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Rigidity Theorems on Spheres and Complex Projective Spaces

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ABSTRACT. This is a short report on two rigidity theorems concerning spheres. One is characterizing Euclidean spheres in terms of the lower bound of the sectional curvature and the length of the shortest closed geodesics. The other is a characterization of complex projective spaces as a smooth Kähler compactification of complex homology cells (which was proved by Van de Ven in dimension ≤ 5 and was conjectured by Brenton and Morrow in general dimensions).

0. Two Rigidity Theorems Concerning Spheres. This note is a report on two rigidity theorems in differential geometry recently obtained by Itokawa and the author [IK] and by the author [K2]:

THEOREM 1 ([IK]). *Let M be an n-dimensional complete Riemannian manifold whose sectional curvature K is bounded below by k^2 with $k > 0$ and the length of the shortest closed geodesics is equal to $\frac{2\pi}{k}$. Then M is isometric to the Euclidean sphere S^n_k of radius $\frac{1}{k}$ in R^{n+1}.*

THEOREM 2 ([K2]). *Let (X, D) be a pair of an n-dimensional compact complex manifold X and a smooth hypersurface D in X. Assume that X is Kähler, or, D is Kähler and X contains no exceptional subvarieties (i.e., subvarieties blown down to a point). Suppose $X - D$ is biholomorphic to a complex homology n-cell. Then (X, D) is biholomorphic to the hyperplane section $(P_n(C), P_{n-1}(C))$.*

Here a (noncompact) complex manifold Y is a *complex homology n-cell* iff $H_{2n-i}(Y, Z)$ $(= H^i_c(Y, Z)) = 0$ for $0 \leq \forall i \leq 2n-1$, where H^*_c denotes the cohomology groups with compact support.

Theorem 1 is completely Riemannian geometric and Theorem 2 is completely complex analytic. There is no logical relationship between two rigidity theorems. But the author wishes to report these results at the same time because he investigated these rigidity phenomena almost at the same time and, which is mathematically more interesting, *both theorems are concerned with characterizations of spheres (with additional structures).* Indeed, Theorem 1 characterizes spheres with a canonical metric structure in terms of the lower bound of sectional curvatures and the length of the shortest closed geodesics. It has a flavor similar to Obata's theorem (see [BGM]) which characterizes Euclidean spheres in terms of the lower bound of Ricci curvatures and the first eigenvalue of the Laplacian. Spheres are not apparent in Theorem 2. However, to prove Theorem 2, we will show that the tubular neighborhood S of D in X together with the standard S^1-action $S^1 \times S \to S$ is isotopic to the sphere of dimension $2n-1$ with the usual S^1-action. Therefore in the proof of Theorem 2 we will characterize odd dimensional spheres with the standard S^1-action, i.e., the Hopf fibration $S^{2n-1} \to P_{n-1}(C)$, from the complex analytical conditions given in Theorem 2. The conditions in Theorem 2 contain no explicit information on curvature and are completely complex analytical. Compare our conditions with curvature conditions in Siu-Yau's theorem [SY] (the positivity of the bisectional curvature on a compact Kähler manifold X implies that $X \cong P_n(C)$ complex analytically).

In this note we explain two examples of the (hopefully new) ideas characterizing spheres in Riemannian geometry and complex algebraic geometry.

Thanks are due to E. Sato for introducing the author to Van de Van-Brenton-Morrow's conjecture.

1. On Theorem 1. Some related rigidity phenomena were known previously. Tsukamoto [Ts] and Sugimoto [Su] proved:

Suppose that M^n satisfies $4k^2 \geq K \geq k^2$. If n is odd, assume that M is simply connected. Then if M has a closed geodesic of length $\frac{2\pi}{k}$, it is isometric to S^n_k.

It follows from Klingenberg's injectivity radius theorem (see [CE] and [Sa2]) that the curvature assumption in the above result and the simple connectivity of M implies that all closed geodesics on M have length $\geq \frac{\pi}{k}$. On the other hand, Fet [F] proved that the curvature assumption in Theorem 1 implies that there exists a closed geodesic on M whose length is $\leq \frac{2\pi}{k}$ and index $\leq n-1$. Note that the condition in the above result on closed geodesics is not the one on the *shortest* closed geodesics. Moreover the upper bound of the sectional curvature is not so natural from the point of view of rigidity theorems in Riemannian geometry. Indeed, for any given k and δ, there is a Riemannian metric on S^2 with $K \geq k^2$ and the length of the shortest closed geodesics δ-close to $\frac{2\pi}{k}$ but whose maximum curvature grows arbitrarily large. In the special case of dimension 2, Toponogov [T] proved

Suppose that M is an abstract surface satisfying $K \geq k^2$. If there exists on M a closed geodesic without self-intersections whose length is $\frac{2\pi}{k}$, then M is isometric to S^2_k.

The condition that the closed geodesics have no self-intersections is not removed. Indeed, for any $k > 0$ there exists an ellipsoid in R^3 which possesses a prime closed geodesic of length $\frac{2\pi}{k}$ and whose curvature is $> k^2$. On the other hand, we assume nothing on the self-intersections of the shortest closed geodesics. As a result, they have no self-intersections. The direct higher dimensional analogue of Toponogov's result does not hold. Indeed, there are lens spaces of constant curvature k^2 so that all geodesics are closed, the prime ones have no self-intersections and they are either homotopic to 0 and have length $\frac{2\pi}{k}$, or, homotopically nontrivial and can be arbitrarily short (see [Sa1]). Of course we have an equivariant version of Theorem 1:

COROLLARY. *If $K \geq k^2$ and the shortest closed geodesics that are homotopic to 0 in M have the length $\frac{2\pi}{k}$, then the universal covering of M must be isometric to S^n_k.*

Under a Ricci curvature assumption, Itokawa [I1,2] proved

If the Ricci curvature of M is $\geq (n-1)k^2$ and if the shortest closed geodesics on M have the length $\geq \frac{\pi}{k}$, then either M is simply connected or else M is isometric with the real projective space all of whose prime closed geodesics have length $\frac{\pi}{k}$.

