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ADJOINT BUNDLES OF AMPLE AND SPANNED VECTOR BUNDLES ON ALGEBRAIC SURFACES

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0. Introduction

Here we carry out a brief survey of [Lanteri-Maeda 91].

The linear system $|K_X + C|$ "adjoint" to a curve C on a surface X has played an important role in understanding the geometry of X since the early days of surface theory. The adjoint bundle $K_X + L$ to a very ample line bundle L on a smooth complex projective surface X was investigated in modern terms by Sommese [Sommese 79] and Van de Ven [Van de Ven 79]. The study of $K_X + L$ was made in [Lanteri-Palleschi 84] when L is simply supposed to be an ample line bundle. Recently, several authors ([Fujita 90], [Wiśniewski 89], [Ye-Zhang 90]) have dealt with a generalized polarized pair $(X, \mathcal{E})$ consisting of a smooth complex projective variety X and an ample vector bundle $\mathcal{E}$ on X, and have investigated the nefness and the ampleness of the adjoint line bundle $K_X + \det \mathcal{E}$. In this note we treat an ample and spanned vector bundle $\mathcal{E}$ of rank $r (r \geq 2)$ on a smooth complex projective surface X, and study some properties of the adjoint bundle $K_X + \det \mathcal{E}$. Precisely, we ask the following

Questions.

(a) *When is* $K_X + \det \mathcal{E}$ *spanned ?*

(b) *When is $K_X + \det \mathcal{E}$ very ample ?*

We can obtain a complete answer to (a) by using Reider's method [Reider 88]. In fact, we will prove the

Theorem A. *Let $\mathcal{E}$ be an ample and spanned vector bundle of rank $r \geq 2$ on a smooth complex projective surface X. Set $L = \det \mathcal{E}$. Then $K_X + L$ is spanned unless $(X, \mathcal{E}) \cong (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}}(1)^{\oplus 2})$.*

The same method also enables us to give a partial but satisfactory answer to (b). The precise statement of our result is as follows:

Theorem B. *Let $\mathcal{E}$ be an ample and spanned vector bundle of rank $r \geq 2$ on a smooth complex projective surface X. Set $L = \det \mathcal{E}$ and assume $L^2 \geq 9$. Then $K_X + L$ is very ample unless $(X, \mathcal{E})$ is one of the following.*

(1) *X is a $\mathbf{P}^1$-bundle over a smooth curve C and $\mathcal{E}_F \cong \mathcal{O}_F(1)^{\oplus 2}$ for any fiber F of $X \longrightarrow C$.*

(2) $(X, \mathcal{E}) \cong (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}}(1)^{\oplus 3})$.

(3) $(X, \mathcal{E}) \cong (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}}(2) \oplus \mathcal{O}_{\mathbf{P}}(1))$.

(4) $(X, \mathcal{E}) \cong (\mathbf{P}^2, \mathcal{T}_{\mathbf{P}})$.

Note that this theorem proves the 2-dimensional part of the conjecture (2.6) in [Lanteri-Palleschi-Sommese 89] since $L^2 = 9$ in the three cases (2), (3) and (4). By the way we notice that the higher dimensional part of it should be restated in the following form.

Conjecture. *Let $\mathcal{E}$ be an ample and spanned vector bundle of rank $n \geq 3$*

on a smooth projective variety X of dimension n. Let $L = \det \mathcal{E}$ and assume $L^n \geq (n+1)^n + 1$. Then $K_X + L$ is very ample unless X is a $\mathbf{P}^{n-1}$-bundle over a smooth curve C and $\mathcal{E}_F \cong \mathcal{O}_F(1)^{\oplus n}$ for any fiber F of $X \longrightarrow C$.

