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Elliptic 3-folds and Non-Kähler 3-folds

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Introduction

The purpose of this report is to study the relationship between Calabi-Yau 3-folds with elliptic fibrations and compact non-Kähler 3-folds with $K = 0$, $b_2 = 0$, $q = 0$. The non-Kähler 3-folds referred to here have firstly appeared in Friedman's paper [3]. In this paper he has shown that if there are sufficiently many (mutually disjoint) (-1,-1)-curves on a Calabi-Yau 3-fold, then one can contract these curves and can deform the resulting variety to a smooth non-Kähler 3-fold with $K_2= 0$, $b_2 = 0$, $q = 0$. For example, in the case of a (general) quintic hypersurface in $\mathbb{P}^4$, one can do this procedure for two lines on it. This phenomenon is analogous to the one for (-2)-curves on a K3 surface. In fact a (-2)-curve on a $K3$ surface often disappears in deformation, and this fact just says that one can contract this (-2)-curve to a point and can deform the resulting variety to a (smooth) $K3$ surface. By this phenomenon, we can explain the varience of the Picard number of $K3$ surface in deformation, and it is well-known that a general point of the moduli space of $K3$ surfaces corresponds to a non-projective (but Kähler) $K3$ surface on which there are no (-2)-curves. Taking such a non-projective surface into consideration, one has a famous theorem that two arbitrary $K3$ surfaces are connected by deformation. But there is a difference between Calabi-Yau 3-folds and $K3$ surfaces, that is, a (-1,-1)-curve never disappears like a (-2)-curve in deformation. This is closely related to the fact that Calabi-Yau 3-folds have a large repertory of topological Euler numbers. For the speculation around this area, one may refer to the paper of M. Reid [12]. The main results of this paper is the following:

Theorem A.

Let X be a Calabi-Yau 3-fold which has an elliptic fibration with a rational section. Then the bimeromorphic class of X is obtained as a semi-stable degeneration of a compact non-Kähler 3-fold with $K = 0$, $b_2 = 0$ and $q = 0$, i.e. there is a surjective proper map f of a smooth 4-dimensional variety $\mathfrak{X}$ to a 1-dimensional disc Δ such that

1) *$f^{-1}(t)$ is a compact non-Kähler 3-fold with $K = 0$, $b_2 = 0$, $q = 0$ for $t \in \Delta^*$,*
2) *$f^{-1}(0) = \sum_{i=0}^{n} X_i$ is a normal crossing divisor of $\mathfrak{X}$, and*
3) *X_0 is bimeromorphic to X.*

Here we will explain the motivation of the formulation in Theorem A. If there are sufficiently many (-1,-1)-curves on X in the Friedman's sense explained above, one has a flat morphism f of a complex analytic variety $\mathfrak{X}$ to a disc Δ whose central fibre is the variety obtained by contraction of these curves, and whose general fibre is a non-Kahler 3-fold with $K = 0$, $b_2 = 0$, $q = 0$. In this situation, $\mathfrak{X}_0 := f^{-1}(0)$ has a number of ordinary double points, but one may assume that the total space $\mathfrak{X}$ is smooth in a suitable condition (e.g. (1.1) in this report). Next blow up these points. Then the central fibre consists of a number of irreducible components, namely, the smooth variety $\tilde{\mathfrak{X}}_0$ obtained by the blowing ups of the ordinary double points on $\mathfrak{X}_0$ and the $\mathbf{P}^3$'s corresonding to each point blown up. However this is not yet a semi-stable degeneration because the multiplicity of each $\mathbf{P}^3$ is two. So taking a suitable base change, one has a semi-stable degeneration. This is a typical example of Theorem A.

The Construction of the Proof

In this report, a Calabi-Yau 3-fold means a smooth projective 3-fold with $c_2 \neq 0$, $q = 0$, K trivial. Since $c_2 \neq 0$, those 3-folds are excluded which are , up to etale covers, Abelian 3-folds or the products of $k3$ surfaces and elliptic curves. Here we will briefly review the Friedman's construction of non-Kahler 3-fold with $K = 0$, $b_2 = 0$. Assume that X is a smooth compact 3-fold with K_X trivial and that mutually disjoint (-1,-1)-curves $C_1, \ldots, C_n$ are given on X. Here a (-1,-1)-curve means a smooth rational curve $\mathbb{P}^1$ whose normal bundle $N_{\mathbb{P}^1/X}$ is ismorphic to $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Then one can contract these curves to points and one has a compact 3-fold $\bar{X}$ with ordinary double points: $\pi\colon X \longrightarrow \bar{X}$. For simplicity we will write $P_i = \pi(C_i)$, $Z = \coprod_i P_i$ and $C = \coprod_i C_i$. We have the following commutative exact diagram:

$$\begin{array}{ccccccccccc} 0 \to & H^1(\pi_*\Theta_X) & \to & H^1(\Theta_X) & \to & H^0(R^1\pi_*\Theta_X) & \to & H^2(\pi_*\Theta_X) & \to & H^2(\Theta_X) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & H^1(T^0_{\bar{X}}) & \to & T^1_{\bar{X}} & \to & H^0(T^1_{\bar{X}}) & \xrightarrow{\alpha} & H^2(T^0_{\bar{X}}) & \to & T^2_{\bar{X}} & \to 0 \end{array}$$

In the above diagram, the map α is interpreted as follows:
First we have an isomorphism $\beta\colon H^0(T^1_{\bar{X}}) \longrightarrow H^2_Z(T^0_{\bar{X}})$ by using the exact sequence defined locally at each P_i :

$$0 \longrightarrow T^0_{\bar{X}} \longrightarrow \Theta_{\mathbb{C}^4}|_{\bar{X}} \longrightarrow \mathcal{O}_{\bar{X}} \longrightarrow T^1_{\bar{X}} \longrightarrow 0.$$

