/\text{rk}(F) < \int_0^1 h^0(E_\alpha(m))d\alpha/\text{rk}(E),$$

2) *if $E_* \in \mathcal{F}^e(H, H_*, \alpha_*)$ is not parabolic stable, then for all $m \geq N$ and all parabolic subsheaves F_* of E_* with $0 \neq F \neq E$,*

$$\int_0^1 h^0(F_\alpha(m))d\alpha/\text{rk}(F) \leq \int_0^1 h^0(E_\alpha(m))d\alpha/\text{rk}(E),$$

and there exists a non-trivial parabolic subsheaf E'_* of E_* such that for all $m \geq N$ and $i > 0$,

$$\int_0^1 h^0(E'_\alpha(m))d\alpha/\text{rk}(E') = \int_0^1 h^0(E_\alpha(m))d\alpha/\text{rk}(E),$$

$h^i(E'(m)) = 0$ and $E'(m)$ is generated by its global sections.

Lemma 2.6 (Fundamental lemma 2.2 in [8]) *Let S be a locally noetherian, connected scheme, $f : X \rightarrow S$ be a smooth, projective, geometrically integral morphism of relative dimension n and let $\mathcal{O}_X(1)$ be an f -very ample invertible sheaf on X . Let a be a rational number, r be a positive integer and $P(m)$ be a polynomial of degree n with the highest term $hm^n/n!$ where h is the degree of $\mathcal{O}_X(1)$ on fibers of f . Then there exist integers L and M such that if F is a torsion free coherent \mathcal{O}_X -module of rank $r' \leq r$ for some geometric point s of S and if F has the properties ;*

1) *for general non-singular curves $C = D_1 \cdot D_2 \cdots D_{n-1}, D_i \in |\mathcal{O}_{X_s}(1)|$, every coherent subsheaf $E(\neq 0)$ of $F \otimes \mathcal{O}_C$ has a degree $\leq \text{rk}(E)a$,*

2) $\mu(F) \leq M$,

then for all $m \geq L$, the following inequality holds ;

$$h^0(F(m)) \leq r'P(m).$$

To prove Lemma 2.6, we need the following lemma

Lemma 2.7 *Assume that u and a are rational numbers with $u \leq a$ and r is a positive integer. Let $\mathcal{F}(a, u, r)$ be the family of classes of coherent sheaves on fibers of X over S such that F is contained in $\mathcal{F}(a, u, r)$ if and only if F is torsion free, $\text{rk}(F) \leq r$, $\mu(F) \leq u$ and F satisfies the condition 1) in Lemma 2.6. Then there exist an integer B , a non-negative number C and a polynomial $\Phi(x)$ of degree $\leq n - 2$ such that*

$$h^0(F(m)) \leq \frac{h \cdot \text{rk}(F)}{n!} \left(m + B + \frac{u + (\text{rk}(F) - 1)a}{h \cdot \text{rk}(F)} \right)^n + \Phi(m) \quad \text{if } m + B \geq -u/h,$$

if $-a/h \leq m + B \leq -u/h$, $h^0(F(m)) \leq C$ and if $m + B \leq -a/h$, $h^0(F(m)) = 0$, B does not depend on u and $\Phi(x)$ has the property ;

$$\text{for all } x \geq -B - u/h, \quad \Phi'(x) \geq 0 \quad \text{and} \quad \Phi(-B - u/h) \geq 0.$$

Proof. We prove the assertion by an induction on n . In the case of $n = 1$, apply Proposition 2.4 to the case $\mu\text{-}\mathcal{F}(0, r)$, then we have an integer B_1 depending only on $X, \mathcal{O}_X(1)$ and r such that for all μ -semi-stable sheaves E of rank $\leq r$, we have

$$h^0(E(m)) \leq h \cdot \text{rk}(E) \left(m + B_1 + \frac{\mu(E)}{h} \right) \quad \text{if } m + B_1 \geq -\mu(E)/h,$$

and $h^0(E(m)) = 0$ if $m + B_1 \leq -\mu(E)/h$.

Let $F = F^1 \supset F^2 \supset \cdots \supset F^t \supset 0$ be the Harder-Narasimhan filtration of F and let $Q^i = F^i/F^{i+1}$. Then we have $\mu(Q^1) \leq \mu(F) \leq u$ and by the assumption

1) in Lemma 2.6, $\mu(Q^i) \leq \mu(Q^i) \leq a$. Since all Q^i are in $\mu\text{-}\mathcal{F}(0, r)$, we have

$$\begin{aligned} h^0(F(m)) &\leq \sum_i h^0(Q^i(m)) \\ &\leq \sum_i h \cdot \text{rk}(Q^i) \max\left\{\left(m + B + \frac{\mu(Q^i)}{h}\right), 0\right\} \\ &\leq h(\text{rk}(F) - 1) \max\left\{\left(m + B + \frac{a}{h}\right), 0\right\} \\ &\quad + h \cdot \max\left\{\left(m + B + \frac{u}{h}\right), 0\right\}. \end{aligned}$$

This is the desired one in the case of $n = 1$.

Let H be a general hyperplane in $|\mathcal{O}_{X_s}(1)|$. By the exact sequence

$$0 \longrightarrow F(m-1) \longrightarrow F(m) \longrightarrow F|_H(m) \longrightarrow 0$$

and the induction hypothesis, there exist B_{n-1}, C_{n-1} and $\Phi_{n-1}(x)$ such that if $m + B_{n-1} \geq -u/h$,

$$\begin{aligned} h^0(F(m)) - h^0(F(m-1)) &\leq \\ &\frac{h \cdot \text{rk}(F)}{(n-1)!} \left(m + B_{n-1} + \frac{u + (\text{rk}(F) - 1)a}{h \cdot \text{rk}(F)}\right)^{n-1} + \Phi_{n-1}(m), \end{aligned}$$

if $-a/h \leq m + B_{n-1} \leq -u/h$, $h^0(F(m)) - h^0(F(m-1)) \leq C_{n-1}$ and if $m + B_{n-1} \leq -a/h$, $h^0(F(m)) = h^0(F(m-1))$. Set $B_n = B_{n-1} + 1$ and $C_n = C_{n-1} \left(\frac{a-u}{h}\right)$. Then, if $m + B_n \leq -a/h$, we have $h^0(F(m)) = 0$ and if $-a/h \leq m + B_n \leq -u/h$, we have $h^0(F(m)) \leq C_n$. Moreover, if $m + B_n \geq -u/h$, then we have

$$\begin{aligned} h^0(F(m)) &\leq C_n + \\ &\int_{-B_{n-1}-u/h}^{m+1} \left(\frac{h \cdot \text{rk}(F)}{(n-1)!} \left(x + B_{n-1} + \frac{u + (\text{rk}(F) - 1)a}{h \cdot \text{rk}(F)}\right)^{n-1} + \Phi_{n-1}(x) \right) dx. \end{aligned}$$

