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A numerical criterion for admissibility of semi-simple elements

SAMPEI USUI

Abstract

In this article, we shall generalize a theorem of Cattani and Kaplan on horizontal representations of SL(2). Their theorem plays an important role in the construction of their partial compactifications of the classifying spaces \( D \) modulo an arithmetic subgroup of Hodge structures of weight 2.

Introduction

A horizontal \( SL_2 \)-representation is a generalization of the notion of "\((H_1)\)-homomorphism" of \( SL_2 \) in the case of the classical theory of Hermitian symmetric domains (cf., e.g., [Sa.2, III]). More precisely, let \( G = G_R := \text{Aut}(H_R, S) \) be the automorphism group of the classifying space \( D \) of Hodge structures of weight \( w \) (see §1). A representation \( \rho : SL_2(R) \to G \) is said to be horizontal at \( r \in D \) if the morphism \( \rho_* : \mathfrak{sl}_2(R) \to \mathfrak{g} \) of the Lie algebras is a morphism of Hodge structures of type \((0,0)\) with respect to the Hodge structures on \( \mathfrak{sl}_2(C) \) and \( \mathfrak{g}_C \) induced by \( i \in U := \text{(upper-half plane)} \) and \( r \in D \) respectively (see Definition (2.1)). In this case, the pair \((\rho, r)\) is uniquely determined by the pair \((Y, r) \in \mathfrak{g} \times D\) with

\[
(0.1) \quad Y := \rho_* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

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Conversely, a pair \((Y, r) \in \mathfrak{g} \times D\) is said to be admissible if there exists a representation \(\rho : \text{SL}_2(\mathbb{R}) \to G\) horizontal at \(r\) and satisfying (0.1). The main result in the present article is a numerical criterion for admissibility of a pair \((Y, r)\) in the case of general weight.

Given a pair \((\rho, r)\) as above, one can refine the Hodge decomposition \(H_C = \bigoplus H^p_\omega\) corresponding to \(r \in D\), under the horizontal action of \(\text{sl}_2(\mathbb{C})\) at \(r\), called a Hodge-\((Z, X_\pm)\) decomposition (see (2.7)). Our proof of the main result is based on an elementary but useful observation (Corollary (2.11), see also Remark (2.12)), which says that the transformation of the Hodge-\((Z, X_\pm)\) decomposition by the inverse \(c^{-1}\) of the Cayley element

\[
c := \rho \left( \exp \frac{\pi i}{4} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)
\]

yields a split mixed Hodge structure, called a mixed Hodge-\((Y, N_\pm)\) decomposition, which is nothing but the limiting mixed Hodge structure of the associated \(\text{SL}_2\)-orbit \(\tilde{\rho} : U \to D\) defined by \(\tilde{\rho}(gi) := \rho(g)r\) for \(g \in \text{SL}_2(\mathbb{R})\) (cf. [Sc, Theorem (6.16)] and its proof). By virtue of this observation, we can view the relationship between the pairs \((\rho, r)\) and \((Y, r)\) from a better perspective, and generalize a numerical criterion [CK, Theorem (2.22)] for admissibility of \((Y, r)\) in the case of weight 2 to the case of general weight.

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§1. Preliminaries

We recall first the definition of a (polarized) Hodge structure of weight \(w\).
Fix a free $\mathbb{Z}$-module $H_{\mathbb{Z}}$ of finite rank. Set $H_{\mathbb{Q}} := \mathbb{Q} \otimes H_{\mathbb{Z}}$, $H := H_{\mathbb{R}} := \mathbb{R} \otimes H_{\mathbb{Z}}$ and $H_{\mathbb{C}} := \mathbb{C} \otimes H_{\mathbb{Z}}$, whose complex conjugation is denoted by $\sigma$. Let $w$ be an integer. A Hodge structure of weight $w$ on $H_{\mathbb{C}}$ is a decomposition

$$
H_{\mathbb{C}} = \bigoplus_{p+q=w} H^{p,q} \quad \text{with} \quad \sigma H^{p,q} = H^{q,p}.
$$

The integers

$$
h^{p,q} := \dim H^{p,q}
$$

are called the Hodge numbers.

A polarization $S$ for a Hodge structure (1.1) of weight $w$ is a non-degenerate bilinear form on $H_{\mathbb{Q}}$, symmetric if $w$ is even and skew-symmetric if $w$ is odd, such that its $\mathbb{C}$-bilinear extension, denoted also by $S$, satisfies

$$
S(H^{p,q}, \sigma H^{p',q'}) = 0 \quad \text{unless} \quad (p, q) = (p', q'),
$$

$$
i^{p-q} S(v, \sigma v) > 0 \quad \text{for all} \quad 0 \neq v \in H^{p,q}.
$$

Remark (1.4) In the geometric case, i.e., the Hodge structure on the $w$-th cohomology group $H^w(X, \mathbb{Q})$ of a smooth projective variety $X \subset \mathbb{P}^N$ of dimension $d$ over $\mathbb{C}$, we take as a polarization

$$
S(u, v) := (-1)^{w(w-1)/2} \int_X u \wedge v \wedge \eta^{d-w}
$$

for primitive classes $u, v \in H^w_{\text{prim}}(X, \mathbb{C}) \simeq H^w_{\text{prim}}(X, \Omega^1_X)$ where $\eta \in H^1(X, \Omega^1_X)$ is the cohomology class of a hyperplane section of $X$.

For fixed $S$ and $\{h^{p,q}\}$, the classifying space $D$ for Hodge structures and its "compact dual" $\check{D}$ are defined by

$$
\check{D} := \{\{H^{p,q}\} \mid \text{Hodge structure on } H_{\mathbb{C}} \text{ with } \dim H^{p,q} = h^{p,q}, \text{ satisfying the first condition in (1.3)}\},
$$

$$
D := \{\{H^{p,q}\} \in \check{D} \mid \text{satisfying also the second condition in (1.3)}\}.
$$
These are homogeneous spaces under the natural actions of the groups

(1.6) \[ G_C := \text{Aut}(H_C, S), \quad G = G_R := \{ g \in G_C \mid gH_R = H_R \}, \]

respectively. Taking a reference point \( r \in D \), one obtains identifications

(1.7) \[ \tilde{D} \cong G_C/B_C, \quad D \cong G/V, \]

where \( B_C \) and \( V \) are the isotropy subgroups of \( G_C \) and of \( G \) at \( r \in D \), respectively. It is a direct consequence of the definition that

(1.8) \[ G \cong \begin{cases} O(2h, k), & \text{if } w = 2t, \\ \text{Sp}(2h, \mathbb{R}), & \text{if } w = 2t + 1, \end{cases} \quad V \cong \begin{cases} U(h^{w,0}) \times \cdots \times U(h^{t+1,t-1}) \times O(h^{t,t}) & \text{if } w = 2t, \\ U(h^{w,0}) \times \cdots \times U(h^{t+1,t}) & \text{if } w = 2t + 1, \end{cases} \]

where \( k := \sum_{|j| \leq [t/2]} h^{t+2j,t-2j} \) and \( h := (\dim H - k)/2 \) if \( w = 2t \), and \( h := \dim H/2 \) if \( w = 2t + 1 \). It is an important observation that \( V \) is compact, but not maximal compact in general. Hence \( D \) is a symmetric domain of Hermitian type if and only if

(1.9) \[ h^{p,q} = 0 \quad \text{unless } (p, q) = \begin{cases} (t + 1, t - 1), (t, t) \text{ or } (t - 1, t + 1), \\ \text{and } h^{t+1,t-1} = 1 & \text{if } w = 2t, \\ (t + 1, t) \text{ or } (t, t + 1) & \text{if } w = 2t + 1. \end{cases} \]