Here we note that (X, P_i) can be embedded into $(\mathbb{C}^4, 0)$ because P_i is an ordinary double point. By the isomorphism β, α is identified with the natural map $H^2_Z(T^0_{\bar{X}}) \longrightarrow H^2(T^0_{\bar{X}})$. In our case it is easily shown that $\pi_*\theta_X = T^0_{\bar{X}}$. Next using the Leray spectral sequences: $H^p_Z(R^q\pi_*\theta_X) \Rightarrow H^{p+q}_C(\theta_X)$ and $H^p(R^q\pi_*\theta_X) \Rightarrow H^{p+q}(\theta_X)$, we have $H^2_Z(T^0_{\bar{X}}) = H^2{}_C(\theta_X)$ and $H^2(T^0_{\bar{X}}) = H^2(\theta_X)$, which implies that the above map is identified with the following maps:

$$\begin{array}{ccc} H^2_C(\theta_X) & \longrightarrow & H^2(\theta_X) \\ \| & & \| \\ H^2_C(\Omega^2_X) & \xrightarrow{\theta} & H^2(\Omega^2_X), \end{array}$$

where the vertical identifications come from the fact that K_X is trivial. If the map θ is surjective, then we have $\mathbb{T}^2_{\bar{X}} = 0$. On the other hand, $H^0(T^1_{\bar{X}}) \simeq H^2_Z(T^0_{\bar{X}}) \simeq H^2_C(\Omega^2_X)$ are isomorphic to a n-dimensional vector space $\bigoplus_{i=1}^{n}\mathbb{C}$, where each factor corresponds to C_i. θ is nothing but the map which associates each basis of the above vector space to the fundamental class of C_i in X. Summing up these results, we have the following fact(1.1):

(1.1) *Let X be a Calabi-Yau 3-fold and $C_1, \ldots, C_n$ mutually disjoint (-1,-1)-curves on X. We employ the same notation as above. Then since $H^2(\Omega^2_X) = H^4(X, \mathbb{C}) = H_2(X, \mathbb{C})$ by the Hodge decomposition and the Poincare duality, the map θ can be identified with the map i_* : $\bigoplus_{i=1}^{n} H_2(C_i, \mathbb{C}) \longrightarrow H_2(X, \mathbb{C})$. In particular, if i_* is surjective and there is an element $(a_1, \ldots, a_n) \in \mathrm{Ker}\, i_*$ such that $a_i \neq 0$ for all i, then $\bar{X}$ is deformed to a smooth compact non-Kähler 3-fold with $K = 0$, $b_2 = 0$ and $q = 0$.*

A typical example of (1.1) is a general quintic hypersurface X in $\mathbb{P}^4$ and two lines on it. In this case, since $Pic(X) = \mathbb{Z}$, it is rather easy to check the conditions in (1.1). But in general it is very difficult to find the curves satisfying the condition in (1.1) even if a Calabi-Yau 3-fold X is given explicitly. In another sense, (1.1) tell us an interesting example where the class $\mathscr{C}$ is not stable by small deformation. In fact, $\bar{X}$ is a Moishezon space, hence is in class $\mathscr{C}$. But the non-Kähler 3-fold V obtained by a small deformation of $\bar{X}$ is not in class $\mathscr{C}$. This is shown as follows. First one has $h^{0,2}(V) = 0$, because $h^{0,1}(V) = 0$ and $K_V = 0$. If V is in class $\mathscr{C}$, then it is bimeromorphic to some compact Kähler manifold Y. Since $h^{0,2}(V) = 0$, $h^{0,2}(Y) = 0$. In fact, by the desingularization theorem [4], we have a complex manifold $\tilde{V}$ which dominates both V and Y, birationally and properly. Using spectral sequences and Chow lemma[5] for $(\tilde{V}, V)$ and $(\tilde{V}, Y)$, we have the result. But $h^{0,2}(Y) = 0$ implies that Y is a projective manifold. Since the algebraic dimension of V equals to 0, this is a contradiction. So V is not in class $\mathscr{C}$. Since $\kappa(\bar{X}) = 0$, this is a counter-example to a question posed in [2].

To return from the digression, we will explain the construction of the proof. First we define a Weierstrass model.

(1.2) Definition

A Weierstrass model $W(\mathscr{L},a,b)$ over a variety S is a closed subvariety in $\mathbb{P}_S(\mathcal{O} \oplus \mathscr{L}^2 \oplus \mathscr{L}^3)$ defined by the equation $Y^2Z = X^3 + aXZ^2 + bZ^3$, where $\mathscr{L} \in \mathrm{Pic}(S)$, $a \in H^0(S, \mathscr{L}^{-4})$, $b \in H^0(S, \mathscr{L}^{-6})$ and

$$Z:\ \mathcal{O} \longrightarrow \mathcal{O} \oplus \mathscr{L}^2 \oplus \mathscr{L}^3$$

$$X:\ \mathscr{L}^2 \longrightarrow \mathcal{O} \oplus \mathscr{L}^2 \oplus \mathscr{L}^3$$

$$Y:\ \mathscr{L}^3 \longrightarrow \mathcal{O} \oplus \mathscr{L}^2 \oplus \mathscr{L}^3$$

are natural injections.

We denote by Σ , the section of $W(\mathscr{L},a,b)$ over S defined by $X = Z = 0$, and denote by π the natural projection of $W(\mathscr{L},a,b)$ to S.

Next let X be a Calabi-Yau 3-fold which has an elliptic fibration with a rational section. Then by [8](Theorem 3.4), X is birational equivalent to a Weierstrass model $W = W(K_S, a, b)$ with only canonical singularities, where S is one of the following: $\mathbb{P}^2$, Σ_i $(0 \leq i \leq 12)$. This is a starting point of our proof. Since W has singularities in the case $S = \Sigma_i$ $(3 \leq i \leq 12)$ if we take a and b generally, we must set up the following definition.