We can take a polynomial $\Phi_n(x)$ so that $\Phi'_n(x) = \Phi(x+1)$, $\Phi_n(-B_n - u/h) \geq 0$ and if $m + B_n \geq -u/h$, the right-hand side of the above inequality is smaller than

$$\frac{h \cdot \text{rk}(F)}{n!} \left(m + B_n + \frac{u + (\text{rk}(F) - 1)a}{h \cdot \text{rk}(F)}\right)^n + \Phi_n(m).$$

Then B_n, C_n and $\Phi_n(x)$ satisfy the desired conditions. \square

Proof of Lemma 2.6. Let $P(m) = h \cdot m^n/n! + a_1 m^{n-1} + (\text{terms of degree } \leq n-2)$ and u be a rational number such that

$$h\left(B + \frac{u + (r' - 1)a}{hr'}\right) < a_1 \quad \text{for all positive integers } r' \leq r.$$

Then there exists an integer $L \geq -B - u/h$ such that for all $m \geq L$ and $0 \leq r' \leq r$, we have

$$\frac{h}{n!} \left(m + B + \frac{u + (r' - 1)a}{hr'} \right)^n + \Phi(m) < P(m).$$

Then for $M = u$ and L , the assertion of Lemma 2.6 holds. \square

Proof of Proposition 2.5. By the boundedness of $\mathcal{F}^e(H, H_*, \alpha_*)$, there exists an integer N_1 such that for all $E_* \in \mathcal{F}^e(H, H_*, \alpha_*)$, $i > 0$, $m \geq N_1$ and $0 \leq \alpha \leq 1$, we have $h^i(E_\alpha(m)) = 0$. Hence $\int_0^1 h^0(E_\alpha(m)) d\alpha / \text{rk}(E) = \text{par-}P_{E_*}(m)$. Let us apply Lemma 2.6 to the case $P(m) = \text{par-}P_{E_*}(m)$, $r = \text{rk}(E)$ and $a = \mu(E) + e$ where E_* is a member of $\mathcal{F}^e(H, H_*, \alpha_*)$. Note that these don't depend on choices of E_* . Then we obtain integers L and M satisfying the conditions in Lemma 2.6. We may assume that $L \geq N_1$. Let F be a coherent subsheaf of E . Since E satisfies the condition 1) in Lemma 2.6, for all α with $0 \leq \alpha \leq 1$, the subsheaf $F_\alpha = E_\alpha \cap F$ of E also satisfies it. If $\mu(F) \leq M$ (hence, $\mu(F_\alpha) \leq M$), then we have

$$h^0(F_\alpha(m)) < \text{rk}(F)P(m) \quad \text{for all } m \geq L.$$

Therefore we have

$$\int_0^1 h^0(F_\alpha(m)) d\alpha / \text{rk}(F) < P(m) \quad \text{for all } m \geq L.$$

Let \mathcal{G} be the family of classes of parabolic sheaves on the fibers of X over S such that F_* is contained in \mathcal{G} if and only if for some member E_* of $\mathcal{F}^e(H, H_*, \alpha_*)$, F_* is a parabolic subsheaf of E_* such that E/F is torsion free, $\mu(F) \geq M$ and F_* has the induced structure. Since $\mathcal{F}^e(H, H_*, \alpha_*)$ is bounded, \mathcal{G} is bounded by Corollary 1.5.1 in [11]. Therefore there exists an integer $L' \geq L$ such that for all $F_* \in \mathcal{G}$, $i > 0$, $m \geq L'$ and $0 \leq \alpha \leq 1$, we have $h^i(F_\alpha(m)) = 0$ and $F(m)$ is generated by its global sections. The set of all parabolic Hilbert polynomials of F_* with $F_* \in \mathcal{G}$ is a finite set. Hence, there exists an integer $N \geq L'$ such that for all $F_* \in \mathcal{G}$ and $m \geq N$,

$$\int_0^1 h^0(F_\alpha(m)) d\alpha / \text{rk}(F) = \text{par-}P_{F_*}(m) \leq P(m)$$

and moreover if F_* is a parabolic subsheaf of a parabolic stable sheaf $E_* \in \mathcal{F}^e(H, H_*, \alpha_*)$, we have

$$\int_0^1 h^0(F_\alpha(m)) d\alpha / \text{rk}(F) < P(m).$$

If F_* is a parabolic subsheaf of $E_* \in \mathcal{F}^e(H, H_*, \alpha_*)$ and $\mu(F) \geq M$, then there exist $F'_* \in \mathcal{G}$ and an injective parabolic homomorphism of F_* to F'_* . Then we have $\int_0^1 h^0(F_\alpha(m)) d\alpha / \text{rk}(F) \leq \int_0^1 h^0(F'_\alpha(m)) d\alpha / \text{rk}(F') \leq P(m)$. Hence,

the assertion 1) and the previous part of the assertion 2) was proved. If $E_* \in \mathcal{F}^e(H, H_*, \alpha_*)$ is not stable, then there exists a non-trivial parabolic subsheaf E'_* of E_* such that $\text{par-}P_{E'_*} = \text{par-}P_{E_*}$. For such E'_* , we have $\mu(E') \geq \text{par-}\mu(E) - \deg D$. We may assume that $\text{par-}\mu(E) - \deg D \geq M$. Hence, E'_* must be contained in \mathcal{G} . Therefore, we have $\int_0^1 h^0(E'_\alpha(m)) d\alpha / \text{rk}(E') = \int_0^1 h^0(E_\alpha(m)) d\alpha / \text{rk}(E)$ and $E'(m)$ is generated by its global sections. \square

Strictly parabolic e -semi-stability is an open property. In fact, we have:

Proposition 2.8 *Let $g : Y \rightarrow T$ be a smooth, projective, geometrically integral morphism of locally noetherian schemes, $\mathcal{O}_Y(1)$ be a g -very ample invertible sheaf, $D \subset Y$ be a relative effective Cartier divisor and (E, F_*, α_*) be a flat family of parabolic sheaves on Y/T . If $H^i(Y_t, \mathcal{O}_Y(1) \otimes k(t)) = 0$ for all $i > 0$ and $t \in T$, then there exist open sets T^{ss} and T^s of T such that for all algebraically closed fields k ,*

$$\begin{aligned} T^{ss}(k) &= \{t \in T(k) \mid (E, F_*, \alpha_*) \otimes k(t) \text{ is strictly parabolic } e\text{-semi-stable}\} \\ T^s(k) &= \{t \in T(k) \mid (E, F_*, \alpha_*) \otimes k(t) \text{ is parabolic } e\text{-stable}\}. \end{aligned}$$

Proof. First, we shall prove that the property that a parabolic sheaf is of type e (or, parabolic semi-stable or, parabolic stable, resp.) is open. Before prove the above, we shall prepare some lemmas.