PROBLEM. *Does Theorem 1 remain true when the assumption on the sectional curvature is weakened to that on the Ricci curvature Ricci $\geq (n-1)\ k^2$?*

This seems to be very difficult. In fact, Itokawa [I1,2] constructed examples so that, for the Ricci curvature assumption, the shortest closed geodesics may have length arbitrarily close to $\frac{2\pi}{k}$ without manifold's even homeomorphic to S^n.

Next we outline the idea of the proof of Theorem 1. For details, see [IK]. We apply the Morse theory to the loop space Ω of M (see [M]). Set $k = 2\pi$ and we characterize $S^n_{2\pi}$. Let $E(\gamma)$ (resp. $L(\gamma)$) be the energy functional (resp. the length functional). Then $L(\gamma)^2 \leq E(\gamma)$ with equality iff γ is parametrized proportional to arclength. Then the critical points of E on Ω are closed geodesics and the constant curves ($\cong M$). Let $\iota(\gamma)$ be the index of the closed geodesic γ. Put

$$\mathcal{C} := \{c \in \Omega;\ c, \text{ is a closed geodesic of length } 1 \text{ and } \iota(c) = n-1\}$$

and

$$\mathcal{C}^* := \{c \in \mathcal{C};\ \text{an unstable simplex of } E \text{ at } c \text{ represents a nontrivial element in } \pi_{n-1}(\Omega, M)\}$$

Fet's theorem and the Morse-Shoenberg index comparison [CE] imply that $\mathcal{C} \neq \emptyset$. In fact we have a stronger assertion:

LEMMA 1.1. *Under the assumption of Theorem 1, $\mathcal{C}^*$ is nonempty and is a closed set in Ω.*

For the proof of Lemma 1.1, we remark that the Morse-Shoenberg index comparison with $S^n_{2\pi}$ implies that M has the homotopy type of the sphere. Then we consider a finite dimensional approximation $'\Omega^{\leq r}$ (r sufficiently large) of the loop space $\Omega^{\leq r}$ and construct a sequence of functionals $\{E_i\}$ s.t. (i) E_i has only nondegenerate critical points in $'\Omega^{1-\epsilon<r<1+\epsilon}$ and (ii) $\lim_{i\to\infty} E_i = E$ in the C^2-topology. Applying the standard Morse theory to $('\Omega^{\leq r}, E_i)$ and taking the limit $i \to \infty$, we get Lemma 1.1.

Now the main step in the proof of Theorem 1 is to show

LEMMA 1.2. *Let $c \in C^*$. Then the set*

$$\mathcal{U}^* := \{u \in UT_{c(0)}M;\ c_u \in \mathcal{C}^*\}$$

is an open set in $UT_{c(0)}M$.

Here UTM denotes the unit tangent bundle of M and c_u denotes the geodesic with u the initial vector. The continuity method then implies that $U^* = UT_{c(0)}M$ and so we get a family in C^* of the shortest closed geodesics in M. The Morse-Shoenberg index comparison implies

LEMMA 1.3. *If* $c \in C$ *then for any* $s \in R$ *and any* $v \in T^{\perp}_{c(s)}c$ *we have* $K(c'(s) \wedge v) = (2\pi)^2$.

Thus we get a family of the shortest closed geodesics along which the curvature is equal to $(2\pi)^2$ which is sufficiently large to construct an explicit isometry of M to $S^n_{2\pi}$ as in Toponogov's maximum diameter theorem ([CE],[Sa2]).

We prove Lemma 1.2 by the Morse theory on Ω. Let $c \in C^*$. Then as in Lemma 1.3 we have

LEMMA 1.4. *Orthogonal Jacobi fields along c is of the form*

$$\textit{const.}\ \sin(2\pi s)V(s) \quad (0 \le s \le 1)$$

where $V(s)$ *is any parallel vector field of elements in* $U(T^{\perp}c|_{[0,1]})$.

Let $\{V_i(s)\}_{i=1}^{n-1}$ be parallel vector fields of orthonormal elements in $U(\perp c|_{[0,1]})$. *These may not close up at* $s = 1$ because the holonomy may not be trivial (in fact, the holonomy turns out to be trivial as we shall see later). Define $2(n-1)$ (discontinuous) vector fields along c:

$$X_i(s) = \begin{cases} V_i(s) & \text{if } 0 \le s \le \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} \le s \le 1 \end{cases}$$

and

$$Y_i(s) = \begin{cases} 0 & \text{if } 0 \le s \le \frac{1}{2} \\ V_i(s) & \text{if } \frac{1}{2} \le s \le 1. \end{cases}$$

Let $(x, y) = (x_1, \cdots, x_{n-1}, y_1, \cdots, y_{n-1}) \in R^{2(n-1)}$ run over a small interval $I \times I \in R^{2(n-1)}$ with center $0 \in R^{2(n-1)}$. Set

$$\tilde{\sigma}(x, y) = \exp_{c(s)}\{\sin(2\pi s)(\sum_{i=1}^{n-1}(x_i X_i(s) + y_i Y_i(s)))\}.$$

Note that the vector field inside exp is continuous. So this will form a $2(n-1)$-simplex in Ω.

REMARK. *If we consider this construction on the model space* $S^n_{2\pi}$, *we get a family of broken geodesics (with corners possibly at* $s = 0$ *and* $s = \frac{1}{2}$*) and these geodesics are smooth iff* $x = y$.