(1.3) Definition

Let $W = W(K_S, a, b)$ be a Weierstrass model over $S = \Sigma_i$ $(3 \leq i \leq 12)$. Then W is called *general* if

(1) W has singularities only on $F = \{ p \in W ; p \in \pi^{-1}(D_0), X = Y = 0\}$, where D_0 is a negative section of S, and

(2) let mD_0 and nD_0 be the fixed components of $|K_S^{-4}|$ and $|K_S^{-6}|$, respectively, then $div(a) = G + mD_0$ and $div(b) = H + nD_0$, where G (resp. H) intersects D_0 transversely and $G.D_0 \cap H.D_0 = \phi$. Here $G.H_0$ (resp. $H.D_0$) denotes the intersection of D_0 and G (resp. H).

We will employ Definitions(1.2),(1.3). Let $W = W(\mathcal{L},a,b)$ be a Weierstrass model over S. Then W is obtained as a double cover of $\mathbb{P}_S(\mathcal{O} \oplus \mathcal{L}^2)$ branched over $B = \{ X^3 + aXZ^2 + bZ^3 = 0 \}$. If W has singularities, then we can use the following.

(1.4) Canonical Resolutions

Let Y be a smooth variety and B a reduced Cartier divisor on it. Assume that $\mathcal{O}_Y(B) = L^{\otimes 2}$ for a line bundle L on Y. Then we have a double cover X of Y branched along B. To resolve the singularities on X, we consider the following process:

Perform a succession of monoidal transformations ν_i $(1 \le i \le m)$ with smooth centres $D_i \subset B_i \subset Y_i$ $(1 \le i \le m)$, where $Y_1 = Y$, $B_1 = B$, $Y_{j+1} \xrightarrow{\nu_j} Y_j$ and $B_{j+1} = \nu_j^* B_j$ for each j. And if we write $B_m = \bar{B} + \sum_{k=1} \mu_k E_k$, where $\bar{B}$ is a proper transform of B by $\nu := \nu_m \circ .. \circ \nu_1$ and E_k's are ν-exceptional divisors, then $\bar{B} + \sum_{k,\ \mu_k:\mathrm{odd}} E_k$ is a smooth divisor. If the above process is possible, then we have a double cover branched along $\bar{B} + \sum_{i,\ \mu_i:\mathrm{odd}} E_i$ and obtain a smooth variety $\tilde{X}$ which is a resolution of X. We call the above process a canonical resolution.

Let $W = W(K_S, a, b)$ be a general Weierstrass model over $S = \Sigma_i (3 \le i \le 12)$ in the sense of Definition(1.3). Then we can perform a canonical resolution on W. In our case, it is easily verified that $\mathrm{Sing}(W) = \{ q \in \mathbb{P} \ ; \ q \in p^{-1}(D_0),\ X = 0 \ \ Y = 0 \}$, where $\mathbb{P} = \mathbb{P}_S(\mathcal{O} \oplus K_S^2 \oplus K_S^3)$, D_0 : negative section, and that the singularities are locally trivial deformations of a rational double points except for a finite number of points which are so-called *dissident* points. So the problem is how to overcome the difficulties which arise at these dissident points. For example, consider the case where $i = 5$. (In the case where i =3, 4, 6, 8, 12 there are no dissident points.) Since G and H never vanish simultaneously on a point q of Sing(W) in Definition(1.3), we may consider two cases: (1) only G vanishes at q , and (2) only H

vanishes at q. But it follows that q is dissident only in the case (2). So we may consider the situation where $q = (0,0,0,0)$, W: $y^2 = x^3 + t^3x + st^4$ in (x,y,s,t)-space$(=\mathbb{C}^4)$. Then the process of a canonical resolution will be found in (Figure 1). As a consequence, we have the following Proposition.

(1.5) Proposition

Let $W = W(K_S, a, b)$ be a Weierstrass model over S, where S is one of the following: $\mathbb{P}^2$, Σ_i $(0 \le i \le 12)$. Then:

(0) $K_W = \mathcal{O}_W$

(1) *In the case $S = \mathbb{P}^2$ or Σ_i $(0 \le i \le 2)$, a general Weierstrass model W is smooth and $Pic(W) = \pi^*Pic(S) \oplus \mathbb{Z}[\Sigma]$. Moreover W is simply-connected.*

(2) *In the case $S = \Sigma_i$ $(3 \le i \le 12)$, a general Weierstrass model W has canonical singularities such that $Sing(W) \simeq \mathbb{P}^1$ and that they are locally trivial deformations of rational double points except for finite number of points. Moreover W has a canonical resolution $\mu: \tilde{W} \longrightarrow W$ such that a) $\tilde{W} \longrightarrow S$ is a flat morphism if $3 \le i \le 8$ or $i = 12$, b) if we view $\tilde{W}$ and W as fibre spaces over $\mathbb{P}^1$ by means of the ruling $S \longrightarrow \mathbb{P}^1$, then, for a general point $t \in \mathbb{P}^1$, $\mu_t: \tilde{W}_t \longrightarrow W_t$ is a minimal resolution of a surface with rational double points, and c) $K_{\tilde{W}} = \mathcal{O}_{\tilde{W}}$.*

Remark. In the case where $9 \le i \le 11$, $\tilde{W}$ is not flat over S. But we can factiorize μ into $\tilde{W} \longrightarrow \bar{\bar{W}} \longrightarrow W$, where $\bar{\bar{W}}$ is a normal variety with the singularities which are locally trivial deformations of a rational double point of A_1-type along C_i $(1 \le i \le r)$, where C_i $(1 \le i \le r)$ denote mutually disjoint smooth rational curves on $\bar{\bar{W}}$. Moreover $\bar{\bar{W}}$ is *flat over S, and* $\tilde{W} \longrightarrow \bar{W}$ *is a trivial resolution of the above singularities. For details, see* (Figures 1,2).