Lemma 2.9 *Let $X \rightarrow S$ be a projective morphism of locally noetherian schemes, $\mathcal{O}_X(1)$ be an f -very ample invertible sheaf, $\phi : E \rightarrow F$ be an \mathcal{O}_X -homomorphism of coherent \mathcal{O}_X -modules flat over S . Then for all polynomials $P(m)$, the following subset of S is closed :*

$$Z(P) = \{s \in S \mid \chi(\ker(\phi_s)(m)) \succeq P(m)\}$$

Moreover, if for all points $s \in S$, $\text{rk}(\ker(\phi_s)) = r$, then

$$W(a) = \{s \in S \mid \deg(\ker(\phi_s)) \geq a\}.$$

is closed.

Proof. We may assume that S is connected and noetherian. Set $C = F/\phi(E)$. For sufficiently large integer m and all points s of S , we have

$$h^0(C(m)|_{X_s}) = \dim f_*(C(m)) \otimes k(s).$$

Hence, for sufficiently large m , the function $h^0(C(m)|_{X_s})$ is upper semicontinuous and since $h^0(F(m)|_{X_s})$ is constant on S , the function $h^0((\phi(m)_s)(E(m)))$ is lower semicontinuous. Moreover, since $h^0(E(m)|_{X_s})$ is constant on S , the function $h^0(\ker(\phi(m)_s))$ is upper semicontinuous. Obviously, $\ker(\phi(m)_s) = \ker(\phi_s)(m)$.

If we choose a large M such that for all points s of S , $h^0(\ker(\phi_s)(M)) = \chi(\ker(\phi_s)(M))$ and $\chi(\ker(\phi_s)(m)) \succeq P(m)$ if and only if $\chi(\ker(\phi_s)(M)) \geq P(M)$, by the upper semi-continuity, we know that $Z(P)$ is closed.

Since the degree of a coherent sheaf K on X_s of rank r is $(n-1)! \times$ (the coefficient of m^{n-1} of the polynomial $\chi(K(m)) - r \cdot \chi(\mathcal{O}_{X_s}(m))$), the closedness of $W(a)$ is the immediate result of that of $Z(P)$. \square

The following lemma is an easy exercise, hence, we leave the proof to the reader.

Lemma 2.10 *Let $\{f_1, f_2, \dots, f_l\}$ be a finite set of upper semi-continuous functions on a topological space T such that for all i and integers N , the set $f_i(T) \cap \{r \in \mathbb{R} | r \leq N\}$ is finite. Then the function $\sum_{i=1}^l f_i$ is upper semi-continuous.*

We may assume that T is connected. Let \mathcal{G} be the family of coherent sheaves on fibers of Y over T such that F is contained in \mathcal{G} if and only if F is a subsheaf of E_t such that E_t/F is torsion free and $\deg F > \text{rk}(F)(\mu(E_t) - \deg D + e)$ where t is a point such that F is on Y_t . By Corollary 1.5.1 in [11], \mathcal{G} is bounded. Let Θ be the set of all Hilbert polynomials of members of \mathcal{G} . Then Θ is a finite set.

For each member θ of Θ , let Q_θ be the quot-scheme $\text{Quot}(E, H - \theta)$ where H is the Hilbert polynomial of E on fibers. The universal quotient on Y_{Q_θ} defines the exact sequence:

$$0 \longrightarrow \tilde{F} \longrightarrow E \otimes \mathcal{O}_Q \longrightarrow \tilde{G} \longrightarrow 0.$$

For every α such that $0 \leq \alpha \leq 1$, let ϕ_α be the canonical homomorphism of \tilde{F} to $E/E_\alpha \otimes \mathcal{O}_{Q_\theta}$. Then by virtue of Lemma 2.9, the function $\deg(\tilde{F}|_{Y_q})_\alpha$ is an upper semi-continuous function on Q_θ . Moreover by Lemma 2.10, the function $\text{par-deg}(\tilde{F}|_{Y_q})_\alpha$ is also upper semi-continuous. Hence, the set

$$Z_\theta = \{q \in Q_\theta | \text{par-}\mu((\tilde{F}|_{Y_q})_\alpha) > \text{par-}\mu(E_*|_{Y_q}) + e\}$$

is a closed subset of Q_θ . Set

$$U = T - \bigcup_{\theta \in \Theta} \pi_\theta(Z_\theta)$$

where π_θ is the projection of Q_θ to T . Then U is an open subset of T and it is the set of all points t of T such that $E_*|_{Y_t}$ is of type e . In fact, for all points q of Z_θ , $E_*|_{Y_q}$ is not of type e . And if t_0 be a point such that $E_*|_{Y_{t_0}}$ is not of type e , then there exists a subsheaf F of E_{t_0} such that E_{t_0}/F is torsion free and $\text{par-}\mu(F_*) > \text{par-}\mu(E_*|_{Y_{t_0}}) + e$, (hence, $\mu(F) > \mu(E) - \deg D + e$), where F_* has the induced structure. So, F is contained in \mathcal{G} . Therefore t_0 is in $\pi_\theta(Z_\theta)$ where θ is the Hilbert polynomial of F .

The proof of the openness of the property that a parabolic sheaf is parabolic semi-stable (or, parabolic stable, resp.) is similar to the above if we use the

(parabolic) Hilbert polynomials instead of (parabolic) degrees. Note that by Lemma 2.9 and 2.10, for large integer m , the function $\text{par-}\chi((\tilde{F}|_{Y_q})_*(m))$ is upper semi-continuous on Q_θ when Q_θ and \tilde{F} defined as above.