A reference Hodge structure \( r = \{ H_r^{p,q} \} \in D \) induces a Hodge structure of weight 0 on the Lie algebra \( g_C := \text{Lie} G_C \) by

(1.10) \[ g_C^{s,-s} := \{ X \in g_C \mid X H_r^{p,q} \subset H_r^{p+s,q-s} \text{ for all } p, q \}. \]

One can define the associated Cartan involution \( \theta_r \) on \( g_C \) by

(1.11) \[ \theta_r(X) := \sum_s (-1)^s X^{s,-s} \quad \text{for } X = \sum_s X^{s,-s} \in g_C = \bigoplus_s g_C^{s,-s}. \]
This can be interpreted in the following way: Set

\[ H^+_r := H^{w,0}_r \oplus H^{w-2,2}_r \oplus H^{w-4,4}_r \oplus \cdots, \]

\[ H^-_r := H^{w-1,1}_r \oplus H^{w-3,3}_r \oplus H^{w-5,5}_r \oplus \cdots. \]

It is clear by definition that the isotropy subgroup of the decomposition \( H_C = H^+_r \oplus H^-_r \) induces the maximal compact subgroup

\[ K \simeq \begin{cases} 
U(h) \times O(k) & \text{if } w = 2t, \\
U(h) & \text{if } w = 2t + 1,
\end{cases} \]

of \( G \) which contains \( V \), and the Cartan involution \( \theta_r \) in (1.11) is the one associated to \( K \). Define a \( \mathbb{C} \)-linear automorphism

\[ E_r : H_C \to H_C \text{ by } E_r := \begin{cases} 
1 & \text{on } H^+_r, \\
-1 & \text{on } H^-_r.
\end{cases} \]

Then the Cartan involution \( \theta_r \) in (1.11) can also be written as

\[ \theta_r X = (\text{Int } E_r)X \text{ for } X \in \mathfrak{g}_C. \]

We recall now well-known results on \( \text{SL}_2 \)-representations. Let \( \xi, \eta \) be two variables, and write

\[ \begin{pmatrix} \xi \\ \eta \end{pmatrix}^{(m)} := \begin{pmatrix} \xi^m \\ \xi^{m-1} \eta \\ \vdots \\ \eta^m \end{pmatrix} \quad (m = 0, 1, 2, \cdots). \]

A representation

\[ \rho_m : \text{SL}_2(\mathbb{R}) \to \text{SL}_{m+1}(\mathbb{R}) \text{ defined by } \rho_m(g) \begin{pmatrix} \xi \\ \eta \end{pmatrix}^{(m)} := \begin{pmatrix} g \begin{pmatrix} \xi \\ \eta \end{pmatrix} \end{pmatrix}^{(m)}. \]
is called a symmetric tensor representation of dimension $m+1$. It is known that the $\rho_m$ ($m = 0, 1, 2, \cdots$) are absolutely irreducible and constitute a full set of representatives for the equivalence classes of finite dimensional irreducible representations of $SL_2(\mathbb{R})$.

We take the standard generators for the Lie algebras $\mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{su}(1, 1)$ which are related by the Cayley transformation $\text{Int} \, c_1$, where

\begin{equation}
(1.18) \quad c_1 := \exp \frac{\pi i}{4} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix},
\end{equation}

as follows:

\begin{align*}
\mathfrak{sl}_2(\mathbb{R}) \ni y := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & \quad n_+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & \quad n_- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\
\text{Int} \, c_1 \downarrow & \quad \downarrow & \quad \downarrow \\
\mathfrak{su}(1, 1) \ni z := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, & \quad x_+ := \frac{1}{2} \begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}, & \quad x_- := \frac{1}{2} \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}.
\end{align*}

The following lemma can be verified directly by using the monomial basis (1.16) and the definition (1.19), and so we omit the proof.

**Lemma (1.20).** (i) In the above notation, $Y_m := \rho_m(y)$ and $N_{m\pm} := \rho_m(n_{\pm})$ satisfy

\begin{align*}
Y_m(\xi^{m-j}\eta^j) &= (m - 2j)\xi^{m-j}\eta^j, \\
N_{m+}(\xi^{m-j}\eta^j) &= (m - j)\xi^{m-j-1}\eta^{j+1}, \\
N_{m-}(\xi^{m-j}\eta^j) &= j\xi^{m-j+1}\eta^{j-1}.
\end{align*}

(ii) For the Cayley element $c_m := \rho_m(c_1) \in SL_{m+1}(\mathbb{C})$,

\begin{align*}
\sigma c_m &= c_m^{-1}\sigma, & \text{where } \sigma & \text{ is the complex conjugation,} \\
c_m^{\pm 2}(\xi^{m-j}\eta^j) &= (\pm i)^m \eta^{-m-j}\xi^j, \\
c_m^A(\xi^{m-j}\eta^j) &= (-1)^m \xi^{m-j}\eta^j.
\end{align*}
Remark (1.21). The Hodge structure on \( g_{1C} := \mathfrak{sl}_2(C) \) induced by \( i \in U := \) (upper-half plane) \( \simeq \text{SL}_2(\mathbb{R})/U(1) \) coincides with the canonical decomposition by the standard “H-element” \( (n_+ - n_-)/2 \) (cf., e.g., [Sa.2, II. §7]):

\[
g_{1C} = g_{1C}^{1,-1} + g_{1C}^{0,0} + g_{1C}^{-1,1} = p_- + eC + p_+ = \{x_-\}c + \{z\}c + \{x_+\}c.
\]

§2. Horizontal \( \text{SL}_2 \)-representations

From now on, we assume that \( w > 0 \) and all Hodge structures of weight \( w \) satisfy \( H^{p,q} = 0 \) unless \( p, q \geq 0 \).

Definition (2.1) (cf. [Sc, p.258]). An \( \text{SL}_2 \)-representation \( \rho : \text{SL}_2(\mathbb{R}) \rightarrow G \) is said to be horizontal at \( r = \{H^p_r\} \in D \) if \( \rho_\ast(z) \in g_{1C}^{-1,1} := \{X \in g_C | X H^p_r \subset H^{p-1,q+1} \} \) for all \( p, q \).

Remark (2.2). It is clear that an \( \text{SL}_2 \)-representation \( \rho \) is horizontal if and only if \( \rho_\ast : \mathfrak{sl}_2(\mathbb{R}) \rightarrow g \) is a morphism of Hodge structures of type \( (0,0) \) with respect to the Hodge structures induced by \( i \in U \) and \( r \in D \), respectively. A horizontal \( \text{SL}_2 \)-representation \( \rho \) induces an equivariant horizontal map \( \bar{\rho} : \mathbb{P}^1 \rightarrow \bar{D} \) with \( \bar{\rho}(i) = r \):

\[
\begin{array}{ccc}
\text{SL}_2(\mathbb{C}) & \xrightarrow{\rho} & G_C \\
\downarrow & & \downarrow \\
\mathbb{P}^1 & \xrightarrow{\bar{\rho}} & \bar{D}
\end{array}
\]

This is a generalization to the present context of the notion of ‘\((H_1)\)-homomorphism’ in the case of symmetric domains of Hermitian type (cf., e.g., [Sa.2, II. (8.5), III. §1]).