We construct a new $2(n-1)$-parameter family $\sigma(x, y)$ of loops by performing a short cut modification to loops with corners (and reparametrizing these by the arclength). Define an $(n-1)$-simplices τ_u and τ_0 by setting

$$\tau_u(x) = \sigma(x,-x) = \exp_{c(s)}\{\sin(2\pi s)(\sum_{i=1}^{n-1}(x_i X_i(s) - x_i Y_i(s)))\}$$

and

$$\tau_0(x) = \tau_0(x,x) = \exp_{c(s)}\{\sin(2\pi s)(\sum_{i=1}^{n-1} x_i V_i(s))\}.$$

These two simplices are transversal at the image of $x = y = 0$ in Ω. Every loop in τ_u has corners at $s = 0$ and $s = \frac{1}{2}$, and a loop in τ_0 has a corner at $s = 0$ if the holonomy is nontrivial. Applying Rauch's second comparison ([CE],[Sa2]), we infer that there is a neighborhood W of $c \in C^*$ in Ω and a positive number ε so that the $2(n-1)$-simplex *$\sigma \cap W$ is contained in the sublevel set $\Omega^{\leq 1}$* and the $(n-1)$-simplex *$\tau_u \cap W$ represents a nontrivial element in $\pi_{n-1}(W, W \cap \Omega^{\leq 1-\epsilon})$*, i.e., *$\tau_u$ is strictly unstable (this is the effect of the presence of nontrivial corners for loops in τ_u).* In particular $\tau \cap W$ cannot be deformed into $\Omega^{<1}$. Now it may be intuitively clear that *the holonomy along c must be trivial on $T^{\perp}_{c(0)}c$, every $\tau_0(x)$ has no corners and the $(n-1)$-simplex τ_0 rides on the level set $\Omega^{=1}$.* Otherwise τ_u may be deformed in W into $\Omega^{<1}$, which is a contradiction. It is now easy to get Lemma 1.2. The new idea in this argument may be the use of the simplex $\tilde{\sigma}(x,y)$ (consisting of "broken geodesics"). Such a simplex was first introduced by Araki in [A] when M is a symmetric space.

2. On Theorem 2. Van de Ven [V] proved Theorem 2 when $\dim X \leq 5$. Van de Ven's method is based on the Riemann-Roch theorem. Brenton and Morrow [BM] conjectured Theorem 2 in general dimensions (see also [PS]). From our point of view, Theorem 2 is a consequence of a general existence theorem for complete Ricci-flat Kähler metrics on certain class of affine algebraic manifolds. This motivates the study in [BK] but we could prove the existence theorem only under an additional condition, i.e., the Kähler-Einstein condition on the divisor at infinity. The existence theorem and its proof in [BK] found some applications ([B], [K1] and [Ye]) but it is too restrictive to be applied to problems in algebraic geometry. Generalizing previous results of [BK] and [TY] by removing the Kähler-Einstein condition at infinity, the author showed the following existence theorem:

EXISTENCE THEOREM ([K2]). *Let X be a Fano manifold and D a smooth hypersurface in X such that $c_1(X) = \alpha[D]$ with $\alpha > 1$. Then $X - D$ admits a complete Ricci-flat Kähler metric.*

To apply this existence theorem to problems in algebraic geometry, we need to know the analytical properties of the resulting metric. This may be described as follows. In the following argument, we always assume that $\dim_C X = n > 1$. As $c_1(X) > 0$, there exists a Hermitian metric for $O_X(D)$ with positive curvature form $\theta > 0$. Let σ be a defining section of $O_X(D)$. Then $\theta = \sqrt{-1}\partial\bar{\partial}t$ with $t = \log \frac{1}{\|\sigma\|^2}$ and

$$\begin{aligned}\omega &= \frac{n}{\alpha - 1}\sqrt{-1}\partial\bar{\partial}\left(\frac{1}{\|\sigma\|^2}\right)^{\frac{\alpha-1}{n}} \\ &= \left(\frac{1}{\|\sigma\|^2}\right)^{\frac{\alpha-1}{n}}\left(\theta + \frac{\alpha-1}{n}\sqrt{-1}\partial t \wedge \bar{\partial} t\right)\end{aligned}$$

defines a complete Kähler metric on $X - D$. Then the Kähler metric $\tilde{\omega}$ in Theorem 2 is obtained by the deformation of ω as follows:

$$\tilde{\omega} = \omega + \sqrt{-1}\partial\bar{\partial}u$$

where u satisfies the *a priori* estimates:

$$\|\nabla_\omega^k u\| \leq C_k \left\{\left(\frac{1}{\|\sigma\|^2}\right)^{\frac{\alpha-1}{2n}}\right\}^{2-k}$$

for $Z \ni \forall k \geq 0$, where ∇_ω is the Levi-Civita connection of ω. In particular, $|u \, \|\sigma\|^{2\frac{\alpha-1}{n}}|$ is bounded above by an *a priori* constant, or in other words, u is at most of quadratic growth relative to the distance function for ω and $\|\nabla_\omega^k u\|$ decays like $dist(o,*)^{2-k}$. Hence the Kähler metrics $\tilde{\omega}$ and ω are equivalent:

$$C\omega < \tilde{\omega} < C^{-1}\omega$$

holds with $C > 0$ an *a priori* constant and geometric properties of ω (at infinity) approximates those of $\tilde{\omega}$. Set

$$\tilde{u} = \frac{n}{\alpha-1}\left(\frac{1}{\|\sigma\|^2}\right)^{\frac{\alpha-1}{n}} + u.$$

Then $\tilde{u}$ is a Kähler potential for a complete Ricci-flat Kähler metric on $X - D$ which is equivalent to the squared distance function from a fixed point in $X - D$.