(3) *For an arbitrary point* $t \in \mathbb{P}^1$ *except for a countable number of points,* $\tilde{W}_t$ *is naturally an elliptic K3 surface and its Mordell Weil group is trivial.*

(4) *Let* E_j $(1 \le j \le m)$ *be* μ*-exceptional divisors. Then* $Pic(\tilde{W}) = (\pi o \mu)^* Pic(S) \oplus \sum_{i=1}^{m} \mathbb{Z}[E_j]$.

(5) $\tilde{W}$ *is simply-connected.*

Theorem A'

Let W and $\tilde{W}$ *be a general Weiertsrass model and its resolution as above. Then we have:*

(1) *In the case* $S = \mathbb{P}^2$ *or* Σ_i $(0 \le i \le 2)$, *there are mutually disjoint* $(-1,-1)$*-curves* $C_1, .., C_4$ *on W such that* $i_* : \bigoplus_{i=1}^{4} H_2(C_i, \mathbb{C}) \longrightarrow H_2(W, \mathbb{C})$ *is surjective and that one can obtain, by the procedure of* (1.1), *a smooth compact non-Kähler 3-fold with* $K = 0$, $b_2 = 0$ *and* $q = 0$.

(2) *In the case* $S = \Sigma_i$ $(3 \le i \le 12)$, *there are mutually disjoint* $(-1,-1)$*-curves* $C_1, ..., C_{n(i)}$ *on the variety* $\tilde{W}'$ *which is obtained from* $\tilde{W}$ *by the composite of flops of* $(-1,-1)$*-curves . For* C_i*'s ,* $i_* : \bigoplus_{j=1}^{n(i)} H_2(C_j, \mathbb{C}) \longrightarrow H_2(\tilde{W}', \mathbb{C})$ *is surjective, and one can obtain , by the procedure of* (1.1), *a smooth compact non-Kähler 3-fold with* $K = 0$, $b_2 = 0$ *and* $q = 0$.

Example (without proof)

Set $S = \Sigma_0$. Let p: $S \longrightarrow \mathbb{P}$ denote one of its ruling. Then $p o \pi$: $W \longrightarrow \mathbb{P}^1$ is a K3-fibration. Let l_1 and l_2 be mutually distinct fibre of p, and let D_1 and D_2 be mutually distinct section of p with $(D_i)^2 = 0$. Note that $\pi^{-1}(l_i) \xrightarrow{g_i} l_i$ (i = 1,2) and $\pi^{-1}(D_i) \xrightarrow{h_i} D_i$ (i = 1,2) are elliptic $K3$ with canonical sections which comes from Σ, respectively. Let us consider the following four mutually disjoint (-1,-1)-curves:

C_1 : a section of g_1 with $(C_1 . \Sigma)_W = 0$

C_2 : a section of g_2 with $(C_2 . \Sigma)_W = 1$

C_3 : a section of h_1 with $(C_3 . \Sigma)_W = 0$

C_4 : a section of h_2 with $(C_4 . \Sigma)_W = 1$.

Then the condition in (1.1) is satisfied.

As for Theorem A' (in particular Theorem A'(2)), it is impossible to give a proof in this report. Details will be found in [10].

The aim of this report is to explain how to derive **Theorem** A from **Theorem** A'. Let X be a Calabi-Yau 3-fold which has an elliptic fibration with a rational section. Then , as is mentioned before, X is birational equivalent to a Weierstrass model W with only canonical singularities. But W is not general in the sense of Definition(1.3). Though W has only canonical singularities, its singularities are possiblly worse than the ones described in Definition(1.3). Let us consider the complete linear system $|\mathcal{L}|$ on $\mathbb{P}_S(\mathcal{O} \oplus K_S^2 \oplus K_S^3)$, where $\mathcal{L} = \mathcal{O}_{\mathbb{P}}(1) \otimes \pi^* K_S^{-6}$ and $\mathcal{O}_{\mathbb{P}}(1)$ is a tautological line bundle of $\mathbb{P}(\mathcal{O} \oplus K_S^2 \oplus K_S^3)$. Let Λ be a sublinear system of $|\mathcal{L}|$ which consists of the elements of the following form :

$$\varphi_1 Y^2 Z + \varphi_2 X^3 + \varphi_3 X Z^2 + \varphi_4 Z^3 = 0 ,$$

where φ_1, $\varphi_2 \in H^0(S, \mathcal{O}_S)$, $\varphi_3 \in H^0(S, K_S^{-4})$ and $\varphi_4 \in H^0(S, K_S^{-6})$. Then consider the universal family over $T = \mathbb{P}(\Lambda)$, $g\colon \mathfrak{W} \longrightarrow T$. Assume that $g^{-1}(t_0) = W$. If we choose a general point t on T, then $\mathfrak{W}_t = g^{-1}(t)$ has the property in Theorem A'. Let C be a curve in T passing through t_0 and t. Then we have a family of Weierstrass models over C, which we denote again by $g : \mathfrak{W} \longrightarrow C$. In the case where $S = \mathbb{P}^2$ or Σ_i $(0 \le i \le 2)$, a general fibre of g is smooth. But in the case where $S = \Sigma_i$ $(3 \le i \le 12)$, a general fibre has singularities by Proposition (1.5)(2). In this case we have the following proposition.