In case $L^2 \leq 8$, we use the adjunction theory developed by Sommese and Van de Ven [Sommese-Van de Ven 87] to make an answer to (b) on the assumption that L is very ample. Our result is the

Theorem C. *Let $\mathcal{E}$ be an ample and spanned vector bundle of rank $r \geq 2$ on a smooth complex projective surface X. Set $L = \det \mathcal{E}$. Assume that L is a very ample line bundle with $L^2 \leq 8$. Then $K_X + L$ is very ample unless $(X, \mathcal{E})$ is one of the following.*

(1) $(X, \mathcal{E}) \cong (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}}(1)^{\oplus 2})$.

(2) *$X \cong Q^2$, a smooth hyperquadric in $\mathbf{P}^3$, and $\mathcal{E} \cong \mathcal{O}_Q(1)^{\oplus 2}$.*

(3) *$X \cong \mathbf{P}_C(\mathcal{F})$ and $\mathcal{E} \cong \rho^*\mathcal{G} \otimes H(\mathcal{F})$ for some indecomposable vector bundles $\mathcal{F}$ and $\mathcal{G}$ of rank two on an elliptic curve C with $c_1(\mathcal{F}) = c_1(\mathcal{G}) = 1$, where $H(\mathcal{F})$ is the tautological line bundle on X and ρ is the projection $X \longrightarrow C$.*

(4) *X is a Del Pezzo surface with $K_X^2 = 2$, and $\mathcal{E} \cong (-K_X)^{\oplus 2}$.*

(5) *X is as in case (4) and $\mathcal{E} \cong f^*\mathcal{F} \otimes (-K_X)$, where $f : X \longrightarrow \mathbf{P}^2$ is the blowing-up of $\mathbf{P}^2$ along seven points and $\mathcal{F}$ is the cokernel of a bundle monomorphism $\mathcal{O}_{\mathbf{P}}(-1)^{\oplus 2} \longrightarrow (\Omega^1_{\mathbf{P}} \otimes \mathcal{O}_{\mathbf{P}}(1))^{\oplus 2}$.*

We will work over the complex number field. Basically we use the standard notation from algebraic geometry. For a vector bundle $\mathcal{E}$ on X, the tautological line bundle on the projective space bundle $\mathbf{P}_X(\mathcal{E})$ associated to $\mathcal{E}$ is denoted by

$H(\mathcal{E})$. A vector bundle is called *spanned* if it is generated by its global sections.

1. Preliminaries

This note relies heavily on Reider's method, which we recall first in the following form.

Lemma 1. [Reider 88] *Let N be a nef line bundle on a smooth projective surface X.*

(1) *If $N^2 \geq 5$ and $K_X + N$ is not spanned, then there exists an effective divisor E satisfying either*

$NE = 0, E^2 = -1$ or $NE = 1, E^2 = 0$.

(2) *If $N^2 \geq 9$ and $K_X + N$ is not very ample, then there exists an effective divisor E satisfying one of the following conditions.*

$NE = 0, E^2 = -1$ or -2;

$NE = 1, E^2 = 0$ or -1;

$NE = 2, E^2 = 0$;

$N \equiv 3E, E^2 = 1$.

Second we use Wiśniewski's idea [Wiśniewski 89, Lemma 3.2] to obtain a result on ample and spanned vector bundles on curves.

Lemma 2. *Let $\mathcal{E}$ be an ample and spanned vector bundle of rank $r \geq 2$ on a projective curve C. Take arbitrary points $p_1, p_2, \cdots, p_{r-1}$ of C with $\mu_i = \mathrm{mult}_{p_i}(C)$.*

(1) *If C is rational, then $c_1(\mathcal{E}) \geq (\sum_{i=1}^{r-1} \mu_i) + 1$.*

(2) *If C is non-rational, then* $c_1(\mathcal{E}) \geq (\sum_{i=1}^{r-1} \mu_i) + 2$.