Now by a similar proof as in Lemma 3.5 of [8], we see that the property that a parabolic sheaf is parabolic e -semi-stable (or, parabolic e -stable, resp.) is open. Moreover, the proof of this Proposition 2.8 is similar to that of Lemma 3.6 in [9] if we use the technic as above. Hence, we omit the rest of the proof. \square

Remark 2.11 By the above proof, we know that the property that a parabolic sheaf is parabolic e -stable is open. Hence, $\text{par-}\Sigma_{D/X/S}^{H_*, \alpha_*, e}$ (or, $\text{par-}\overline{\Sigma}_{D/X/S}^{H_*, \alpha_*, e}$, resp.) is an open subfunctor of $\text{par-}\Sigma_{D/X/S}^{H_*, \alpha_*}$ (or, $\text{par-}\overline{\Sigma}_{D/X/S}^{H_*, \alpha_*}$, resp.) consisting of all flat families of parabolic e -stable (or, strictly parabolic e -semi-stable, resp.) sheaves.

3 Construction of Moduli Space

From now on, we shall fix the following situation:

(3.0) Let S be a scheme of finite type over a universally Japanese ring Ξ and let $f : X \rightarrow S$ be a smooth, projective, geometrically integral morphism such that the dimension of each fiber of X over S is n . Let $\mathcal{O}_X(1)$ be an f -very ample invertible sheaf such that for all points s in S and $i > 0$, $H^i(X_s, \mathcal{O}_X(1) \otimes \mathcal{O}_{X_s}) = 0$. Let $D \subset X$ be a relative effective Cartier divisor.

Fix a non-empty family $\mathcal{F}^e(H, H_*, \alpha_*)$. By the boundedness of $\mathcal{F}^e(H, H_*, \alpha_*)$, there exists an integer N_0 such that for every member E_* of $\mathcal{F}^e(H, H_*, \alpha_*)$, the conditions 1), 2) in Proposition 2.5 and the following conditions are satisfied:

(3.0.1) For all i and $m \geq N_0$, $F_i(E)(m)$ and $(E/F_i(E))(m)$ are generated by its global sections.

(3.0.2) $H^j(F_i(E)(m)) = 0$ and $H^j((E/F_i(E))(m)) = 0$ for all $j \geq 1$, i and $m \geq N_0$.

Fix an integer $m \geq N_0$ and let V_m be a free Ξ -module of rank $H(m)$. Let Q be the Quot-scheme $\text{Quot}(V_m \otimes \mathcal{O}_X, H[m])$ and let $V_m \otimes \mathcal{O}_{X_Q} \rightarrow \tilde{E}(m)$ be the universal quotient. Moreover, let Q_i be the Quot-scheme $\text{Quot}(V_m \otimes \mathcal{O}_X, H_i[m])$ and let $V_m \otimes \mathcal{O}_{X_{Q_i}} \rightarrow \tilde{E}_i(m)$ be the universal quotient for $i = 1, \dots, l$.

Let $\nu : Q \rightarrow \text{Pic}_{X/S}$ be the morphism determined by the invertible sheaf $\det \tilde{E}(m)$ (cf. [8]) and let P be a finite union of connected components which contain $\nu(Q)$. Then we have a P -morphism τ of Q to the Gieseker space Z defined in [8]. Z/P is the \mathbf{P}^N -bundle in étale topology and for a K -valued

geometric point x of P , a fiber Z_x is $\mathbf{P}(\mathrm{Hom}(\wedge^r V_m \otimes k(x), H^0(L_x)))^\vee$ where L_x is the invertible sheaf corresponding to x and r is the rank of \tilde{E} on fibers. (For details, see §4 in [8].)

$$\begin{array}{ccc} Q & \xrightarrow{\tau} & Z \\ & \searrow \nu & \swarrow \rho \\ & & P \end{array}$$

We shall define a sequence of morphisms

$$R_1 \rightarrow R_2 \rightarrow \cdots \rightarrow R_l \rightarrow Q,$$

coherent $\mathcal{O}_{X_{R_i}}$ -module $\tilde{E}_i(m)$ and surjections $\tilde{E}_{i+1}(m) \otimes \mathcal{O}_{X_{R_i}} \rightarrow \tilde{E}_i(m)$, inductively. Let R_l be the Quot-scheme $\mathrm{Quot}(\tilde{E}(m), H_l[m])$ and let $\tilde{E}(m) \otimes \mathcal{O}_{X_{R_l}} \rightarrow \tilde{E}_l(m)$ be the universal quotient. Assume that those are defined for $j \geq i+1$. Let R_i be the Quot-scheme $\mathrm{Quot}(\tilde{E}_{i+1}(m), H_i[m])$ and let $\tilde{E}_{i+1}(m) \otimes \mathcal{O}_{X_{R_i}} \rightarrow \tilde{E}_i(m)$ be the universal quotient.

Then we have the sequence of surjections of $\mathcal{O}_{X_{R_1}}$ -modules which are flat over R_1 .

$$V_m \otimes \mathcal{O}_{X_{R_1}} \xrightarrow{\phi} \tilde{E}(m) \otimes \mathcal{O}_{X_{R_1}} \xrightarrow{\phi_l} \tilde{E}_l(m) \otimes \mathcal{O}_{X_{R_1}} \xrightarrow{\phi_{l-1}} \cdots \xrightarrow{\phi_1} \tilde{E}_1(m).$$

Note that R_i is a closed subscheme of $Q \times \prod_{j=i}^l Q_j$. (Use the lemma 4.3 in [23].) Let $\phi'_l : \tilde{E}(m) \otimes \mathcal{O}_{X_{R_1}}(-D) \rightarrow \tilde{E}_l(m) \otimes \mathcal{O}_{X_{R_1}}$ be the restriction of ϕ_l to $\tilde{E}(m) \otimes \mathcal{O}_{X_{R_1}}(-D)$. By the lemma 4.3 in [23], there exists a unique closed subscheme $\bar{\Gamma}$ of R_1 such that for all morphisms $g : T \rightarrow R_1$, $g^*(\phi'_l) = 0$ if and only if g factors through $\bar{\Gamma}$. Let Q^0 be an open subscheme of Q such that for every algebraically closed field K , $Q^0(K) = \{x \in Q(K) \mid \tilde{E}(m) \otimes k(x) \text{ is torsion free}\}$ (cf.[11]). Note that the restriction of ν to Q^0 is an immersion (see Proposition 4.9 in [8]). Set $R_i^0 = R_i \times_Q Q^0$ and $\Gamma^0 = \bar{\Gamma} \times_Q Q^0$.