Now let (X, D) be as in Theorem 2. Then Brenton-Morrow [BM] proved the following

LEMMA 2.1. *Let (X, D) be as in Theorem 2. Then X is a Fano manifold (hence projective algebraic) and $c_1(X) = \alpha[D]$ with $\alpha > 1$. Moreover there is a smooth map $\psi : X \to P_n(C)$ taking D into a hyperplane $P_{n-1}(C)$ which induces ring isomorphisms*

$$\psi^* : H^*(P_n(C), Z) \to H^*(X, Z)$$

$$\psi_D^* : H^*(P_{n-1}(C), Z) \to H^*(D, Z).$$

Hence (X, D) in Theorem 2 satisfies the conditions in the Existence Theorem. Therefore $X - D$ admits a complete Ricci-flat Kähler metric $\tilde{\omega} = \sqrt{-1}\partial\bar{\partial}\tilde{u}$ which has properties described above. Write $c_i(\omega)$ (resp. $p_i(\omega)$) for the i-th Chern form (resp. the i-th Pontrjagin form) computed from the Kähler metric ω. Now we look at the following "equality" (both sides may diverge):

$$\int_{X-D} c_2(\tilde{\omega}) \wedge \tilde{\omega}^{n-2} = \int_{X-D} c_2(\theta) \wedge \tilde{\omega}^{n-2} + \int_{X-D} (c_2(\tilde{\omega}) - c_2(\theta)) \wedge \tilde{\omega}^{n-2}.$$

By this equality, we compare the growth of these curvature integrals. We consider the secondary characteristic class on large geodesic balls in the computation of the second term in the right side. Since $\tilde{\omega}$ is a Ricci-flat Kähler metric, we have

$$\int_{X-D} c_2(\tilde{\omega}) \wedge \tilde{\omega}^{n-2} \geq 0.$$

(Note that this property is used in the proof of the fact that a compact Kähler manifold with $c_1 = c_2 = 0$ is covered holomorphically by a complex torus.) We can compute (the growth of) the integrals in the right hand side of the above "equality" explicitly. Since $\tilde{\omega}$ is a Ricci-flat Kähler metric, there occurs no change in the left hand side if we replace c_2 by $c_2 - \frac{1}{2}c_1^2 = -\frac{1}{2}p_1$. We thus have

$$0 \leq -\int_{X-D} p_1(\theta) \wedge \tilde{\omega}^{n-2} + \int_{X-D} (-p_1(\tilde{\omega}) + p_1(\theta)) \wedge \tilde{\omega}^{n-2}.$$

From Lemma 2.1 and [MS, Lemma 20.2, pp. 232-233] (the theory of the combinatorial Pontrjagin classes), we infer that the the growth rate of the first integral in the right side is given by the Pontrjagin number $-\frac{1}{2}(p_1(P_n(C)) \cup h^{n-2})([P_n(C)])$, where h is the positive generator of $H^2(P_n(C))$. Thus, the above inequality amounts to the following surprising estimate on α (recall that $c_1(X) = \alpha[D]$):

$$\alpha \geq n + 1.$$

Indeed, the first term in the right side is computed on $P_n(C)$ and α appears in the second term with a positive coefficient. Now we recall Kobayashi-Ochiai's characterization of complex projective spaces [KO]: If $\alpha \geq n+1$ then X is biholomorphic to $P_n(C)$. We thus have $(X, D) = (P_n(C), P_{n-1}(C))$, i.e., the hyperplane section. We can even prove Kobayashi-Ochiai's characterization [KO] using complete Ricci-flat Kähler metrics [K2]. Indeed, since $\alpha = n+1$, we can construct n nontrivial holomorphic functions $(z_1, \cdots, z_n)$ on $X - D$ with at most linear growth (with respect to the distance function of the metric $\tilde{\omega}$). These holomorphic functions will give an isomorphism

$$z = (z_1, \cdots, z_n) : X - D \to C^n$$

and $|dz_1 \wedge \cdots \wedge dz_n|^2$ coincides with the volume form $\tilde{\omega}^n$ (after a scale change). We thus have

LEMMA 2.2. *There exists holomorphic functions $z_1, \cdots, z_n$ on $X - D$ which give an isomorphism $X - D \cong C^n$ and the Ricci-flat Kähler potential $\tilde{u}$ grows like $|z_1|^2 + \cdots + |z_n|^2$.*

Thus $\tilde{u}$ satisfies the equation

$$\det\left(\frac{\partial^2 \tilde{u}}{\partial z_i \partial \bar{z}_j}\right) = 1$$

and $\tilde{u}$ grows like a squared distance function of the standard flat metric on C^n. Using Calabi's third order estimate [C] (see also [Au]), we infer that $\tilde{u}$ is in fact a quadratic function and thus $\tilde{\omega}$ turns out to be a flat metric on C^n. It follows from this and the definition of $\tilde{u}$ that any tubular neighborhood S of D in X is diffeomorphic to the sphere S^{2n-1} and that the natural S^1-action (induced from the complex structure) on S is isotopic to that on the Hopf fibration $S^{2n-1} \to P_{n-1}(C)$. It follows that D is diffeomorphic to $P_{n-1}(C)$ and (X, D) is diffeomorphic to the hyperplane section $(P_n(C), P_{n-1}(C))$. Finally, Hirzebruch-Kodaira's characterization of $P_n(C)$ ([HK], see also [Y1]) implies that (X, D) is biholomorphic to the hyperplane section $(P_n(C), P_{n-1}(C))$.