(1.6) Proposition

Let S be a surface isomorphic to Σ_i $(3 \le i \le 12)$ and C a curve. Consider the following flat family of Weierstrass models over S:

$$\begin{array}{ccccc}
\mathbb{P}_S(\mathcal{O} \oplus K_S^2 \oplus K_S^3) \times C & & & & \\
\cup & \searrow & & & \\
\mathfrak{W} & \longrightarrow & S \times C & & \\
& g \searrow & \swarrow p_1 & p_2 \searrow & \\
& & C & & S
\end{array}$$

$$\mathfrak{W}\colon Y^2 Z = X^3 + aXZ^2 + bZ^3 \quad , \quad a \in H^0(S \times C,\, p_2^* K_S^{-4}),$$
$$b \in H^0(S \times C,\, p_2^* K_S^{-6})$$

Assume that $\mathfrak{W}_t$ is general for every $t \in C$ except for a finite number of points $\{t_1, \ldots, t_n\}$. Then there is a projective resolution $\Xi\colon \tilde{\mathfrak{W}} \longrightarrow \mathfrak{W}$ such that $\Xi_t\colon \tilde{\mathfrak{W}}_t \longrightarrow \mathfrak{W}_t$ becomes the resolution in Proposition(1.5) for every $t \notin \{t_1, \ldots, t_n\}$.

So we have a flat projective morphism $\tilde{g} : \tilde{\mathcal{W}} \longrightarrow C$ whose general fibre is smooth. For a general point $t \in C$, $\tilde{\mathcal{W}}_t$ satisfies Theorem A'(2), that is, there is a sequence of flops of (-1,-1)-curves $D_j \subset \mathcal{W}_t^{(j)}$:

$$\begin{array}{ccccccc} \mathcal{W}_t^{(0)} & \dashrightarrow & \mathcal{W}_t^{(1)} & \dashrightarrow & \mathcal{W}_t^{(2)} \cdots\cdots & \dashrightarrow & \mathcal{W}_t^{(m)} \\ \| & & & & & & \| \\ \tilde{\mathcal{W}}_t & & & & & & \tilde{\mathcal{W}}'_t \end{array}$$

and there are (-1,-1)-curves on $\tilde{\mathcal{W}}'_t$ to be contracted. Let us consider the irreducible component H of $Hilb_{\tilde{\mathcal{W}}/C}$ which contains $[D_0]$. Note that $Hilb_{\tilde{\mathcal{W}}/C}$ is etale over C at $[D_0]$ because D_0 is a (-1,-1)-curve on $\tilde{\mathcal{W}}_t$. Hence H is determined uniquely, and H is etale over C at $[D_0]$. Taking a suitable finite cover of C, we may assume that H_{red} is birational to C. Then we have the following diagram:

$$\begin{array}{ccc} \mathcal{D} & \subset & \tilde{\mathcal{W}} \\ & \searrow \quad \swarrow & \\ & C & \end{array},$$

where $\mathcal{D}_t$ is a (-1,-1)-curve on $\tilde{\mathcal{W}}_t$ for every point $t \in C^*$: a Zariski open subset of C. Restrict $\tilde{g} : \tilde{\mathcal{W}} \longrightarrow C$ to C^*, and consider $\tilde{g}^* : \tilde{\mathcal{W}}^* \longrightarrow C^*$. Then by [1] (Corollary 6.10), we can perform a flop of $\mathcal{D}^*$ relatively over C^*, and get $\tilde{g}^{(1)*} : \tilde{\mathcal{W}}^{(1)*} \longrightarrow C^*$. Here $\tilde{\mathcal{W}}^{(1)*}$ is in general not a scheme, but an algebraic space. We can compactify $\tilde{\mathcal{W}}^{(1)*}$ by [11] and have a proper surjective map $\tilde{g}^{(1)}: \tilde{\mathcal{W}}^{(1)} \longrightarrow C$. $\tilde{\mathcal{W}}^{(1)}$ is assumed to be smooth by [4], and $\tilde{\mathcal{W}}^{(1)}$ is birational to $\tilde{\mathcal{W}}$ over C. Since $\tilde{\mathcal{W}}_{t_0}$ contains an irreducible component birational to W and both $\tilde{\mathcal{W}}$ and $\tilde{\mathcal{W}}^{(1)}$ are smooth, $\tilde{\mathcal{W}}^{(1)}_{t_0}$ also contains an irreducible component birational to W. As a consequence, by repeating this process, we may assume from the first that there are (-1,-1)-curves to be contracted on $\tilde{\mathcal{W}}_t$. In the case where $S = \Sigma_i (3 \le i \le 12)$, we consider $\tilde{g} : \tilde{\mathcal{W}} \longrightarrow C$ which is obtained by repeating above process. And in the case $S = \mathbb{P}^2$ or $\Sigma_i (0 \le i \le 2)$, we consider the original $g : \mathcal{W} \longrightarrow C$. Then we can use the following, if necessary, after a base change by a finite cover of C:

(1.7) Proposition

Let h: $\mathcal{Y} \longrightarrow C$ be a proper flat morphism with connected fibres of an irreducible smooth 4-dimensional algebraic space $\mathcal{Y}$ to a smooth curve C. Let $t_0 \in C$ be a fixed point and W an irreducible component of $\mathcal{Y}_{t_0}$. Assume that there is a proper flat family of curves in $\mathcal{Y}$:

$$\begin{array}{ccc} \mathcal{C}_i & \subset & \mathcal{Y} \\ {\scriptstyle \text{flat proper}} \searrow & & \swarrow \\ & C & \end{array} \qquad (1 \le i \le n)$$

such that for a general point $t \in C$, (1) $\mathcal{C}_{i,t} (1 \le i \le n)$ are mutually disjoint (-1,-1)-curves on $\mathcal{Y}_t$, (2) these curve satisfy the condition in (1.1), and (3) we can obtain from $\mathcal{Y}_t$ a non-Kähler 3-fold with $K = 0$, $b_2 = 0$ and $q = 0$ by the process in (1.1). Then there is a proper surjective map of a 4-dimensional complex manifold $\mathfrak{X}$ to a 1-dimensional disc Δ such that

1) *$f^{-1}(t)$ is a compact non-Kähler 3-fold with $K = 0$, $b_2 = 0$ and $q = 0$, for $t \in \Delta^*$,*