$$\begin{array}{ccccccc} \Gamma^0 & \hookrightarrow & R_1^0 & \hookrightarrow & Q^0 \times \prod_{i=1}^l Q_i & \longrightarrow & Q^0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \bar{\Gamma} & \hookrightarrow & R_1 & \hookrightarrow & Q \times \prod_{i=1}^l Q_i & \longrightarrow & Q \end{array}$$

On X_{Γ^0} , we have the sequence of surjections:

$$V_m \otimes \mathcal{O}_{X_{\Gamma^0}} \xrightarrow{\phi} \tilde{E}(m) \otimes \mathcal{O}_{X_{\Gamma^0}} \xrightarrow{\phi_l} \tilde{E}_l(m) \otimes \mathcal{O}_{X_{\Gamma^0}} \xrightarrow{\phi_{l-1}} \cdots \xrightarrow{\phi_1} \tilde{E}_1(m) \otimes \mathcal{O}_{X_{\Gamma^0}}.$$

Let $\bar{F}_{i+1}(\tilde{E}(m))$ be the kernel of the homomorphism

$$\phi_i \circ \phi_{i-1} \circ \cdots \circ \phi_l : \tilde{E}(m) \otimes \mathcal{O}_{X_{\Gamma^0}} \longrightarrow \tilde{E}_i(m) \otimes \mathcal{O}_{X_{\Gamma^0}}$$

for $i = 1, \dots, l$ and let $\bar{F}_1(\tilde{E}(m)) = \tilde{E}(m) \otimes \mathcal{O}_{X_{\Gamma^0}}$. Since $\bar{F}_1(\tilde{E}(m))$ is $(f \times 1_{\Gamma^0})$ -torsion free, the homomorphism $\tilde{E}(m) \otimes \mathcal{O}_{X_{\Gamma^0}}(-D) \rightarrow \tilde{E}(m) \otimes \mathcal{O}_{X_{\Gamma^0}}$ is injective and hence, $\bar{F}_{l+1}(\tilde{E}(m)) = \tilde{E}(m) \otimes \mathcal{O}_{X_{\Gamma^0}}(-D)$. Then $(\tilde{E}(m) \otimes \mathcal{O}_{X_{\Gamma^0}}, \bar{F}_*, \alpha_*)$ is a flat family of parabolic sheaves on X_{Γ^0} over Γ^0 .

Let U_i be the maximal open subscheme of Q_i such that for all points x of U_i and $j \geq 1$, $H^j(\tilde{E}_i(m) \otimes k(x)) = 0$ and $V_m \otimes \mathcal{O}_{U_i} \rightarrow f_{i*}(\tilde{E}_i(m) \otimes \mathcal{O}_{X_{U_i}})$ is surjective where f_i is the projection of X_{U_i} to U_i . Then $f_{i*}(\tilde{E}_i(m) \otimes \mathcal{O}_{X_{U_i}})$ is a locally free \mathcal{O}_{U_i} -module of rank $H_i(m)$. Hence we have a morphism $\gamma_i : U_i \rightarrow G_i$, where G_i is the Grassmann scheme $\text{Grass}(V_m \otimes \mathcal{O}_S, H_i(m))$. Set

$$\Gamma = \Gamma^0 \times_{(Q^0 \times \prod_{i=1}^l Q_i)} (Q^0 \times \prod_{i=1}^l U_i).$$

Then, we obtain a homomorphism $\Psi : \Gamma \rightarrow Z \times \prod_{i=1}^l G_i$.

$$\begin{array}{ccc} \Gamma & \xrightarrow{\Psi} & Z \times \prod_{i=1}^l G_i \\ \searrow \text{closed} & & \nearrow \tau \times \prod \gamma_i \\ & & Q^0 \times \prod_{i=1}^l U_i \end{array}$$

By virtue of Proposition 2.8, there exists an open subscheme Γ^{ss} (or, Γ^s) of Γ such that a geometric point x of Γ is contained in Γ^{ss} (or, Γ^s) if and only if the corresponding parabolic sheaf $(\tilde{E}(m) \otimes \mathcal{O}_{X_{\Gamma}}, \bar{F}_*, \alpha_*) \otimes k(x)$ is strictly parabolic e -semi-stable (or, parabolic e -stable, respectively) and the homomorphism

$$(3.0.1) \quad H^0(\phi \otimes k(x)) : V_m \otimes k(x) \rightarrow H^0(\tilde{E}(m) \otimes k(x))$$

is an isomorphism.

Proposition 3.1 *The morphism $\Psi : \Gamma^{ss} \rightarrow Z \times \prod_{i=1}^l G_i$ is an immersion.*

Proof. Let $V_m \otimes \mathcal{O}_{G_i} \rightarrow \mathcal{H}_i$ be the universal quotient on G_i and let J_i be the kernel of the quotient.

$$0 \rightarrow J_i \rightarrow V_m \otimes \mathcal{O}_{G_i} \rightarrow \mathcal{H}_i \rightarrow 0.$$

Moreover, the universal quotient sheaf on X_Q defines the following exact sequence.

$$0 \rightarrow \tilde{K} \rightarrow V_m \otimes \mathcal{O}_{X_Q} \rightarrow \tilde{E}(m) \rightarrow 0.$$

From these sequences, we have the following exact commutative diagram of $\mathcal{O}_{X_Q \times G_i}$ -modules. (Set $\tilde{X} = X_Q \times G_i$.)

$$\begin{array}{ccccccc}
& & & 0 & & & \\
& & & \downarrow & & & \\
& & & \widetilde{K} \otimes \mathcal{O}_{\widetilde{X}} = \widetilde{K} \otimes \mathcal{O}_{\widetilde{X}} & & & \\
& & & \downarrow & & \downarrow & \\
0 & \longrightarrow & J_i \otimes \mathcal{O}_{\widetilde{X}} & \longrightarrow & V_m \otimes \mathcal{O}_{\widetilde{X}} & \longrightarrow & \mathcal{H}_i \otimes \mathcal{O}_{\widetilde{X}} \longrightarrow 0 \\
& & \parallel & & \downarrow & & \downarrow \\
& & J_i \otimes \mathcal{O}_{\widetilde{X}} & \longrightarrow & \widetilde{E}(m) \otimes \mathcal{O}_{\widetilde{X}} & \longrightarrow & \mathcal{F} \longrightarrow 0 \\
& & & & \downarrow & & \downarrow \\
& & & & 0 & & 0
\end{array}$$

where $\mathcal{F} = V_m \otimes \mathcal{O}_{\widetilde{X}} / (\widetilde{K} \otimes \mathcal{O}_{\widetilde{X}} + J_i \otimes \mathcal{O}_{\widetilde{X}})$ that is not flat over $Q \times G_i$. Let us consider the Quot-scheme $\text{Quot}(\mathcal{H}_i \otimes \mathcal{O}_{\widetilde{X}}, H_i[m])$. Set $\widetilde{X} = X \times \text{Quot}(\mathcal{H}_i \otimes \mathcal{O}_{\widetilde{X}}, H_i[m])$. On \widetilde{X} , we have the universal quotient:

$$(3.1.1) \quad \mathcal{H}_i \otimes \mathcal{O}_{\widetilde{X}} \longrightarrow \widetilde{\mathcal{F}} \longrightarrow 0.$$