We now outline the proof of the Existence Theorem. The geometric idea is this: We consider the family $\{\gamma_\varepsilon\}$ of the Chern forms of $O_X(D) = K_X^{-\frac{1}{\alpha}}$ such that the support of γ_ε concentrates along D in the limit $\varepsilon \to 0$. Then we solve the complex Monge-Ampère equations (the prescribed Ricci form equations) $\{E_\varepsilon\}$ under suitable scaling conditions. Yau's solution to Calabi's conjecture [Y1] implies that there exists a unique solution at each stage. We introduce suitable weight functions and derive uniform weighted C^0 and C^2 estimates for solutions of $\{E_\varepsilon\}$. Finally we take the limit $\varepsilon \to 0$ to get a complete Ricci-flat Kähler metric $\tilde{\omega} = \omega + \sqrt{-1}\partial\bar{\partial}\tilde{u}$. The weighted C^0 estimates and their limit imply that $\tilde{u}$ is at most of quadratic growth relative to the metric ω.

We consider the family of smooth Kähler metrics on X defined by

$$\omega_\varepsilon = \left(\frac{1}{||\sigma||^2 + \varepsilon}\right)^{\frac{\alpha-1}{n}} \left(\theta + \frac{\alpha - 1}{n} \frac{||\sigma||^2}{||\sigma||^2 + \varepsilon} \sqrt{-1}\partial t \wedge \bar{\partial} t\right).$$

It is easy to see that $[\omega_\varepsilon] \propto c_1(X)$ and $\lim_{\varepsilon \to 0} \omega_\varepsilon = \omega$. Let V be a Ricci-flat volume form on $X - D$ with poles of order 2α along D and set

$$V_\varepsilon = \left(\frac{||\sigma||^2}{||\sigma||^2 + \varepsilon}\right)^\alpha V.$$

By a suitable scale change, we may assume that

$$\int_X V_\varepsilon = \int_X \omega_\varepsilon^n.$$

Define f_ε by $f_\varepsilon = \log \frac{\omega_\varepsilon^n}{V_\varepsilon}$. Then $\lim_{\varepsilon\to 0} f_\varepsilon = \frac{\omega^n}{V}$. We introduce the following family of the complex Monge-Ampère equations on X with weighted normalization conditions:

$$(E_\varepsilon) \qquad \begin{cases} (\omega_\varepsilon + \sqrt{-1}\partial\bar\partial u_\varepsilon)^n \; (= V_\varepsilon) \; = \; e^{-f_\varepsilon}\omega_\varepsilon^n, \\ \displaystyle\int_X \frac{u_\varepsilon}{\phi_\varepsilon}\omega_\varepsilon^n = 0. \end{cases}$$

Here, ϕ_ε is a smooth weight function which is approximately a squared distance function relative to ω_ε from a fixed point (independent of ε) in $X - D$. By Yau's solution to Calabi's conjecture [Y1], the above equation has a unique solution u_ε for a fixed ε. We want to prove that there is a constant $C > 0$ such that

$$\|\frac{u_\varepsilon}{\phi_\varepsilon}\|_{C^0} \le C$$

holds for all sufficiently small ε. The existence of the Sobolev inequalities with a uniform constant is most important in doing so. Set $\gamma = \frac{n}{n-1}$. We then have

LEMMA 2.3 (CF. [L]). *There exists a constant $c > 0$ independent of (sufficiently small) ε such that for each ε the Sobolev inequality*

$$\left(\int_X |f|^{2\gamma}\omega_\varepsilon^n\right)^{\frac{1}{\gamma}} \le c \int_X |df|^2_{\omega_\varepsilon}\omega_\varepsilon^n + Vol(\omega_\varepsilon)^{-\frac{1}{n}} \int_X |f|^2\omega_\varepsilon^n$$

holds for all C^1-functions f on X.

Deriving weighted *a priori* estimates independent of ε is quite complicated. Details can be found in [K2]. The outline is as follows. We use the *continuity method* in the following way. Replacing f_ε by τf_ε with $0 \le \tau \le 1$, we get a two parameter family of complex Monge-Ampère equations $\{E_{\varepsilon,\tau}\}$. Let $\mathcal{C} \subset [0,1]$ be a set of τ such that the solutions $u_{\varepsilon,\tau}$ have weighted C^0 estimates uniform relative to ε. Clearly $0 \in \mathcal{C}$. Showing the openness is reduced to a linear problem. Here we only mention the following two remarks: (i) the C^0 estimate needed for the proof of the openness is shown by the argument in the C^0 estimate in the proof of the closedness (cf. [BK]), and (ii) For the C^2 estimate, we will use Cheng-Yau's gradient estimate [CY, Theorem 6] and the standard Schauder estimates. The main difficulty lies in showing the closedness. We need uniform weighted *a priori* estimates for the two-parameter family of Monge-Ampère equations. In the following argument, we set $\tau = 1$. Combining the nonlinear equation

$$(1 - e^{-f_\varepsilon})\omega_\varepsilon^n = (-\sqrt{-1}\partial\bar\partial u_\varepsilon) \wedge (\sum_{i=1}^{n-1} \omega_\varepsilon^{n-1-i}\tilde\omega_\varepsilon^i)$$