Corollary 1. *Let $\mathcal{E}$ be an ample and spanned vector bundle of rank $r \geq 2$ on a projective variety X. Put $L = \det \mathcal{E}$. Then X has no effective 1-cycles E such that $LE < r$.*

Corollary 2. *Let $X, \mathcal{E}$ and L be as above. If an effective 1-cycle E on X satisfies $LE = r$, then $E \cong \mathbf{P}^1$.*

We need also the following lemma.

Lemma 3. *Let $\mathcal{E}$ be an ample and spanned vector bundle of rank $r \geq 2$ on a smooth projective variety X of dimension $n \geq 2$. Then $H(\mathcal{E})^{n+r-1} \geq 3$.*

Furthermore we can prove a slight strengthening of Wiśniewski's theorem [Wiśniewski 89, Theorem 3.4] which will be used later on.

Lemma 4. *Let X be a smooth projective variety of dimension $n \geq 1$ and $\mathcal{E}$ an ample and spanned vector bundle on X of rank $r \geq n$. Assume $c_n(\mathcal{E}) = 1$. Then $(X, \mathcal{E}) \cong (\mathbf{P}^n, \mathcal{O}_{\mathbf{P}}(1)^{\oplus n})$.*

2. Proof of Theorem A

Theorem A. *Let $\mathcal{E}$ be an ample and spanned vector bundle of rank $r \geq 2$ on a smooth projective surface X. Set $L = \det \mathcal{E}$. Then $K_X + L$ is spanned unless $(X, \mathcal{E}) \cong (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}}(1)^{\oplus 2})$.*

Proof. Assume that $(X, \mathcal{E}) \not\cong (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}}(1)^{\oplus 2})$. Then we have $c_2(\mathcal{E}) \geq 2$ by

Lemma 4, since $c_2(\mathcal{E}) > 0$ by [Bloch-Gieseker 71]. Combining the formula $L^2 = c_2(\mathcal{E}) + H(\mathcal{E})^{r+1}$ with Lemma 3 gives $L^2 \geq 5$, so that Lemma 1 applies; but the exceptions to the spannedness of $K_X + L$ are excluded in view of Corollary 1, and we are done. Q.E.D.

3. Proof of Theorem B

Theorem B. *Let $\mathcal{E}$ be an ample and spanned vector bundle of rank $r \geq 2$ on a smooth projective surface X. Set $L = \det \mathcal{E}$ and assume $L^2 \geq 9$. Then $K_X + L$ is very ample unless $(X, \mathcal{E})$ is one of the following.*

(1) *X is a $\mathbf{P}^1$-bundle over a smooth curve C and $\mathcal{E}_F \cong \mathcal{O}_F(1)^{\oplus 2}$ for any fiber F of $X \longrightarrow C$.*

(2) $(X, \mathcal{E}) \cong (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}}(1)^{\oplus 3})$.

(3) $(X, \mathcal{E}) \cong (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}}(2) \oplus \mathcal{O}_{\mathbf{P}}(1))$.

(4) $(X, \mathcal{E}) \cong (\mathbf{P}^2, \mathcal{T}_{\mathbf{P}})$.

Proof. (outline) Assume that $K_X + L$ is not very ample. Then by Lemma 1 and Corollary 1, there exists an effective divisor E satisfying one of the following.

(i) $LE = 2, E^2 = 0$;

(ii) $L \equiv 3E, E^2 = 1$.

(3.1) In case (i), combining $LE = 2$ with Corollary 1 and Corollary 2 gives $r = 2$ and $E \cong \mathbf{P}^1$. Since $E^2 = 0$, X is ruled and E is a fiber of the ruling. We use Corollary 1 again to see that every fiber F is irreducible and reduced. Thus

X is a $\mathbf{P}^1$-bundle over a smooth curve C and $\mathcal{E}_F \cong \mathcal{O}_F(1)^{\oplus 2}$.