Let δ be the following homomorphism of $\widetilde{K} \otimes \mathcal{O}_{\widetilde{X}}$ to $\widetilde{\mathcal{F}}$.

$$\widetilde{K} \otimes \mathcal{O}_{\widetilde{X}} \longrightarrow V_m \otimes \mathcal{O}_{\widetilde{X}} \longrightarrow \mathcal{H}_i \otimes \mathcal{O}_{\widetilde{X}} \longrightarrow \widetilde{\mathcal{F}}.$$

Then by Lemma 4.3 in [23], we have a closed subscheme Δ_i of $\text{Quot}(\mathcal{H}_i \otimes \mathcal{O}_{\widetilde{X}}, H_i[m])$ such that for every morphism $g : T \rightarrow \text{Quot}(\mathcal{H}_i \otimes \mathcal{O}_{\widetilde{X}}, H_i[m])$, $(1_X \times g)^*(\delta) = 0$ if and only if g factors through Δ_i . Then, on X_{Δ_i} , the surjection (3.1.1) is factored as follows:

$$\mathcal{H}_i \otimes \mathcal{O}_{X_{\Delta_i}} \longrightarrow \mathcal{F} \otimes \mathcal{O}_{X_{\Delta_i}} \xrightarrow{\lambda} \widetilde{\mathcal{F}} \otimes \mathcal{O}_{X_{\Delta_i}}.$$

Set

$$\Delta_i^0 = \{x \in \Delta_i \mid \lambda|_{X_x} \text{ is isomorphism}\}.$$

Since $\Delta_i^0 = \Delta_i - \pi_{\Delta_i}(\text{Supp}(\ker \lambda))$, Δ_i^0 is an open subset of Δ_i . Let η be the projection of $\text{Quot}(\mathcal{H}_i \otimes \mathcal{O}_{\widetilde{X}}, H_i[m])$ to $Q \times G_i$ and let $\widetilde{\Delta}_i$ be the scheme theoretic image of Δ_i by η . Since η is proper, $\eta|_{\Delta_i} : \Delta_i \rightarrow \widetilde{\Delta}_i$ is surjective. Set

$$\widetilde{\Delta}_i^0 = \{x \in \widetilde{\Delta}_i \mid \text{The Hilbert polynomial of } \mathcal{F}|_{X_x} \text{ is } H_i[m]\}.$$

Then $\widetilde{\Delta}_i^0$ is an open subset of $\widetilde{\Delta}_i$. In fact, for all points x of $\Delta_i - \Delta_i^0$, the Hilbert polynomial of $\mathcal{F}|_{X_x}$ is not equal to that of $\widetilde{\mathcal{F}}|_{X_x}$. Hence,

$$\begin{aligned}
\widetilde{\Delta}_i^0 &= \widetilde{\Delta}_i - \eta(\Delta_i - \Delta_i^0) \\
&= \eta(\Delta_i^0).
\end{aligned}$$

Let $\iota : \coprod_j \tilde{\Delta}_{ij}^0 \rightarrow \tilde{\Delta}_i^0$ be the flattening stratification of $\mathcal{F} \otimes \mathcal{O}_{X_{\Delta_i^0}}$. (For the definition, see for example Lecture 8 in [15].) Then by the universality of $\coprod_j \tilde{\Delta}_{ij}^0$, $\eta : \Delta_i^0 \rightarrow \tilde{\Delta}_i^0$ factors through $\coprod_j \tilde{\Delta}_{ij}^0 \rightarrow \tilde{\Delta}_i^0$. Moreover, by the universality of Δ_i^0 , the morphism $\iota : \coprod_j \tilde{\Delta}_{ij}^0 \rightarrow \tilde{\Delta}_i^0$ factors through η . Therefore there exists an isomorphism of Δ_i^0 to $\coprod_j \tilde{\Delta}_{ij}^0$ and we obtain the following commutative diagram:

$$\begin{array}{ccc} \Delta_i^0 & \xrightarrow{\text{isomorphism}} & \coprod_j \tilde{\Delta}_{ij}^0 \\ \eta \searrow & & \swarrow \iota \\ & \tilde{\Delta}_i^0 & \end{array}$$

In particular, ι is proper. Therefore, all $\tilde{\Delta}_{ij}^0$ are closed subscheme of $\tilde{\Delta}_i^0$ and those images are open. Note that ι may not be an isomorphism, since, in general, $\tilde{\Delta}_i^0$ is not reduced. But we have known that $\eta : \Delta_i^0 \rightarrow \tilde{\Delta}_i^0$ is a closed immersion.

By Lemma 4.3 in [23], we know that $\text{Quot}(\mathcal{H}_i \otimes \mathcal{O}_{\tilde{X}}, H_i[m])$ is a closed subscheme of $Q \times Q_i$. Hence, Δ_i^0 is a subscheme of $Q \times Q_i$ and we obtain an immersion:

$$\Delta_1^0 \times_Q \Delta_2^0 \times_Q \cdots \times_Q \Delta_l^0 \longrightarrow Q \times \prod_{i=1}^l Q_i.$$

Moreover, we have an immersion determined by $\eta : \Delta_i^0 \rightarrow \tilde{\Delta}_i^0$:

$$\Delta_1^0 \times_Q \Delta_2^0 \times_Q \cdots \times_Q \Delta_l^0 \longrightarrow Q \times \prod_{i=1}^l G_i.$$

We know that Γ^{ss} is a subscheme of $Q \times \prod_{i=1}^l Q_i$ and the projection of Γ^{ss} to $Q \times Q_i$ factored by Δ_i^0 . Hence, Γ^{ss} is a subscheme of $\Delta_1^0 \times_Q \Delta_2^0 \times_Q \cdots \times_Q \Delta_l^0$. Therefore, Γ^{ss} is a subscheme of $Q \times \prod_{i=1}^l G_i$. Since $\nu : Q^0 \rightarrow Z$ is an immersion and Γ^{ss} is contained in $Q^0 \times \prod_{i=1}^l Q_i$, $\Psi : \Gamma^{ss} \rightarrow Z \times \prod_{i=1}^l G_i$ is an immersion. \square

From now on, we assume that all α_i are rational numbers. Set $G = \text{SL}(V_m)$ which acts on Γ, Q, Q_i, Z and G_i . Ψ is the G -morphism and Γ^{ss} (or, Γ^s , respectively) is the G -invariant open set of Γ .