Let \( \rho : \text{SL}_2(\mathbb{R}) \rightarrow G \) be a representation horizontal at \( r = \{H^p_r\} \in D \), and set

\[
(2.3) \quad Y := \rho_\ast(y), \quad N_\pm := \rho_\ast(n_\pm); \quad Z := \rho_\ast(z), \quad X_\pm := \rho_\ast(x_\pm).
\]
Notice that by (1.19) these are related under the Cayley transformation:

(2.4) \[ Z = (\text{Int } c)Y, \quad X_{\pm} = (\text{Int } c)N_{\pm}, \quad c := \rho(c_1). \]

\((Y, N_{\pm})\) and \((Z, X_{\pm})\) define direct sum decompositions of \(H\) and \(H_C\) whose summands are

(2.5) \[ P_{\lambda}^{(\lambda+2k)} := N_{\pm}^k(H(Y; \lambda + 2k) \cap \text{Ker } N_{\pm}), \]

(2.6) \[ Q_{\lambda}^{(\lambda+2k)} := X_{\pm}^k(H_C(Z; \lambda + 2k) \cap \text{Ker } X_{\pm}), \]

for all eigenvalues \(\lambda \in \{0, \pm 1, \pm 2, \cdots \pm w\}\) of \(Y\) and \(Z\) and for \(k \geq \max \{-\lambda, 0\}\), respectively. Here we denote by \(H(Y; \lambda + 2k)\) etc. the eigenspace of an endomorphism \(Y\) of \(H\) with eigenvalue \(\lambda + 2k\). Since \(\rho\) is horizontal at \(r = \{H^p,q\}\), (2.6) is compatible with this Hodge structure and we set

(2.7) \[ Q_{\lambda}^{(\lambda+2k)} a+k,b+\lambda+k := Q_{\lambda}^{(\lambda+2k)} \cap H^a+k,b+\lambda+k \quad (a, b \geq 0). \]

These form a refined direct sum decomposition which we call the \textit{Hodge-} \((Z, X_{\pm})\) \textit{decomposition of} \((\rho, r)\) (cf. Remark (2.12) below). Transforming this by the inverse \(c^{-1}\) of the Cayley element, we define

(2.8) \[ P_{\lambda}^{(\lambda+2k)} a+k,b+k := c^{-1}Q_{\lambda}^{(\lambda+2k) a+k,b+\lambda+k}. \]

Lemma (2.9). (i) \[ \sigma Q_{\lambda}^{(\lambda+2k) a+k,b+\lambda+k} = Q_{\lambda}^{(-\lambda+2(\lambda+k)) b+\lambda+k,a+k}. \]

(ii) \[ c Q_{\lambda}^{(\lambda+2k) a+k,b+\lambda+k} = c^2 P_{\lambda}^{(\lambda+2k) a+k,b+k} = P_{\lambda}^{(-\lambda+2(\lambda+k)) a+\lambda+k,b+\lambda+k}. \]

Proof. It is easy to see, by definition, that \(cP_{\lambda}^{(\lambda+2k)} = Q_{\lambda}^{(\lambda+2k)}\). Hence, by the first equality in (1.20.ii), we have

\[ \sigma Q_{\lambda}^{(\lambda+2k)} = \sigma cP_{\lambda}^{(\lambda+2k)} = c^{-1} \sigma P_{\lambda}^{(\lambda+2k)} = c^{-1} P_{\lambda}^{(\lambda+2k)} = c^{-2} Q_{\lambda}^{(\lambda+2k)}. \]
On the other hand, by the second equality in (1.20.ii), the third and the second equalities in (1.20.i), we see that on $P_{A+2k}^{(\lambda+2k)}$

$$c^{-2} = \begin{cases} 
\frac{i^{\lambda+2k} \frac{k!}{(\lambda+k)!} N_{-\lambda}^{\lambda}}{+} & \text{if } \lambda \geq 0, \\
\frac{i^{\lambda+2k} (\lambda+k)!}{k!} N_{+}^{\lambda} & \text{if } \lambda < 0. 
\end{cases}$$

Taking their Cayley transforms, we see that on $Q_{\lambda}^{(\lambda+2k)}$

(2.10) $$c^{-2} = \begin{cases} 
i^{\lambda+2k} \frac{k!}{(\lambda+k)!} X_{-\lambda}^{\lambda} & \text{if } \lambda \geq 0, \\
i^{\lambda+2k} (\lambda+k)! X_{+}^{\lambda} & \text{if } \lambda < 0. 
\end{cases}$$

Thus, by the definition of the $Q_{\lambda}^{(\lambda+2k)}$, we have in both cases that

$$\sigma Q_{\lambda}^{(\lambda+2k)} = c^{-2} Q_{\lambda}^{(\lambda+2k)} = X_{\mp \lambda}^{\pm \lambda} Q_{\lambda}^{(\lambda+2k)} = Q_{-\lambda}^{(\lambda+2k)} = Q_{-\lambda}^{(-\lambda+2(\lambda+k))}.$$ 

This together with $\sigma H_r^{b+k, b+\lambda+k} = H_r^{b+k, b+\lambda+k}$ yields the assertion (i).

By horizontality, $X_- \in g_{C}^{1,-1}$ and $X_+ \in g_{C}^{-1,1}$, hence $X_{\mp \lambda}^{\pm \lambda} \in g_{C}^{\lambda,-\lambda}$. This together with (2.10) shows that

$$c^{-2} Q_{\lambda}^{(\lambda+2k)a+k, b+\lambda+k} = X_{\mp \lambda}^{\pm \lambda} Q_{\lambda}^{(\lambda+2k)a+k, b+\lambda+k} = Q_{-\lambda}^{(-\lambda+2(\lambda+k))a+k, b+k}.$$ 

Thus we obtain the second equality in (ii). The first equality in (ii) follows from the second. \(\blacksquare\)

**Corollary (2.11).** Let $(\rho, r)$ be as above. For each eigenvalue $\lambda$ of $Y$ and for $k \geq \max \{-\lambda, 0\}$, we see that

$$\mathbb{C} \otimes P_{A}^{(\lambda+2k)} = \bigoplus_{a+b+2k = \omega - \lambda, a, b \geq 0} P_{\lambda}^{(\lambda+2k)a+k,b+k}$$
is a Hodge structure of weight \( w - \lambda \). Moreover, in the case \( \lambda = k = 0 \), this is \( S \)-polarized.

**Proof.** We should observe the behavior under the complex conjugation \( \sigma \):

\[
\sigma P_{\lambda}^{(\lambda+2k)a+b+k} = c^{-1} Q_{\lambda}^{(\lambda+2k)a+b+\lambda+k} = c Q_{-\lambda}^{(-\lambda+2(\lambda+k))b+\lambda+k,a+k} = c^2 P_{-\lambda}^{(-\lambda+2(\lambda+k))b+\lambda+k,a+\lambda+k} = P_{\lambda}^{(\lambda+2k)b+k,a+k}.
\]

This shows the first assertion.

The representation \( \rho \) is trivial on \( Q_0^{(0)} \), hence \( P_0^{(0)}a,b = c^{-1} Q_0^{(0)}a,b = Q_0^{(0)}a,b \), and so the second assertion trivially holds. \( \square \)

We call a direct sum decomposition in (2.11) the mixed Hodge-(\( Y, N_\pm \)) decomposition of \((\rho, r)\).

**Remark (2.12).** We remark here some observations which are verified easily by (1.20.i), their Cayley transforms and horizontality of \( \rho \) at \( r \). A Hodge-(\( Z, X_\pm \)) decomposition and a mixed Hodge-(\( Y, N_\pm \)) decomposition form “nests of diamonds”, respectively. For example, in the case of weight \( w = 3 \), these nests of diamonds are illustrated respectively as in Figures 1 and 2.