($\tilde{\omega}_\varepsilon = \omega_\varepsilon + \sqrt{-1}\partial\bar{\partial}u_\varepsilon$) and the Sobolev inequality on (X, ω_ε)

$$\left(\int_X |\frac{u_\varepsilon}{\phi_\varepsilon}|^{p\gamma}\right)^{\frac{1}{\gamma}} \le c\int_X |\partial|\frac{u_\varepsilon}{\phi_\varepsilon}|^{\frac{p}{2}}|^2 + \mathrm{Vol}(\omega_\varepsilon)^{-\frac{1}{n}}\int_X |\frac{u_\varepsilon}{\phi_\varepsilon}|^p,$$

we have

$$\text{(1)}\qquad \begin{aligned}\left(\int_X |\frac{u_\varepsilon}{\phi_\varepsilon}|^{p\gamma}\right)^{\frac{1}{\gamma}} &\le cp\int_X |\frac{u_\varepsilon}{\phi_\varepsilon}|^{p-1}\frac{|1-e^{-f_\varepsilon}|}{\phi_\varepsilon} \\ &\quad + cpa_n\int_X |\frac{u_\varepsilon}{\phi_\varepsilon}|^p\frac{1}{\phi_\varepsilon} + c\int_X |\frac{u_\varepsilon}{\phi_\varepsilon}|^p\frac{1}{\phi_\varepsilon}\end{aligned}$$

on (X, ω_ε). Here $a_n = a^{2n(n-1)}$ and a is a constant such that $tr_{\tilde{\omega}_\varepsilon}\omega_\varepsilon \le a$. Of course a should be estimated independently. Although the above inequality involves an unknown constant a, we are able to derive an *a priori* C^0 estimate for $\frac{u_\varepsilon}{\phi_\varepsilon}$ in the following way. Let fix an ε. We choose a sequence of weight functions $\{\phi_\varepsilon(i)\}_{i=0}^\infty$ in the following way:

$$\phi_\varepsilon(i) \approx \begin{cases} \left(\frac{D_\varepsilon}{2^i}\right)^2, & \text{if dist}\,(o,*) \le \frac{D_\varepsilon}{2^i} \\ dist\,(o,*)^2, & \text{if dist}\,(o,*) \ge \frac{D_\varepsilon}{2^i}.\end{cases}$$

Here D_ε denotes the diameter of (X, ω_ε). First of all we let $\phi = \phi_\varepsilon(0) = D_\varepsilon$ (constant weight function). Then we have no second term in the right hand side of (1) (but we do have the third term). Set $0 < v_n = \sup_\varepsilon \frac{D_\varepsilon^{2n}}{\mathrm{Vol}(X,\omega_\varepsilon)} < \infty$ and $a' = \sup|\frac{u_\varepsilon}{D_\varepsilon^2}|$. Then (1) becomes

$$\text{(2)}\qquad \begin{aligned}\left(\int_X |\frac{u_\varepsilon}{\phi_\varepsilon}|^{p\gamma}\right)^{\frac{1}{\gamma}} &\le cp\int_X |\frac{u_\varepsilon}{\phi_\varepsilon}|^{p-1}\frac{|1-e^{-f_\varepsilon}|}{\phi_\varepsilon} + c\int_X |\frac{u_\varepsilon}{\phi_\varepsilon}|^p\frac{1}{\phi_\varepsilon} \\ &\le cp\int_X |\frac{u_\varepsilon}{\phi_\varepsilon}|^{p-1}\left(\frac{|1-e^{-f_\varepsilon}|}{\phi_\varepsilon} + \frac{a'}{p}\right)\end{aligned}$$

if $\phi = \phi_\varepsilon(0)$. We use the following well-known inequality:

$$\text{(3)}\qquad px^{p-1}y \le \lambda(p-1)x^p + \lambda^{1-p}y^p$$

valid with any positive numbers x, y and λ. We will use this inequality to the right hand side of (2) with

$$\text{(4)}\qquad x = |\frac{u_\varepsilon}{\phi}|, \quad y = \frac{|1-e^{-f_\varepsilon}|}{\phi} \text{ and } \lambda = \frac{1}{2v_n cpD_\varepsilon^2},$$

where we determine p by setting

(5) $$p = p_\varepsilon = \log D_\varepsilon$$

for a fixed ε (note that $p \to \infty$ as $\varepsilon \to 0$). Here $n = \dim_C X > 1$. We then have from (2),(3),(4) and (5) the folowing estimate:

$$\|\frac{u_\varepsilon}{\phi}\|_p \leq C \log D_\varepsilon$$

for $p_\varepsilon \gamma \leq p \leq \infty$. This seems to be very unsatisfactory because the right hand side becomes infinity as $\varepsilon \to 0$. But we are able to improve this estimate step by step by means of the weighted normaliozation condition $\int u_\varepsilon/\phi = 0$, the inequality (2) (which is essentially the consequence of the Sobolev inequality and the Monge-Ampère equation $(E)_\varepsilon$) and Young's inequality with approapriately chosen λ at each step. Applying Moser's iteration technique (this gives C^0-estimate from a given L^p-estimate for solutions of differential inequalities of some kind; see, for instance, [BK]), we will get an *a priori* C^0-estimate:

$$\|\frac{u_\varepsilon}{\phi}\|_{C^0} \leq C$$

as desired. Continuing the similar arguments with weight functions ϕ_i and estimating the sum of errors arising from the normalization conditions in $(E)_\varepsilon$ with different choice of weight functions, we finally get an *a priori* C^0-estimate for $\frac{u_\varepsilon}{\phi_\varepsilon}$. Imposing weighted normalization conditions in $(E)_\varepsilon$ is essential in showing that the sum of errors converges to some finite number independent of ε. As there exists a uniform Sobolev constant for Sobolev inequalities on (X, ω_ε) (Lemma 2.3), the above estimate is independent of sufficiently small ε. Inductions step goes as follows. The first step in particular gives an *a priori* C^0 estimate for $\frac{u_\varepsilon}{\phi_\varepsilon(1)}$ in the region $\mathrm{dist}(o, *) \leq \frac{D_\varepsilon}{2}$. Next we set $\phi = \phi_\varepsilon(1)$. In the region $\mathrm{dist}(o, *) \geq \frac{D_\varepsilon}{2}$, we already have a good weighted *a priori* C^0 estimate. This time we argue as above and get a good weighted *a priori* C^0 estimate in the region $\mathrm{dist}(o, *) \geq \frac{D_\varepsilon}{4}$. Iterating this process about $\log_2 D_\varepsilon$-times, we get a desired C^0 estimate for $\frac{u_\varepsilon}{\phi_\varepsilon}$ (although we have errors coming from the normalization process with different weight functions, the sum of all errors remain bounded above by a constant independent of ε):

$$\|\frac{u_\varepsilon}{\phi_\varepsilon}\|_{C^0} \leq C(\text{Sobolev const.})(a_n^{2n(n-1)})^{\frac{1}{\log D_\varepsilon}} \sup|f_\varepsilon|$$

where D_ε denotes the diameter of (X, ω_ε).