We shall define a G -linearized invertible sheaf on $Z \times \prod_{i=1}^l G_i$. Let E_* be a member of $\mathcal{F}^e(H, H_*, \alpha_*)$ and let $\varepsilon_i = \alpha_{i+1} - \alpha_i$ for $i = 1, \dots, l$, where we put $\alpha_{l+1} = 1$. Let $\mathcal{O}_Z(1)$ be a tautological invertible sheaf which has the canonical G -linearization and let $\mathcal{O}_{G_i}(1)$ be a tautological invertible sheaf on G_i which has also the canonical G -linearization. Now, let L be the G -linearized invertible sheaf:

$$\mathcal{O}_Z(\text{par-}P_{E_*}(m)) \otimes \bigotimes_{i=1}^l \mathcal{O}_{G_i}(\varepsilon_i).$$

The open set consisting of all semi-stable (or, stable) points with respect to this G -linearization is denoted by $(Z \times \prod_{i=1}^l G_i)^{ss}$ (or, $(Z \times \prod_{i=1}^l G_i)^s$, respectively). Those are G -invariant open sets of $Z \times \prod_{i=1}^l G_i$.

Recall some facts on stable points of Z or G_i . Let x be a K -valued geometric point of $Z \times \prod_{i=1}^l G_i$. We denote the point of $Z(K)$ (or, $G_i(K)$) determined by x by T_x (or, $g_{i,x}$, respectively). We use the same symbol $g_{i,x}$ for the surjection $g_{i,x} : V_m \otimes K \rightarrow J_{i,x}$ which corresponds to x and moreover, we denote its kernel by $W_{i,x}$. T_x is identified with the homomorphism $T_x : \bigwedge^r V_m \otimes K \rightarrow H^0(L_x)$ where L_x is the invertible sheaf corresponding to $\rho(x) \in P$. Each subspace W of $V_m \otimes K$ has the maximal (or, minimal) T_x -dimension which is denoted by $\overline{\dim}_{T_x} W$ (or, $\underline{\dim}_{T_x} W$, respectively). (See for the definitions of these dimensions, Definition 3.1 in [23].) In general, $\underline{\dim}_{T_x} W \leq \overline{\dim}_{T_x} W$ and if equality holds, then it is denoted by $\dim_{T_x} W$.

Lemma 3.2 *Let x be a K -valued geometric point of $Z \times \prod_{i=1}^l G_i$. Assume that the point T_x in $Z(K)$ has the following property:*

$$(3.2.1) \quad \text{For all subspaces } W \text{ of } V_m \otimes K, \underline{\dim}_{T_x} W = \overline{\dim}_{T_x} W.$$

Then the point x is semi-stable (or, stable) with respect to the G -linearized invertible sheaf L if and only if for all non-trivial vector subspaces W of $V_m \otimes K$, the following inequality holds:

$$\begin{aligned} & \text{par-}P_{E_*}(m)(\dim_{T_x} W \cdot \dim(V_m \otimes K) - r \dim W) \\ & + \sum_{i=1}^l \varepsilon_i (\dim W \cdot \dim W_{i,x} - \dim(V_m \otimes K) \cdot \dim(W_{i,x} \cap W)) \geq 0 \end{aligned}$$

(or, > 0 , respectively).

Let $\sigma(W, x)$ be the left-hand side of the above inequality. Since $\dim(V_m \otimes K) = H(m)$ and $\dim W_{i,x} = H(m) - H_i(m)$. Hence, we have the following description of $\sigma(W, x)$:

$$\begin{aligned} \sigma(W, x) &= \dim W \left(\sum_{i=1}^l \varepsilon_i (H(m) - H_i(m)) - \text{par-}\chi(E_*(m)) \right) \\ & \quad + H(m)(\text{par-}P_{E_*}(m) \cdot \dim_{T_x} W - \sum_{i=1}^l \varepsilon_i \dim(W_{i,x} \cap W)) \\ &= \dim W \left((1 - \alpha_1)H(m) - \sum_{i=1}^l \varepsilon_i H_i(m) - H(m) + \sum_{i=1}^l \varepsilon_i H_i(m) \right) \\ & \quad + H(m)(\text{par-}P_{E_*}(m) \cdot \dim_{T_x} W - \sum_{i=1}^l \varepsilon_i \dim(W_{i,x} \cap W)) \\ &= H(m)(\text{par-}P_{E_*}(m) \cdot \dim_{T_x} W - \sum_{i=1}^l \varepsilon_i \dim(W_{i,x} \cap W) - \alpha_1 \dim W). \end{aligned}$$

Proposition 3.3 1) $\Psi(\Gamma^{ss}) \subseteq (Z \times \prod_{i=1}^l G_i)^{ss}$.

2) $\Psi(\Gamma^s) \subseteq (Z \times \prod_{i=1}^l G_i)^s$.

3) If a point x is in Γ^{ss} but not in Γ^s , then $\Psi(x)$ is not in $(Z \times \prod_{i=1}^l G_i)^s$.