(Figure 1)
On these nests of diamonds, the complex conjugation by $\sigma$ sends respectively a summand $Q^{(\lambda+2k)a+k,b+\lambda+k}_\lambda$ to a summand $Q^{(-\lambda+2(\lambda+k))b+\lambda+k,a+k}_{-\lambda}$ which are symmetric with respect to the origin of the diamonds, and a summand $P^{(\lambda+2k)a+k,b+k}_\lambda$ to a summand $P^{(\lambda+2k)b+k,a+k}_{\lambda}$ which are symmetric with respect to the vertical axis. The operator $X_+$ (resp. $X_-$) sends a summand $Q^{(\lambda+2k)a+k,b+\lambda+k}_\lambda$ one step down (resp. up) to a summand $Q^{(\lambda+2+2(k-1))a+k-1,b+\lambda+2+k-1}_{\lambda+2}$ (resp. $Q^{(\lambda-2+2(k+1))a+k+1,b+\lambda-2+k+1}_{\lambda-2}$), and $X_{\pm}$ are inverse to each other up to non-zero constant between these summands whenever both summands actually appear in the nest of diamonds. Similarly, the operator $N_+$ (resp. $N_-$) sends a summand $P^{(\lambda+2k)a+k,b+k}_\lambda$ one step down (resp. up) to a summand $P^{(\lambda+2+2(k-1))a+k-1,b+k-1}_{\lambda+2}$ (resp. $P^{(\lambda-2+2(k+1))a+k+1,b+k+1}_{\lambda-2}$), and $N_{\pm}$ are inverse to each other up to non-zero constant between these summands whenever both summands actually appear in the nest of diamonds. The Cayley element $c$ transforms the second nest of diamonds together with the action of the operators $Y, N_{\pm}$ to the first nest of diamonds together with the action of the operators $Z, X_{\pm}$:

$$cP^{(\lambda+2k)a+k,b+k}_\lambda = Q^{(\lambda+2k)a+k,b+\lambda+k}_\lambda.$$

By using these operators, we can explain why the summands outside the nests of diamonds vanish in the following way. We claim first that $Q^{(\lambda+2k)a+k,b+\lambda+k}_\lambda = 0$ for $\lambda > 0$ and $b < 0$. Indeed, $X_{\lambda+k}$ is injective on this summand by the Cayley transform of the third equality in (1.20.i). On the other hand, looking at the Hodge type, we see
that \( X^{\lambda+k} Q_{\lambda}^{(\lambda+2k)a+k,b+\lambda+k} \subset Q_{-\lambda-2k}^{(-\lambda-2k+2(\lambda+2k))a+\lambda+2k,b} = 0 \) by horizontality. Thus we get our claim. It follows by symmetry under the complex conjugation \( \sigma \) that \( Q_{\lambda}^{(\lambda+2k)a+k,b+\lambda+k} = 0 \) for \( \lambda < 0 \) and \( a < 0 \). Finally, by the inverse of the Cayley transformation, we have \( P_{\lambda}^{(\lambda+2k)a+k,b+k} = 0 \) for \( \lambda > 0 \) and \( b < 0 \), and for \( \lambda < 0 \) and \( a < 0 \).

We call the length of the side of the biggest diamond in a nest the size of the nest of diamonds.

Another remark is that a mixed Hodge-(\( Y, N \pm \)) decomposition is nothing but the limiting split mixed Hodge structure of the associated \( \text{SL}_2 \)-orbit \( \tilde{\rho} : U \rightarrow D \), \( \tilde{\rho}(g) := \rho(g) \text{tr} \) \( g \in \text{SL}_2(\mathbb{R}) \), and the monodromy weight filtration \( L \) is described as \( L_i = \bigoplus_{\lambda \leq i} \bigoplus_k P_{-\lambda}^{(\lambda+2k)} \) (cf. [Sc, (6.16)] and its proof, [CK, pp. 13–14]).

In the above notation, for all \( \lambda, \ a, \) and \( b \), put

\[
\begin{align*}
n_\lambda &:= \dim_R H(Y; \lambda) = \dim_C H_C(Z; \lambda), \\
p_{\lambda}^{a,b} &:= \dim_C P_{-\lambda-2k}^{(\lambda+2k)a+k,b+k} = \dim_C Q_{-\lambda-2k}^{(\lambda+2k)a+k,b+\lambda-k}.
\end{align*}
\]

Notice that, by construction, the middle terms and the terms on the extreme right hand side of the second equality in (2.13) are independent of \( k \) (cf. Remark (2.12)).

**Lemma (2.14).** For \( (\rho, r) \) as above, the following hold:

(i) \( \sum_{a+b = w-\lambda} p_{\lambda}^{a,b} = n_\lambda - n_{\lambda+2} \) for all \( 0 \leq \lambda \leq w \).

(ii) \( p_{\lambda}^{b,a} = p_{\lambda}^{a,b} \) for all \( \lambda, a, b \) with \( 0 \leq \lambda \leq w, a \geq 0, b \geq 0 \) and \( a + b = w - \lambda \).

(iii) \( h^{a,b} = h^{a+1,b-1} - (p_0^{a+1,b-1} + p_1^{a+1,b-2} + \cdots + p_{b-1}^{a+1,0}) \)

\[+ (p_0^{a,b} + p_1^{a-1,b} + \cdots + p_0^{a,b}) \]

for all \( a, b \) with \( a \geq 0, b \geq 0 \) and \( a + b = w \).

**Proof.** We first observe that there is an exact sequence
for every \( \lambda \geq 0 \) (and \( N_- \) yields a right splitting). (i) and (ii) follow from this and (2.11).

In order to show (iii), we look at the morphism \( X_+ : H^{a+1,b-1} \to H^{a,b} \) and its kernel and cokernel:

\[
\ker = Q_0^{(0)a+1,b-1} \oplus Q_1^{(1)a+1,b-1} \oplus \cdots \oplus Q_{b-1}^{(b-1)a+1,b-1} \\
\cong P_0^{(0)a+1,b-1} \oplus P_1^{(1)a+1,b-2} \oplus \cdots \oplus P_{b-1}^{(b-1)a+1,0},
\]

\[
\coker \cong Q_0^{(0)a,b} \oplus Q_1^{(1)a,b} \oplus \cdots \oplus Q_{-a}^{(a)a,b} \\
\cong Q_0^{(0)a,b} \oplus Q_1^{(1)a-1,b+1} \oplus \cdots \oplus Q_{a}^{(a)0,b+a} \\
\cong P_0^{(0)a,b} \oplus P_1^{(1)a-1,b} \oplus \cdots \oplus P_{a}^{(a)0,b}.
\]

Looking at the dimension, we get (iii). \( \blacksquare \)

**Definition (2.15).** We call a set of integers \( \{p_{a,b}^\lambda\} \), which satisfies the conditions (i), (ii) and (iii) of (2.14), a set of primitive Hodge numbers belonging to \( \{h^p,q,n_\lambda\} \).

§3. Admissible \( \mathbb{R} \)-semi-simple elements

We continue to use the notation in the previous sections.

**Proposition (3.1).** Given a pair \( (Y,r) \in g \times D \), there exists at most one representation \( \rho : \text{SL}_2(\mathbb{R}) \to G \) which is horizontal at \( r \) and \( \rho_* y = Y \).