(3.2) In case (ii), E is ample and so E is irreducible and reduced. By Corollary 1 $LE = 3$ implies $r \leq 3$. If $r = 3$, then from Corollary 2, $E \cong \mathbf{P}^1$. By the classification theory of polarized surfaces of sectional genus zero [Lanteri-Palleschi 84, Corollary 2.3], we have two possibilities:

(3.2.1) $(X, \mathcal{O}_X(E)) \cong (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}}(i)), i = 1, 2$.

(3.2.2) $(X, \mathcal{O}_X(E))$ is a scroll over $\mathbf{P}^1$.

In case (3.2.1), $i = 1$ and $L = \mathcal{O}_{\mathbf{P}}(3)$. Consider the vector bundle $\mathcal{E} \otimes \mathcal{O}_{\mathbf{P}}(-1)$. This is trivial when restricted to any line in $\mathbf{P}^2$. Therefore itself is trivial, and hence $\mathcal{E} \cong \mathcal{O}_{\mathbf{P}}(1)^{\oplus 3}$. In case (3.2.2), we may assume $X = \mathbf{P}_{\mathbf{P}}(\mathcal{O}_{\mathbf{P}} \oplus \mathcal{O}_{\mathbf{P}}(-e))$ for some $e \geq 0$. Thus E is very ample. Since $E^2 = 1$, we have $(X, \mathcal{O}_X(E)) \cong (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}}(1))$. This is absurd.

(3.3) In the following we can assume $r = 2$. Then we can prove that the arithmetic genus $g(E) \leq 1$. Therefore the classification theory of polarized surfaces of sectional genus ≤ 1 applies.

(3.4) Now suppose $g(E) = 0$. Then the same argument as in (3.2) shows $(X, L) \cong (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}}(3))$, hence $\mathcal{E}$ is a uniform bundle of splitting type (2,1). By the classification theory of uniform bundles on $\mathbf{P}^2$ [Van de Ven 72], $\mathcal{E}$ is either the direct sum of two line bundles or the twisted tangent bundle. Consequently $\mathcal{E} \cong \mathcal{O}_{\mathbf{P}}(2) \oplus \mathcal{O}_{\mathbf{P}}(1)$ or $T_{\mathbf{P}}$.

(3.5) To complete the proof of Theorem B, we discuss the case $g(E) = 1$. There are two possibilities [Lanteri-Palleschi 84, Corollary 2.4]:

(3.5.1) X is a Del Pezzo surface and $\mathcal{O}_X(E) = -K_X$.

(3.5.2) $(X, \mathcal{O}_X(E))$ is a scroll over an elliptic curve C.

In case (3.5.1), $K_X^2 = 1$ and $L = -3K_X$. In case (3.5.2), we can write $X = \mathbf{P}_C(\mathcal{F})$ for some *normalized* vector bundle $\mathcal{F}$ of rank two on C. Moreover, $\mathcal{O}_X(E) = H(\mathcal{F}) + \rho^* A$ for some line bundle A on C, where ρ is the projection. Set $e = -c_1(\mathcal{F})$ and $a = \deg A$. Then $e \geq -1$ and $E^2 = 2a - e = 1$. By the criterion for an ample line bundle, we have $e = -1$ and $a = 0$. Thus $\mathcal{F}$ is indecomposable and $L = 3H(\mathcal{F}) + \rho^* B$ for some line bundle B of degree 0 on C. In sum, (X, L) is one of the following:

(1) X is a Del Pezzo surface with $K_X^2 = 1$, and $L = -3K_X$.

(2) $X \cong \mathbf{P}_C(\mathcal{F})$ for some indecomposable vector bundle $\mathcal{F}$ of rank two on an elliptic curve C with $c_1(\mathcal{F}) = 1$. $L = 3H(\mathcal{F}) + \rho^* B$ for some line bundle B of degree 0 on C, where ρ is the projection $X \longrightarrow C$.

However, we can show that neither (1) nor (2) occurs. Q.E.D.

For the proof of Theorem C, we refer to [Lanteri-Maeda 91].

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