Proof. Let x be a K -valued geometric point of Γ^{ss} and W be a non-trivial vector subspace of $V_m \otimes K$. Let the surjection $\phi_x : V_m \otimes \mathcal{O}_{X_x} \rightarrow E(m)$ and $E_*(m)$ correspond to $x \in \Gamma^{ss}$. By the definition of Γ^{ss} , $H^0(\phi_x) : V_m \otimes K \rightarrow H^0(E(m))$ is an isomorphism. Set $F(m) = \phi_x(W \otimes \mathcal{O}_{X_x})$. Then we have

$$(3.3.1) \quad \dim_{T_{\Psi(x)}} W = \text{rk}(F).$$

Since W and $H^0(F(m))$ are subspaces of $V_m \otimes K \cong H^0(E(m))$, $W \subset H^0(F(m))$, hence, we have

$$(3.3.2) \quad \dim W \leq h^0(F(m)).$$

The subspace $W_{i,\Psi(x)}$ of $V_m \otimes K$ is identified with $H^0(F_{i+1}(E)(m))$ by $H^0(\phi_x)$. Since

$$H^0((F_{i+1}(E) \cap F)(m)) = H^0(F_{i+1}(E)(m)) \cap H^0(F(m)),$$

we have

$$(3.3.3) \quad \dim W_{i,\Psi(x)} \cap W \leq h^0((F_{i+1}(E) \cap F)(m)).$$

Therefore, we obtain

$$\begin{aligned} \sigma(W, \Psi(x)) &\geq H(m)(\text{par-}P_{E_*}(m) \cdot \text{rk}(F) \\ &\quad - (\alpha_1 h^0(F(m)) + \sum_{i=1}^l \varepsilon_i h^0((F_{i+1}(E) \cap F)(m)))) \\ &= H(m)(\text{par-}P_{E_*}(m) \cdot \text{rk}(F) - \int_0^1 h^0(F_\alpha(m)) d\alpha). \end{aligned}$$

By virtue of Proposition 2.5 and Lemma 3.2, the assertions 1) and 2) have been proved. Let E'_* be a parabolic subsheaf of E_* which satisfies the last condition in Proposition 2.5. Set $W = H^0(E'(m))$. Then we have

$$H^0((E' \cap F_{i+1}(E))(m)) = W \cap W_{i,\Psi(x)}.$$

Since $E'(m)$ is generated by its global sections, $\dim_{T_{\Psi(x)}} W = \text{rk}(E')$. Hence,

$$\sigma(W, \Psi(x)) = H(m)(\text{par-}P_{E_*}(m) \cdot \text{rk}(E') - \int_0^1 h^0(E'_\alpha(m)) d\alpha) = 0.$$

So, $\Psi(x)$ is not in $(Z \times \prod_{i=1}^l G_i)^s$. \square

Let $\tilde{\Gamma}^{ss}$ be the scheme theoretic image of Γ^{ss} in $(Z \times \prod_{i=1}^l G_i)^{ss}$. Then by virtue of Theorem 4 of [18], there exists a good quotient $\xi : \tilde{\Gamma}^{ss} \rightarrow Y$ and Y is projective over S . Set $\bar{M}_e = Y - \xi(\tilde{\Gamma}^{ss} - \Gamma^{ss})$. Then \bar{M}_e is quasi-projective over S . Moreover \bar{M}_e contains $M_e = \xi(\Gamma^s)$ as an open subscheme. By the similar arguments as in §5 in [8], we have

Theorem 3.4 M_e is a coarse moduli scheme of $\text{par-}\Sigma_{D/X/S}^{H_*,\alpha_*,e}$, that is, M_e has the following properties.

1) For each geometric point s of S , there exist a natural bijection :

$$\theta_s : \text{par-}\Sigma_{D/X/S}^{H_*,\alpha_*,e}(k(s)) \longrightarrow M_e(k(s)).$$

2) For $T \in (\text{Sch}/S)$ and a flat family of parabolic sheaves E_* on X_T/T such that E_* has the property (1.12.1) and for every geometric point t of T , $E_*|_{X_t}$ is parabolic e -stable, then there exists a morphism

$$f_{E_*}^e : T \longrightarrow M_e$$

such that for all points t in $T(k(s))$, $f_{E_*}^e(t) = \theta_s(E_*|_{X_t})$. Moreover, for a morphism $g : T' \rightarrow T$ in (Sch/S) ,

$$f_{E_*}^e \circ g = f_{(1_X \times g)^*(E_*)}^e.$$

3) If $M' \in (\text{Sch}/S)$ and maps

$$\theta'_s : \text{par-}\Sigma_{D/X/S}^{H_*,\alpha_*,e}(k(s)) \longrightarrow M'(k(s))$$

have the above property 2), then there exists a unique S -morphism Υ of M_e to M' such that $\Upsilon(k(s)) \circ \theta_s = \theta'_s$ and $\Upsilon \circ f_{E_*}^e = f'_{E_*}$ for all geometric points s of S and for all E_* , where f'_{E_*} is a morphism given by the property 2) for M' and θ'_s .

By the similar arguments as in §5 of [8], we know that there exists a unique morphism $\nu_{e,e'}$ of M_e to $M_{e'}$ if $e \leq e'$. Moreover, M_e can be regarded as an open subscheme of $M_{e'}$ through $\nu_{e,e'}$. Taking inductive limit of $\{M_e\}$, an S -scheme $M_{D/X/S}^{H_*,\alpha_*}$ is obtained. By the same proof as Theorem 5.6 of [8], we have

Theorem 3.5 The functor $\text{par-}\Sigma_{D/X/S}^{H_*,\alpha_*}$ has a coarse moduli scheme $M_{D/X/S}^{H_*,\alpha_*}$ in (Sch/S) . Moreover, $M_{D/X/S}^{H_*,\alpha_*}$ is separated and locally of finite type over S .

For the functor $\overline{\text{par-}}\Sigma_{D/X/S}^{H_*,\alpha_*,e}$, by the similar argument as in §4 of [9], we have

Theorem 3.6 \bar{M}_e has the following properties.

1) For each geometric point s of S , there exists a natural surjection :

$$\bar{\theta}_s : \text{par-}\overline{\Sigma}_{D/X/S}^{H_*,\alpha_*,e}(k(s)) \longrightarrow \bar{M}_e(k(s)).$$

2) For $T \in (\text{Sch}/S)$ and a flat family of parabolic sheaves E_* on X_T/T such that E_* has the property (1.12.1) and for every geometric point t of T , $E_*|_{X_t}$ is strictly parabolic e -semi-stable, then there exists a morphism

$$f_{E_*}^e : T \longrightarrow \bar{M}_e$$

such that for all points t in $T(k(s))$, $f_{E_*}^e(t) = \bar{\theta}([E_*|_{X_t}])$ where $[\cdot]$ means the equivalence class defined by (1.12.2). Moreover, for a morphism $g : T' \rightarrow T$ in (Sch/S) , we have

$$f_{E_*}^e \circ g = f_{(1_X \times g)^*(E_*)}^e.$$

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