**Proof.** Since \( y \) and \( z \) generate \( \mathfrak{sl}_2(\mathbb{C}) \), it is enough to show that if such a representation \( \rho \) exists then the eigenspaces of \( Z \), and hence \( Z \) itself, are determined by the pair \( (Y,r) \). Actually, we shall show by induction on the size \( w \) of the nest of diamonds of the Hodge-(\( Z, X_\pm \)) decomposition (2.7) (cf. Remark (2.12)) that this nest of diamonds is completely determined by \( (Y,r) \).
First notice that

\( (3.2) \quad Y = i(X_+ - X_-). \)

For a subspace \( M \) of \( H_C \), we put, throughout this proof,

\[
M^\perp := \{ v \in H_C \mid S(v, \sigma u) = 0 \text{ for all } u \in M \},
\]

\[
\text{projection} \{ M \to H_{P,q}^p \} := \text{Im} \{ M \subset H_C = \bigoplus_{p' + q' = w} H_{p',q'} \to H_{p,q} \}.
\]

Then we see that

\[
Q_w^{(w-1)_k, w-1} = \text{projection} \left\{ Y^{w-1} \left( \bigoplus_{0 \leq \lambda \leq w-1} Q_{-\lambda}^{(w)_k, w} \right) \to H_{r, w-1}^{1, w-1} \right\},
\]

\[
Q^{(w-1)_k, w-1-k} = \text{projection} \left\{ Y^{k} Q_{w-1}^{(w-1)_k, w} \to H_{r, w-1-k}^{1+k, w-1-k} \right\} (0 \leq k \leq w-1),
\]

\[
Q^{(w-2)_k, w-2} = \text{projection} \left\{ Y^{w-2} \left( \bigoplus_{0 \leq \lambda \leq w-2} Q_{-\lambda}^{(w)_2, w} \right) \to H_{r, w-2}^{2, w-2} \right\},
\]

\[
Q^{(w-2)_k, w-2-k} = \text{projection} \left\{ Y^{k} Q_{w-2}^{(w-2)_k, w-2} \to H_{r, w-2-k}^{2+k, w-2-k} \right\} (0 \leq k \leq w-2),
\]

\[
\cdots \cdots
\]

Thus \( Q_{-\lambda}^{(w-\lambda-k, \lambda-k} (0 \leq \lambda \leq w, 0 \leq k \leq \lambda) \) are determined. Taking the complex conjugation by \( \sigma \) of these, we get \( Q_{-\lambda}^{(\lambda, \lambda-k, \lambda-k} = \sigma Q_{-\lambda}^{(\lambda, \lambda-k, \lambda-k} (0 \leq \lambda \leq w, 0 \leq k \leq \lambda) \). Applying the induction hypothesis to the nest of diamonds of size \( \leq w-2 \) in

\[
\left( \bigoplus_{0 \leq \lambda \leq w} Q_{-\lambda}^{(w-\lambda-k, \lambda-k} \oplus Q_{-\lambda}^{(\lambda, \lambda-k, \lambda-k} \right)^\perp
\]

(cf. Remark (2.12)), we get our assertion. \( \blacksquare \)
Definition (3.3). A pair \((Y, r) \in \mathfrak{g} \times D\) is admissible if there exists a representation \(\rho : \text{SL}_2(\mathbb{R}) \to G\) which is horizontal at \(r\) and \(\rho_\ast(y) = Y\).

The set of primitive Hodge numbers \(\{p^a_b\}\) belonging to \(\{h^{p,q}, n_\lambda\}\) is called the type of an admissible pair \((Y, r)\).

\(Y \in \mathfrak{g}\) is said to be admissible if \((Y, r)\) is an admissible pair for some \(r \in D\).

Now we prove the following numerical criterion for admissibility:

Theorem (3.4). \(Y \in \mathfrak{g}\) is admissible if and only if \(Y\) is semi-simple over \(\mathbb{R}\) whose eigenvalues are contained in \(\{0, \pm 1, \pm 2, \ldots, \pm w\}\) and there exists a set of primitive Hodge numbers \(\{p^a_b\}\) belonging to \(\{h^{p,q}, n_\lambda\}\), where \(n_\lambda := \dim H(Y; \lambda)\) (cf. Definition (2.15)).

Proof. Since \(Y\) is semi-simple over \(\mathbb{R}\), the eigenspaces \(H(Y; \lambda)\) are defined over \(\mathbb{R}\) and \(H(Y; \lambda)\) and \(H(Y; \mu)\) are \(S\)-orthogonal unless \(\lambda + \mu = 0\). Therefore \(H(Y; \lambda)\) and \(H(Y; -\lambda)\) are \(S\)-dual.

Since \(n_\lambda - n_{\lambda + 2} \geq 0\) for \(\lambda \geq 0\) by the condition (2.14.i), we can take a direct sum decomposition

\[
H(Y; \lambda) = P_\lambda^{(\lambda)} \oplus P_\lambda^{(\lambda + 2)} \oplus P_\lambda^{(\lambda + 4)} \oplus \cdots \quad \text{for } \lambda \geq 0
\]

with \(\dim P_\lambda^{(\lambda + 2k)} = n_{\lambda + 2k} - n_{\lambda + 2k + 2}\). Moreover, in the case \(\lambda = 0\), the decomposition (3.5) can be taken to be \(S\)-orthogonal. We denote the \(S\)-dual decomposition by

\[
H(Y; -\lambda) = P_{-\lambda}^{(\lambda)} \oplus P_{-\lambda}^{(\lambda + 2)} \oplus P_{-\lambda}^{(\lambda + 4)} \oplus \cdots \quad (\lambda \geq 0),
\]
i.e., \(P_{-\lambda}^{(\lambda + 2k)}\) and \(P_\lambda^{(\lambda + 2m)}\) are \(S\)-orthogonal unless \(k = m\).

By the conditions (i) and (ii) of (2.14), we can choose a Hodge decomposition

\[
\mathbb{C} \otimes P_\lambda^{(\lambda + 2k)} = \bigoplus_{a+b+2k = w-\lambda, a, b \geq 0} P_\lambda^{(\lambda + 2k)a+b+k} \quad \text{for } \lambda \geq 0, k \geq 0,
\]
with \( \dim P^{(\lambda+2k)a+k,b+k}_\lambda = p^{a,b}_{\lambda+2k} \). Moreover, in the case \( \lambda = k = 0 \), the Hodge structure (3.7) can be chosen to be \( S \)-polarized. We denote the \( S(\cdot, \sigma \cdot) \)-orthogonal decomposition by

\[
(3.8) \quad \bigoplus_{0 \leq k \leq \lambda} \bigoplus_{a+b+2\lambda+2k = w+\lambda} P^{(\lambda+2k)a+k,b+k}_\lambda = P^{a+k,b+k}_{\lambda+2k} \quad (\lambda \geq 0, k \geq 0),
\]

i.e., \( S(P^{(-\lambda+2(\lambda+k))a+b+\lambda+k}_\lambda, \sigma P^{(\lambda+2k)a'+b'+k}_\lambda) = 0 \) unless \( (a, b) = (a', b') \). Notice that \( P^{(-\lambda+2(\lambda+k))a+b+\lambda+k}_\lambda = P^{(\lambda+2k)a+b+\lambda+k}_\lambda \).

Now we consider the cases \( \lambda \geq 0 \) and \( \lambda < 0 \) altogether. For \( k \geq \max\{-\lambda, 0\} \) and \( a \geq b \), let

\[
(3.9) \quad \{ v^{(\lambda+2k)a+k,b+k}_\lambda, j \mid 1 \leq j \leq p^{a,b}_{\lambda+2k} \}
\]

be a \( \mathbb{C} \)-basis of \( P^{(\lambda+2k)a+k,b+k}_\lambda \), such that

\[
(3.10) \quad S(v^{(-\lambda+2(\lambda+k))a+b+\lambda+k}_\lambda, \sigma v^{(\lambda+2k)a+k,b+k}_\lambda, j') = \delta_{jj'}(-1)^{a+b-k} \frac{(\lambda+2k)}{k}.
\]

In the case \( a = b \), we can moreover take the above basis (3.9) to consist of real elements. Put

\[
(3.11) \quad v^{(\lambda+2k)b+k,a+k}_\lambda, j = \sigma v^{(\lambda+2k)a+k,b+k}_\lambda, j \quad (a \geq b).
\]