We now proceed to showing *a priori* estimates for $\mathrm{tr}_{\tilde{\omega}_\varepsilon} \omega_\varepsilon$ (which impliy the estimates for the second order derivatives of mixed type). Let $m > 0$ be a large integer. If we put $K = -(m+1)a' < 0$ with $a' = \sup|\frac{u_\varepsilon}{\phi_\varepsilon}$, we get $\frac{m+2}{m+1}K \leq \frac{u_\varepsilon + K\phi_\varepsilon}{\phi_\varepsilon} \leq \frac{m}{m+1}K < 0$. Set

$u'_\varepsilon = u_\varepsilon + KG_\varepsilon$, where G_ε is the Kähler potential for ω_ε defined by

$$\int \left(\frac{1}{e^{-t}+\varepsilon}\right)^{\frac{\alpha-1}{n}} dt$$

with $t = \log \frac{1}{\|\sigma\|^2}$ with a suitable normalization. We then have

$$\frac{m+2}{m+1} KG_\varepsilon \leq u'_\varepsilon \leq \frac{m}{m+1} K\phi_\varepsilon < 0.$$

Let $\delta = \frac{1}{2N+1}$ with $N > 0$ a large positive integer. Then $u'^{\delta}_\varepsilon < 0$, i.e., the negative $(2N+1)$-st root of $u'_\varepsilon < 0$ is well defined. Set

$$\rho_0 = \left(\frac{n}{\alpha-1}\right)^{\frac{1}{2}} \exp\left(\frac{\alpha-1}{2n}\right).$$

Then $\psi_0 = \rho_0^2$. Direct computation shows

$$\begin{aligned}
\Delta_{\tilde{\omega}_\varepsilon}\left(\frac{u'_\varepsilon}{\rho_0^2}\right)^\delta &= \delta(\delta-1)\left(\frac{u'_\varepsilon}{\rho_0^2}\right)^{\delta-2} \frac{\mathrm{tr}_{\tilde{\omega}_\varepsilon}(\sqrt{-1}\partial u'_\varepsilon \wedge \bar{\partial} u'_\varepsilon)}{\rho_0^4} \\
&+ 4\delta(\delta-1)\left(\frac{u'_\varepsilon}{\phi_0}\right)^{\delta} \frac{\mathrm{tr}_{\tilde{\omega}_\varepsilon}(\sqrt{-1}\partial\rho_0 \wedge \bar{\partial}\rho_0)}{\rho_0^2} \\
&+ \delta(\delta-1)\left(\frac{u'_\varepsilon}{\rho_0^2}\right)^{\delta-1} \mathrm{tr}_{\tilde{\omega}_\varepsilon}\left(\sqrt{-1}\partial u'_\varepsilon \wedge \bar{\partial}\frac{1}{\rho_0^2} + \sqrt{-1}\partial\frac{1}{\rho_0^2} \wedge \bar{\partial} u'_\varepsilon\right) \\
&+ \delta\left(\frac{u'_\varepsilon}{\rho_0^2}\right)^{\delta-1} \frac{\Delta_{\tilde{\omega}_\varepsilon} u'_\varepsilon}{\rho_0^2} \\
&+ \delta\left(\frac{u'_\varepsilon}{\rho_0^2}\right)^{\delta-1} \mathrm{tr}_{\tilde{\omega}_\varepsilon}\left(\sqrt{-1}\partial u'_\varepsilon \wedge \bar{\partial}\frac{1}{\rho_0^2} + \sqrt{-1}\partial\frac{1}{\rho_0^2} \wedge \bar{\partial} u'_\varepsilon\right) \\
&+ \delta\left(\frac{u'_\varepsilon}{\rho_0^2}\right)^{\delta-1} \frac{u'_\varepsilon}{\rho_0^2} \mathrm{tr}_{\tilde{\omega}_\varepsilon}\left(-\frac{\sqrt{-1}\partial\bar{\partial}\rho_0^2}{\rho_0^2} + \frac{8\sqrt{-1}\partial\rho_0 \wedge \bar{\partial}\rho_0}{\rho_0^2}\right).
\end{aligned}$$

Let U_ε be a region in X defined by the following properties:

$$\sqrt{-1}\partial G_\varepsilon \wedge \bar{\partial} G_\varepsilon \geq (const.)\sqrt{-1}\rho_0^2 \partial\rho_0 \wedge \bar{\partial}\rho_0 \quad \text{and} \quad \sqrt{-1}\partial\bar{\partial}\rho_0 \leq (const.)\omega_\varepsilon$$

and

$$\phi_\varepsilon \geq (const.)\rho_0^2.$$

If δ is sufficiently small (in fact we let $\delta \to 0$) and $|K|$ is sufficiently large (but independent of ε), the above equality implies the following:

$$\Delta_{\tilde{\omega}_\varepsilon}\left(\frac{u_\varepsilon}{\rho_0^2}\right)^\delta \leq \delta\left(\frac{m|K|}{m+1}\right)^{\delta-1} \frac{(1+c|K|)n - \frac{1}{2}\mathrm{tr}_{\tilde{\omega}_\varepsilon}\omega_\varepsilon - c'n(e^{-f_\varepsilon}-1)}{\rho_0^2}$$

on U_ε, where c and c' are positive constants independent of ε. Let A be a positive number such that

$$\frac{A\delta}{2}\left(\frac{m+1}{m|K|}\right)^{1-\delta} = 1 + \sup_{U_\varepsilon} |\rho_0^2(\text{bisectional curvature of } \omega_\varepsilon)| =: 1 + C.$$