2) *$f^{-1}(0) = \sum_{i=1} W_i$ is a normal crossing divisor of $\mathfrak{X}$, and*

3) *W_0 is bimeromorphic to W.*

Proof) Let C^* be a suitable Zariski open subset in C. Then by [1](Corollary 6.10), we can contract $\mathcal{E}_i^*$'s on $\mathcal{Y}^*$ relatively over C^*, and obtain $\bar{\mathcal{Y}}^*$. We can compactify $\bar{\mathcal{Y}}^*$ and have proper flat map of a normal algebraic space $\bar{\mathcal{Y}}$ to C. Then $\bar{\mathcal{Y}}$ is birational to $\mathcal{Y}$ over C. Consider the function field K of $\mathcal{Y}$, and let v be a discrete valuation ring which corresponds to W. Let L be a suitable Galois extension of K, then the normalizations of $\mathcal{Y}$ and $\bar{\mathcal{Y}}$ in L become schemes by the argument of [11](proposition 1). Denote them by $\mathfrak{X}$ and $\bar{\mathfrak{X}}$, respectively. Then $\mathcal{Y}$ (resp. $\bar{\mathcal{Y}}$) is the quotient of $\mathfrak{X}$ (resp. $\bar{\mathfrak{X}}$) by the Galois group $G = Gal(L/K)$. Let $v_1, \ldots, v_k$ be the extension of v in L. Then each element $g \in G$ induces a permutation of v_i's. If g sends v_i to v_j , then we will write $j = g(i)$. By [7](P.153), for v_1 , there is a variety $\bar{\mathfrak{X}}_{v_1}$ birational to $\bar{\mathfrak{X}}$ such that (1) $\bar{\mathfrak{X}}_{v_1}$ is projevtive over $\bar{\mathfrak{X}}$, (2) $\bar{\mathfrak{X}}_{v_1} \supset \{ z \in \bar{\mathfrak{X}} ; \bar{\mathfrak{X}}$ is isomorphic to $\mathfrak{X}$ at $z \}$, and (3) if v_1 dominates a point y of $\bar{\mathfrak{X}}_{v_1}$ and a point y' on $\mathfrak{X}$, then $\mathcal{O}_y$ dominates $\mathcal{O}_{y'}$. In this case we may assume that $\bar{\mathfrak{X}}_{v_1}$ and $\bar{\mathfrak{X}}$ are isomorphic at every point except for points over $t_0 \in C$. We denote $\bar{\mathfrak{X}}_{v_1}$ by $\bar{\mathfrak{X}}_1$, and define $\bar{\mathfrak{X}}_g$ for each $g \in G$ by the following fibre product:

$$\begin{array}{ccc} \bar{\mathfrak{X}}_1 & \overset{\sim}{\longleftarrow} & \bar{\mathfrak{X}}_g \\ \downarrow & & \downarrow \\ \bar{\mathfrak{X}} & \underset{g_*^{-1}}{\overset{\sim}{\longleftarrow}} & \bar{\mathfrak{X}} \end{array} .$$

On the other hand, viewing $\bar{\mathfrak{X}}_g$'s as $\bar{\mathfrak{X}}$ - schemes, we have :

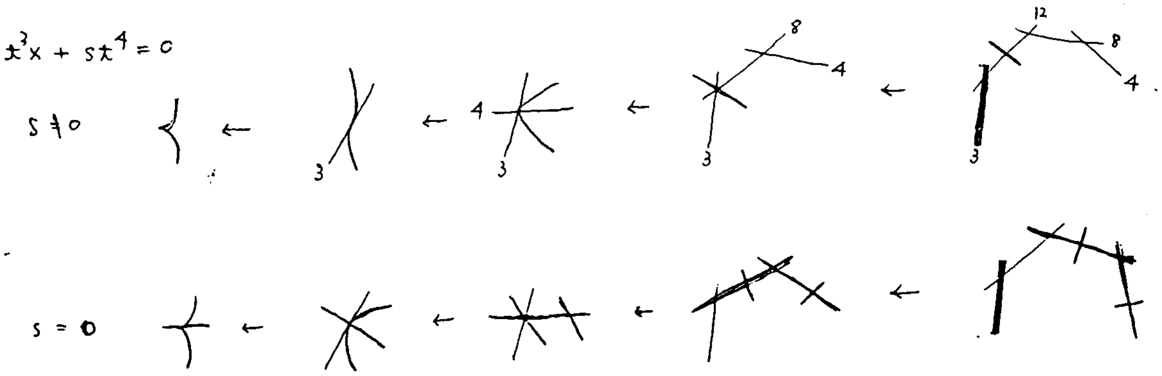
$$(*)\qquad \begin{array}{ccccccc} \bar{\mathfrak{X}}_1 & \overset{\text{birat.}}{\dashrightarrow} & \bar{\mathfrak{X}}_{g_1} & \dashrightarrow & \cdots\cdots & \dashrightarrow & \bar{\mathfrak{X}}_{g_l} \\ & \searrow & & \searrow & & \swarrow & \\ & & & \bar{\mathfrak{X}} & & & G = \{1, g_1, \ldots, g_l\}. \end{array}$$