Define now \( \mathbb{C} \)-linear endomorphisms \( N_\pm \) of \( \mathbb{H}_C \) by

\[
(3.12) \quad N^\pm v^{(\lambda+2k)a+k,b+k}_\lambda, j = \begin{cases} \lambda v^{(\lambda+2k)(a+k-1),b+k-1}_\lambda, j & \text{if } a \geq b, \\ (\lambda-k) v^{(\lambda-2k)(a+k+1),b+k+1}_\lambda, j & \text{if } a < b. \end{cases}
\]
for all $\lambda$, non-negative $a, b$ and $k \geq \max\{-\lambda, 0\}$. By construction, it is easy to see that $N_\pm$ commute with the complex conjugation $\sigma$ and satisfy the commutation relations: 
$[N_+, N_-] = Y$, and $[Y, N_\pm] = \pm 2N_\pm$, respectively. It is also easy to verify that $S(N_\pm, \cdot \cdot) + S(\cdot \cdot, N_\pm) = 0$, respectively. Indeed, for example, one can compute as

$$S(N_\pm v_{-\lambda, j}, v_{\lambda+k,b+\lambda+k}) + S(v_{-\lambda, j}, N_\pm) = 0.$$ 

Thus we see that $N_\pm \in \mathfrak{g}$ and hence there exists a unique representation

$$\rho : SL_2(\mathbb{R}) \rightarrow G \text{ such that } \rho \cdot Y = Y \text{ and } \rho \cdot N_\pm = N_\pm, \text{ respectively.}$$

By using the Cayley element $c := \rho(c_1) \in G_C$, we define

$$Q_\lambda^{(\lambda+2k)a+k,b+\lambda+k} := cP_\lambda^{(\lambda+2k)a+k,b+\lambda+k}, \quad H_{p, q} := \bigoplus_{a+k=p, b+\lambda+k=q} Q_\lambda^{(\lambda+2k)a+k,b+\lambda+k},$$

where, on the right hand side of the second equality, the summation is taken over all the eigenvalues $\lambda$ of $Y$, all integers $k \geq \max\{-\lambda, 0\}$ and all non-negative integers $a, b$ with $a + b + \lambda + 2k = w$. This defines a Hodge structure. Indeed, by using (1.20.ii), one sees that

$$\sigma Q_\lambda^{(\lambda+2k)a+k,b+\lambda+k} = \sigma cP_\lambda^{(\lambda+2k)a+k,b+\lambda+k} = c^{-1} \sigma P_\lambda^{(\lambda+2k)a+k,b+\lambda+k}$$

and hence $\sigma H_{p, q} = H_{q, p}$. One can moreover verify that (3.14) is $S$-polarized. Indeed, the direct sum in (3.14) is $S$-orthogonal by construction and, for

$$cP_\lambda^{(\lambda+2k)a+k,b+\lambda+k}, cP_\lambda^{(\lambda+2k)a+k,b+\lambda+k} \in Q_\lambda^{(\lambda+2k)a+k,b+\lambda+k} \subset H_{p, q},$$
one can compute as

\[
q_{\nu}^{-q} S(c^\lambda_{\nu}, j, \sigma c^\lambda_{\nu}, j') = q_{\nu}^{-a-b} S(c^\lambda_{\nu}, j, \sigma c^\lambda_{\nu}, j')
\]

Thus we have \( \{H^p,q\} \in D \).

Finally, we claim that the representation \( \rho \) in (3.13) is horizontal at \( \{H^p,q\} \in D \). Indeed, since \( Z = (\text{Int} c) Y, X_\pm = (\text{Int} c) N_\pm \), one can compute, by (1.20), as

\[
Z Q_{\lambda}^{(\lambda+2k)} a+b+\lambda k = c Y P_{\lambda}^{(\lambda+2k)} a+b+k = Q_{\lambda}^{(\lambda+2k)} a+b+\lambda k
\]

This completes the proof of the theorem. 

We remark that the condition on \( \{n_\lambda\} \) in Theorem (3.4) coincides with the one in [CK, (2.29)] in the case of weight 2.

Fix identifications \( D \simeq G/V \) and \( R \simeq G/K \), where \( K \) is a maximal compact subgroup of \( G \) containing \( V \) and \( R \) is the associated Riemannian symmetric domain, and let \( \theta_K \) be the associated Cartan involution. We denote the projection by

(3.15) \[ \pi : D \simeq G/V \rightarrow G/K \simeq R. \]

**Proposition (3.16).** We use the notation in Theorem (3.4). Let \( Y \in g \) be an admissible element.
(i) If \( r \in D \) forms an admissible pair \((Y, r)\), then \( \theta_r Y = -Y \), where \( \theta_r \) is the Cartan involution on \( g \) induced from \((1.11)\).

(ii) If \( \theta_K Y = -Y \), then there exists \( r \in \pi^{-1}([K]) \) such that \((Y, r)\) is an admissible pair.

(iii) For each set of primitive Hodge numbers \( \{p_{\lambda}^{a, b}\} \) belonging to \( \{h^{p,q}, n_\lambda\} \), \( G_Y := \{g \in G \mid (Ad g)Y = Y\} \) acts transitively on the set \( \{r \in D \mid (Y, r) \text{ is an admissible pair of type } \{p_{\lambda}^{a, b}\}\} \).

Proof. (i) follows from (3.2) and (1.11).

(ii): Assume \( \theta_K Y = -Y \). Take a point \( r' \in D \) at which \( Y \) is admissible and let \( K' \) be the maximal compact subgroup of \( G \) associated to the Cartan involution \( \theta_r \). By the result in (i) for \((Y, r')\) and the assumption, \( Y \) can be viewed as a tangent vector to \( R \) at \([K']\) as well as at \([K]\): \( Y \in T_R([K']), Y \in T_R([K]) \). By the transitivity of tangent spaces of a Riemannian symmetric domain, there exists \( g \in G \) such that \((\text{Int} g)K' = K\) and \((Ad g)Y = Y\). Hence the admissibility of \((Y, r')\) implies that of \(( (Ad g)Y, gr' ) = (Y, gr') , \) where \( gr' \in \pi^{-1}([K]) \).

(iii): Suppose that \( r, r' \in D \) are points at which \( Y \) is admissible of the same type \( \{p_{\lambda}^{a, b}\} \). Let \( \rho, \rho' : SL_2(\mathbb{R}) \rightarrow G \) be the corresponding representations. It is enough to show that there exists \( g \in G \) such that \( \rho' = (\text{Int} g)\rho \). Indeed, if this is the case, then \((Ad g)Y = (Ad g)(\rho_*(y)) = \rho'_*(y) = Y\) and \( gr = g\rho(i) = \rho'(i) = r' \).

Since \((Y, r)\) and \((Y, r')\) have the same type \( \{p_{\lambda}^{a, b}\} \), the types of the irreducible decompositions of \( \rho \) and \( \rho' \) coincide. Now we use the well-known fact that any finite-dimensional irreducible representation of \( SL_2 \) is isomorphic to a suitable symmetric tensor power representation (cf. §1). Thus we get our assertion.

Appendix

In this appendix, we shall generalize the results in [CK] and construct a partial
compactification of the classifying space $\Gamma \setminus D$, $\Gamma = G_\mathbb{Z} := \{ g \in G | g H_\mathbb{Z} = H_\mathbb{Z} \}$, of Hodge structures in general weight, adding those points which correspond one-parameter degenerations of type II. Since the arguments are analogous to those in [CK], we shall only indicate the outline of the construction. We use the notation in previous sections.

An admissible semi-simple element $Y \in g$ is of type II if its eigenvalues are $0$ or $\pm 1$.

A horizontal $SL_2$-representation $\rho$ is of type II if so is $Y = \rho(y)$.

A period map $\varphi : \Delta^* \rightarrow \Gamma \setminus D$ from the punctured disc, i.e., a holomorphic map with horizontal local liftings, is of type II if its monodromy logarithm $N := (1/m) \log \gamma^m$ satisfies $N^2 = 0$, where $\gamma$ is the monodromy of $\varphi$ and $m$ is the least positive integer such that the eigenvalues of $\gamma^m$ are all unity.

Throughout this appendix, we shall consider only those $Y, \rho$ and $\varphi$ of type II.