Set $C' = n + n\max\{c, c'\}(1 + |1 - e^{-f_\varepsilon}|)$. Now we recall Chern-Lu's infinitesimal Schwarz lemma ([Ch],[Y3]):

$$\Delta_{\tilde{\omega}_\varepsilon} \log \operatorname{tr}_{\tilde{\omega}_\varepsilon} \omega_\varepsilon \geq -\frac{C}{\phi_\varepsilon} \operatorname{tr}_{\tilde{\omega}_\varepsilon} \omega_\varepsilon.$$

We thus have

$$\Delta_{\tilde{\omega}_\varepsilon}\left\{\log \operatorname{tr}_{\tilde{\omega}_\varepsilon} \omega_\varepsilon - A\left(\frac{u'_\varepsilon}{\rho_0^2}\right)^\delta\right\} \geq \frac{\operatorname{tr}_{\tilde{\omega}_\varepsilon} \omega_\varepsilon}{\rho_0^2} - \frac{A\delta\left(\frac{m+1}{m|K|}\right)^{1-\delta}(n + C'|K|)}{\rho_0^2}. \tag{6}$$

Since the function

$$-A\left(\frac{u'_\varepsilon}{\rho_0^2}\right)^\delta = -A\left(\frac{u'_\varepsilon}{\phi_\varepsilon}\frac{\phi_\varepsilon}{\rho_0^2}\right)^\delta > 0$$

assumes its local minimum along D and its derivative is ∞ along D, the function $\log \operatorname{tr}_{\tilde{\omega}_\varepsilon} \omega_\varepsilon - A\left(\frac{u'_\varepsilon}{\rho_0^2}\right)^\delta$ never takes its local maximum value along D and also near D. If we take ε sufficiently small then we can apply the maximum principle to the inequality (6). Finally, letting $\delta \to 0$, we get a desired uniform estimates for $\operatorname{tr}_{\tilde{\omega}_\varepsilon} \omega_\varepsilon$. This implies that there exists a constant c such that

$$c\omega_\varepsilon < \tilde{\omega}_\varepsilon < c^{-1}\omega_\varepsilon$$

holds for all sufficiently small ε. The estimation of higher derivatives $D^k u_\varepsilon$ follows from the interior Schauder estimates.

COROLLARY. *Let (X, D) be as above and let $N_{D/X}$ be the normal bundle of D in X. Then there exists a Ricci-flat Kähler metric on $N_{D/X}$ which is complete toward D and blows down the infinity section (i.e., the zero-section of $N_{D/X}^{-1}$ which appears at infinity of $N_{D/X}$).*

We get this corollary by scaling the resulting Ricci-flat complete Kähler metric on $X - D$ in Theorem 2 by constant $\varepsilon > 0$ and let $\varepsilon \to 0$. Such a metric defined on certain kind of isolated singularities will be useful in studying degeneration of heat kernels ([Yo]). We end this note by gathering some problems for future study.

PROBLEM 1. *Which compactifications of C^n are rational ?*

PROBLEM 2. *Suppose that X is a Kähler compactification of C^n. Let $D = \sum_{i=1}^{r} D_i$ be a divisor at infinity with reduced structure. If $c_1(X) = \sum_{i=1}^{r} \alpha_i[D_i] > 0$ with $\forall \alpha_i > 1$, is X a rational variety ?*

PROBLEM 3. *Generalize the Existence Theorem in [K2] to (X, D) in which D has at worst normal crossings.*

Recently Azad and the author [AK] showed that there exists a complete Ricci-flat Kähler metric on symmetric varieties (in the sense of [DP]). This is a special case of Problem 3. Indeed, the symmetric variety G^C/K^C associated to the Riemannian symmetric space G/K of compact type is equivariantly compactified to a Fano manifold X and the divisor D at infinity consists of $r = \text{rank}(G/K)$ smooth hypersurfaces with normal crossings (DeConcini-Procesi's compactification [DP]). In this case $c_1(X) = \sum_{i=1}^{r} d_i[D_i]$ with $d_i > 1$.

PROBLEM 4. *Find a characterization of $(Q_n(C), CQ_{n-1}(C))$ in the spirit of Theorem 2, where $CQ_{n-1}(C)$ is a quadric cone and $Q_n(C) - CQ_{n-1}(C) = C^n$.*

PROBLEM 5. *Find a characterization of Kähler C-spaces as compactifications of C^n.*

PROBLEM 6. *The construction of a family of smooth Kähler metrics ω_ε on X approximating a complete one ω on $X - D$ will be generalized to any pair (X, D) where X is a smooth projective variety and D is a smooth ample hypersurface. Suppose we are given a vector bundle on X stable with respect to $c_1(O_X(D))$. Then by Donaldson's theorem [D1,2] there exists a unique Hermitian-Einstein metric H_ε on E over (X, ω_ε). What is the limitting object as ε tends to 0 ? It is hoped that this limitting procedure will give a generalization of [D3] (the relative Kobayashi-Hitchin correspondence of framed instantons on S^4 (equivalently, Hermitian-Einstein connections on C^2 framed at infinity) and the holomorphic bundles on $P_2(C)$ trivialized along a line). For this problem, see [LOS].*

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