Take a closure $\tilde{\mathfrak{X}}$ of the *graph* of (*) in $\prod_g \bar{\mathfrak{X}}_g$, and embed $\bar{\mathfrak{X}}$ into $\prod_g \bar{\mathfrak{X}}$ in such a way that $z \longrightarrow \prod_g z$. Then the natural projection from $\prod_g \bar{\mathfrak{X}}_g$ to $\prod_g \bar{\mathfrak{X}}$ induces a pojective morphism of $\tilde{\mathfrak{X}}$ to $\bar{\mathfrak{X}}$. Consider the action of G to $\prod_g \bar{\mathfrak{X}}$ defined in such a way that g sends g_i-th factor $\bar{\mathfrak{X}}$ of $\prod_g \bar{\mathfrak{X}}$ to gg_i-th factor $\bar{\mathfrak{X}}$ of $\prod_g \bar{\mathfrak{X}}$ and that this map of $\bar{\mathfrak{X}}$ to itself coincides with the natural g-action of $\bar{\mathfrak{X}}$. Clearly $\bar{\mathfrak{X}}$ is stable by this G-action, and this action coincides with the original G-action of $\bar{\mathfrak{X}}$. So, the natural G-action is induced on $\tilde{\mathfrak{X}}$. . If we take a normalization of $\tilde{\mathfrak{X}}$, then the action of G naturally extends. So we may assume that $\tilde{\mathfrak{X}}$ is normal. Then the quotient $\tilde{\mathfrak{Y}}$ of $\tilde{\mathfrak{X}}$ by G is an algebraic space by [6](P.183 l.8) [15], and we have a birational morphism of $\tilde{\mathfrak{Y}}$ to $\bar{\mathfrak{Y}}$. This morphism is an isomorphism over a general point t C. But by the construction, $\tilde{\mathfrak{Y}}_{t_0}$ contains an irreducible component birational to W. So, from the first, we may assume that $\bar{\mathfrak{Y}}_{t_0}$ has an irreducible component birational to W. Now let us consider the Kuranishi space $(\mathfrak{U}, u_0)$ of $\bar{\mathfrak{Y}}_{t_0}$, which is a complex space and has versal property at every point u near u_0 [13][14]. On the other hand, $\bar{\mathfrak{Y}}_t$ can be deformed to a non-Kähler 3-fold with $K = 0$, $b_2 = 0$ and $q = 0$ for every point t near t_0, which implies that ther is a flat deformation $f\colon \mathfrak{X} \longrightarrow \Delta$ such that $f^{-1}(0) = \bar{\mathfrak{Y}}_{t_0}$ and that $f^{-1}(t)$ is a non-Kähler 3-fold with $K = 0$, $b_2 = 0$ and $q = 0$. Then the semi-stable reduction for f is a desired one. *Q. E. D.*

Figure 1.

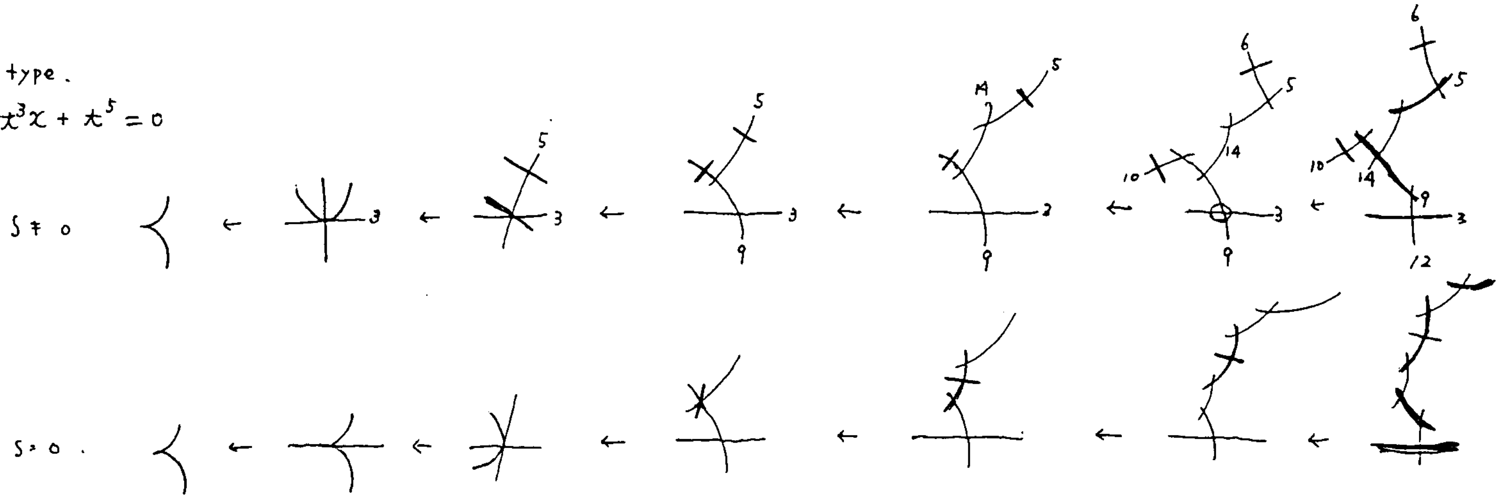
E_6-type

B: $x^3 + t^3 x + s t^4 = 0$



E_7-type.

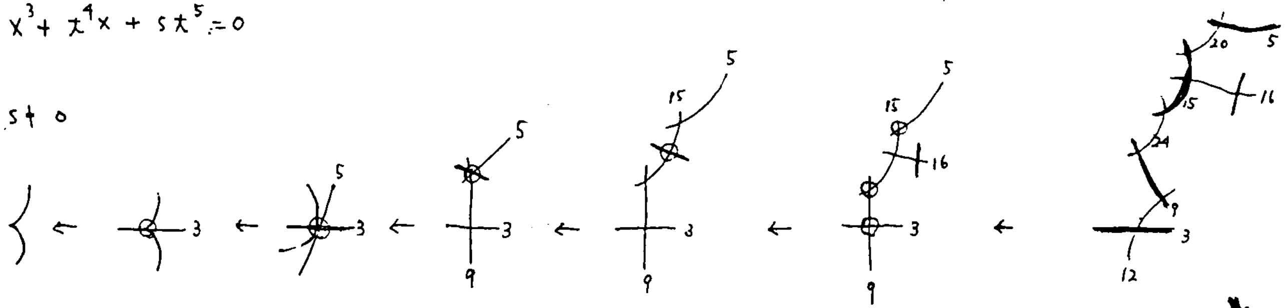
B: $x^3 + s t^3 x + t^5 = 0$



E_8 - type

B: $x^3 + t^4 x + s t^5 = 0$

$s \neq 0$



$s = 0$.