(A.1) Let $H$ be an $\mathbb{R}$-vector space underlying Hodge structures of weight $w$, and $S$ the polarizing bilinear form on $H$ (see §1). For an isotropic subspace $W_1$, of $H$, we have a filtration $0 =: W_2 \subset W_1 \subset W_0 := W_1^\perp \subset W_{-1} := H$, where $W_1^\perp$ means the subspace of $H$ perpendicular to $W_1$ with respect to $S$. Set $n_\lambda := \dim W_\lambda/W_{\lambda+1}$. We assume that there exists a set of primitive Hodge numbers $\{ p^{a,b}_\lambda \}$ belonging to $\{ h^{p,q}, n_\lambda \}$ (see Definition (2.15)). We denote by $\tilde{S}$ the non-degenerate bilinear form on $W_0/W_1$ induced by $S$. Let $\phi$ be a polarizing bilinear form of a Hodge structure on $W_1 \mathbb{C}$ of type $\{ p^{a,b}_\lambda \}$. Two such forms are considered to be equivalent if they are different only up to a positive multiplicative constant.

Definition(A.1.1). Given $W_1, \phi := \{ p^{a,b}_\lambda \}$ and $\phi$ as above. The associated
boundary component \( B = B(W_1, p, \phi) = B^w(W_1, p) \times B^{w-1}(W_1, p, \phi) \) is defined by

\[
B^w(W_1, p) : \text{classifying space for } \tilde{S}\text{-polarized Hodge structures on } (W_0/W_1)_\mathbb{C}
\]
of type \( \{p_0^{a,b}\} \).

\[
B^{w-1}(W_1, p, \phi) : \text{classifying space for } \phi\text{-polarized Hodge structures on } W_1_{\mathbb{C}}
\]
of type \( \{p_1^{a,b}\} \).

The boundary bundle \( B = B(W_1, p) = B^w(W_1, p) \times B^{w-1}(W_1, p, \phi) \) is defined as the disjoint union of all boundary components \( B(W_1, p, \phi) \) where \( \phi \) runs over all equivalence classes of polarizing forms on \( W_1 \) of type \( \{p_1^{a,b}\} \).

Theorem(3.4) shows that for every boundary bundle \( B(W_1, p) \) there exists an admissible element \( Y \in \mathfrak{g} \) with the set of primitive Hodge numbers \( p \) and \( W_1 = H(Y; 1) \). Theorem(3.4) and Proposition(3.16,iii) (and some argument in linear algebra) show that for every boundary component \( B(W_1, p, \phi) \) there exists a horizontal \( SL_2 \)-representation \( \rho \) such that \( W_1 = H(Y; 1), p_1^{a,b} = \dim P_\lambda^{(\Lambda)}(a,b) \) and \( \phi = S(N_{-,-}) \).

**Definition (A.1.2).** A boundary bundle \( B(W_1, p) \) is rational if the isotropic subspace \( W_1 \subset H \) is defined over \( \mathbb{Q} \). A boundary component \( B(W_1, p, \phi) \) is rational if \( W_1 \) and the form \( \phi \) are defined over \( \mathbb{Q} \). We denote by \( D^{**} \subset D^* \) the union of all rational boundary components and the union of all rational boundary bundles, respectively.

**A.2** Let \( (W_1, p, \phi) \) be a polarized isotropic subspace, and \( \{P_\lambda^{(\Lambda)}(a,b)\} \in B(W_1, p, \phi) \) a point in the associated boundary component. Then these are transformed by \( g \in G \) to the polarized isotropic subspace \((gW_1, p, (g^{-1})^*\phi)\) and the point \( \{gP_\lambda^{(\Lambda)}(a,b)\} \in (gW_1, p, (g^{-1})^*\phi) \), respectively. This defines a natural action of \( G \) on the union of all boundary components, which restricts to an action of \( G_\mathbb{Q} \) on \( D^* \) and \( D^{**} \).

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Notice also that \( g \in G \) transforms an admissible pair \((Y, r) \in g \times D\) to the admissible pair \((\text{Ad}(g)Y, gr)\), and an \( SL_2 \)-representation \( \rho : SL_2(\mathbb{R}) \to G \) horizontal at \( r \in D \) to the \( SL_2 \)-representation \( \text{Int}(g)\rho \) horizontal at \( gr \).

**Definition (A.2.1).** For a boundary bundle \( B = B(W_1, p) \), we define its normalizer \( N(B) := \{ g \in G \mid gB = B \} \) and its centralizer \( Z(B) := \{ g \in G \mid g|_B = \text{id} \} \). Let \( Y \in g \) be an admissible semi-simple element such that \((W_1, p) = (H(Y; 1), p)\). We denote by \( G_Y \) the isotropy subgroup of \( Y \) in the adjoint action of \( G \), and \( G(Y) := \{ g \in G_Y \mid \det(g|_{W_1}) = 1 \} \).

In order to express these groups by matrices, we take a basis of \( H = H_{-1} \oplus H_0 \oplus H_1 \), \( H_\lambda := H(Y; \lambda) \), subjected to the decomposition so that the bilinear form \( S \) becomes

\[
S = \begin{pmatrix}
O & J \\
-J & O
\end{pmatrix}
\]

if \( w \) is odd, \( \begin{pmatrix}
O & 0 & J \\
0 & \pm I & 0 \\
J & 0 & 0
\end{pmatrix}
\)

if \( w \) is even,

where \( J \) is an antidiagonal matrix \( J = \text{antidiag}(1, \ldots, 1) \), Then the matrices of \( N(B) \) are of the form

\[
\begin{pmatrix}
g_1 & O & O \\
* & g_0 & O \\
* & * & g_1
\end{pmatrix}
\]

where \( g_0 \in \text{Aut}(H_0, S|_{H_0}), g_1 \in \GL(n_1, \mathbb{R}), g_{-1} = J^t g_1^{-1} J, \)

and

\[
Z(B) = \{ g \in N(B) \mid g_0 = \pm I_0, g_1 = a I_1 (a \in \mathbb{R}^*) \},
\]

\( G_Y = \{ g \in N(B) \mid \text{the } * \text{'s are } 0 \}, \)

\( G(Y) = \{ g \in G_Y \mid \det g_1 = 1 \}. \)

For these expressions, one can see easily
Proposition (A.2.2). (i) $N(B), Z(B), G_Y, G(Y)$ are all independent of the choice of a set of primitive Hodge numbers $p$.

(ii) $N(B)$ is a parabolic subgroup of $G$ preserving the filtration $0 = W_2 \subset W_1 \subset W_0 = W_1^\perp \subset W_{-1} = H$.

(iii) $Z(B)$ is a closed normal subgroup of $N(B)$.

(iv) $G(Y)$ is a semi-simple group, acting transitively on $B$ with compact isotropy subgroup.

(v) $N(B) = G(Y) \cdot Z(B)$ is an almost direct product, i.e., $G(Y) \cap Z(B)$ is finite.

Let $g = \mathfrak{t} + \mathfrak{p}$ be a Cartan decomposition with $Y \in \mathfrak{p}$, $\mathfrak{t}$ a maximal abelian subspace, containing $Y$ (i.e., $\mathfrak{t}$ is the intersection of $g$ with a maximal $\mathbb{R}$-split Cartan subalgebra of $g_\mathbb{C}$), $m$ the centralizer of $\mathfrak{t}$ in $g$, and $\Phi \subset \mathfrak{t}^*$ the system of restricted roots for the adjoint action of $\mathfrak{t}$ on $g$. Then we have a root space decomposition $g = \mathfrak{t} \oplus m \oplus \sum_{\alpha \in \Phi} g_\alpha$, where $g_\alpha := \{ X \in g \mid \text{Ad}(H)X = \alpha(H)X \ (H \in \mathfrak{t}) \}$.