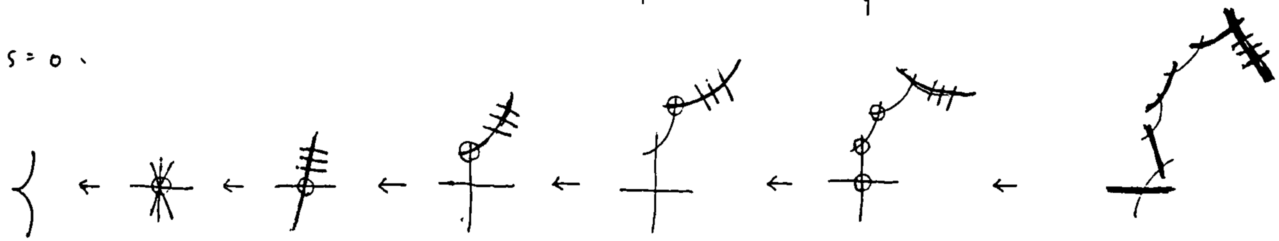
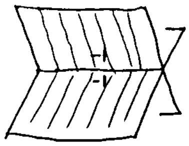
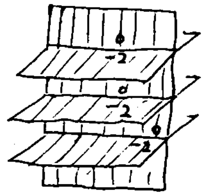


Figure 2. (μ-exceptional divisors).

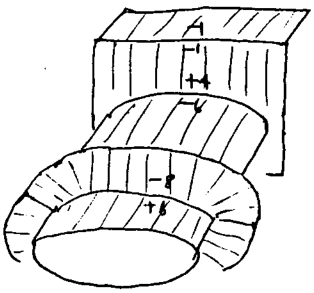
$i = 3$. (A_2-type).



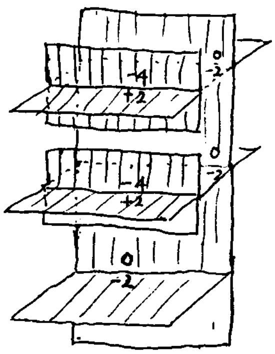
$i = 4$ (D_4-type).



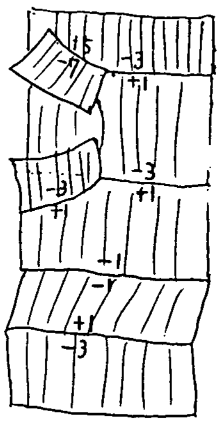
$i = 5$ (E_6-type).



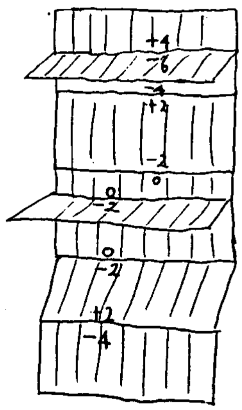
$i = 6$. (E_6-type).



$i = 7$. (E_7-type).

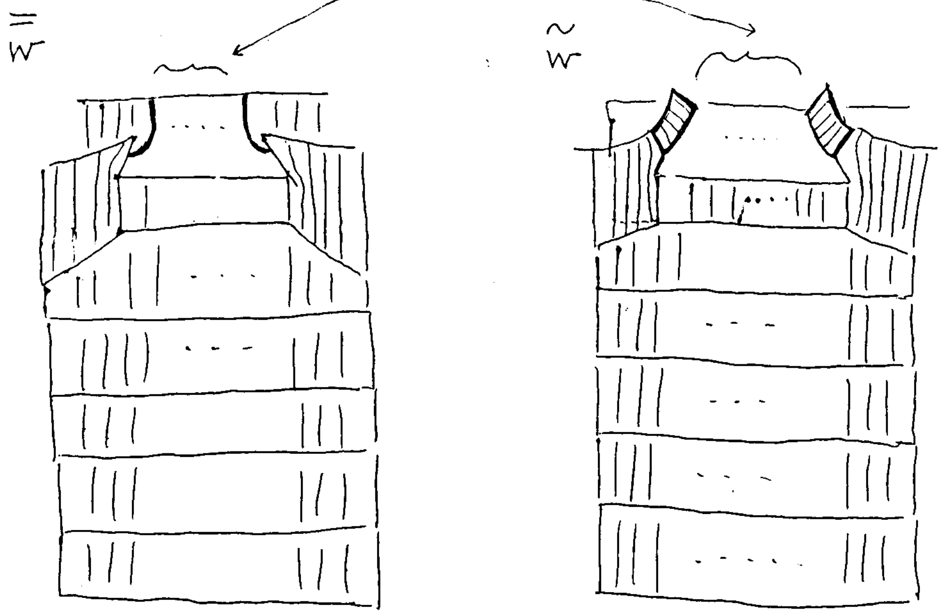


$i = 8$. (E_7-type).



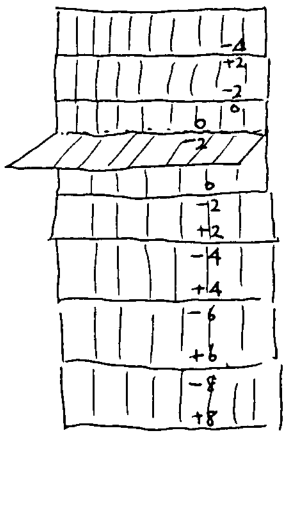
$i = 9, 10, 11.$ (E_8-type)

There are 3, 2 or 1 dissident points according to whether i = 9, 10 or 11.



$i = 12.$ (E_8-type)

$\overline{\overline{W}} = \tilde{W}$



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