If we select a basis of $H$, compatible with the choice in (A.1), with respect to which the matrices of $\mathfrak{t}$ are of the diagonal form

$$H(\lambda_1, \ldots, \lambda_r) := \begin{cases} \text{diag}(-\lambda_1, \ldots, -\lambda_r) \oplus \text{diag}(\lambda_r, \ldots, \lambda_1), w: \text{odd}, \\ \text{diag}(-\lambda_1, \ldots, -\lambda_r) \oplus \text{diag}(0, \ldots, 0) \oplus \text{diag}(\lambda_r, \ldots, \lambda_1), w: \text{even}, \end{cases}$$

where $r$ is the $\mathbb{R}$-rank of $g$ and $\lambda_i \in \mathbb{R}$. Notice that $Y = (1, \ldots, 1, 0, \ldots, 0)$, and that the elements $H_i := H(\lambda_1, \ldots, \lambda_r)$ with $\lambda_j = \delta_{ij}$ form a basis of $\mathfrak{t}$ and define the lexicographic order of the roots in which the system of the positive roots $\Phi^+$ contains $\{ \alpha \in \Phi \mid \alpha(Y) > 0 \}$. Let $\xi_i$ ($1 \leq i \leq r$) be the basis of $\mathfrak{t}^*$ dual to $H_i$ ($1 \leq i \leq r$). Then the positive roots are calculated as

$$\text{Let us denote } r := \sum_{\alpha \in \Phi^+} g_\alpha, R := \exp r, T := \exp t \text{ and by } K \text{ the maximal compact subgroup of } G \text{ with } \mathfrak{k} := \text{Lie } K. \text{ Then one has the Iwasawa decomposition } G = RTK. \text{ This induces the corresponding decompositions:}$$
Proposition (A.2.3). (i) $N(B) = RTK_Y$, where $K_Y = K \cap G_Y$.

(ii) $Z(B) = (R \cap Z)(T \cap Z)(K \cap Z)$, where $Z = Z(B)$.

(iii) Let $g(Y) := \text{Lie} G(Y)$. Then $t \cap g(Y)$ is a maximal abelian subspace of $p \cap g(Y)$, $\Phi^+(Y) := \{\alpha \in \Phi^+ | \alpha(Y) = 0\}$ is the system of positive roots for the adjoint action of $t \cap g(Y)$ on $g(Y)$ and $t \cap g(Y) = \sum_{\alpha \in \Phi^+(Y)} g_{\alpha}$ (because of the compatibility of the orders), whence one has $G(Y) = (R \cap G(Y))(T \cap G(Y))(K \cap G(Y))$.

(iv) Let $r \in D$ be a point with which $Y$ forms an admissible pair and whose isotropy subgroup $V_r$ of $G$ is contained in $K$ (cf. Proposition (3.16.ii)), and let $I_b$ be the isotropy subgroup of $N(B)$ at $b = b(Y, r) := \{P^{(\lambda)a,b}\} \in B(Y, r) \subset B$. Then one has $V_r \cap G(Y) \subset I_b \cap G(Y) \subset K \cap G(Y), I_b \cap RT \subset Z(B), I_b = (RT \cap Z(B))(K \cap I_b)$.

The proof is similar to those for [CK, (3.28), (3.36), (3.40)]. Our present assumption 'type II' will be used in the proof of (iv) of the above proposition.

(A.3) Now we choose as $t$ a maximal $Q$-split Cartan subalgebra of $g$ and choose a maximal compact subgroup $K$ of $G$ such that $t \subset p$ for the associated Cartan decomposition. Let $t^+$ be the positive Weyl chamber, and $\overline{t^+}$ its closure. We denote by $\mathcal{S}$ the set of complete representatives of the $G_Q$-equivalent classes of $Q$-rational admissible element $Y$ of type II in $\overline{t^+}$. It is easy to see by definition that $\mathcal{S}$ is a finite set and that, for any admissible element $Y \in g_Q$ of type II, there exists $g \in G_Q$ satisfying $\text{Ad}(g)Y \in \mathcal{S}$.

Definition (A.3.1). The boundary bundles $B(Y, p)$ for $Y \in \mathcal{S}$ and $p$ being a set of primitive Hodge numbers compatible with $Y$ will be called the standard rational boundary bundles.

Let $\pi : D \to G/K$ be the canonical projection. By Proposition (3.16.ii) and Remark (2.12), one can choose a reference point $r_{Y,p} \in \pi^{-1}([K]) \subset D$ for each
compatible pair \((Y, p)\) with \(Y \in \mathcal{S}\), so that \((Y, r_Y, p)\) is an admissible pair. Let \(B(Y, r_Y, p)\) be the boundary component contained in the boundary bundle \(B(Y, p)\).

Let \(G = RTK\) be the Iwasawa decomposition in the present context.

**Definition (A.3.2).** A Siegel set in \(G\) is defined as \(\sigma = \omega_R T_\lambda K\), where \(\omega_R \subset R\) is a compact subset and \(T_\lambda := \{t \in T \mid e^\alpha(t) \geq \lambda (\alpha \in \Phi^+)\}\) for a positive real number \(\lambda\).

The extended Siegel set in \(D^*\) is the subset \(\sigma^* = \bigcup_{Y \in \mathcal{S}, p} \sigma_Y b_Y, p \subset D^*\), where \(\sigma_Y := \sigma \cap N(B(Y, p))\), \(b_Y, r_Y, p := \{P^{(\lambda), a, b}\} \subset B(Y, r_Y, p) \subset B(Y, p)\), and the union is taken over the finite set of all compatible pairs \((Y, p)\) with \(Y \in \mathcal{S}\).

Notice that the extended Siegel set \(\sigma^*\) in \(D^*\) is independent of the choice of a set of complete representatives of the reference points \(r_Y, p \in \pi^{-1}([K]) \subset D\). It is known that a Siegel set \(\sigma\) in \(G\) has the Siegel property: for any \(g \in G_Q\), \(\{\gamma \in \Gamma \mid \gamma \sigma \cap g\sigma \neq \emptyset\}\) is a finite set. Moreover, if the subset \(\omega_R\) and the constant \(\lambda\) are adequately chosen, then there exists a finite subset \(C \subset G_Q\) containing 1 such that \(G = C \sigma\) and \(D^* = C \sigma^*\).

Let \(\pi_{Y, p} : N(B(Y, p)) \rightarrow B(Y, p)\), sending \(g\) to \(gb_Y, p\), be the natural projection.

**Definition (A.3.3).** Let \(U_1 \subset B(Y, p)\) be an open set, \(U_2\) an open neighborhood of 1 \(\in K\), and \(\lambda\) a positive real number. Then the open set, in \(D\), \(V(U_1, U_2, \lambda) := \{g \pi_{Y, p}^{-1}(U_1) \cap U_2, e^\alpha(g) > 0 (\alpha \in \Phi, \alpha(Y) > 0)\}\) will be called a tube over \(U_1 \subset B(Y, p)\).

**Theorem (A.3.4).** (i) The sets \(U(U_1, U_2, \lambda) := (U_1 \cup V(U_1, U_2, \lambda)) \cap \sigma^*\), together with the natural topology on \(\sigma' := \sigma r_{Y, p} \subset D\), form a basis of a Hausdorff topology \(\tau^*\) on the extended Siegel set \(\sigma^*\).

(ii) Let \(g \in G\) and \(x \in \sigma\). If \(g\sigma \in \sigma^*\), then, for any \(\tau^*\)-neighborhood \(U'\) of \(gx \in \sigma^*\), there exists a \(\tau^*\)-neighborhood \(U\) of \(x \in \sigma^*\) such that \(gU \cap \sigma^* \subset U'\). If
$gx \notin \sigma^*$, then there exists a $\tau^*$-neighborhood $U$ of $x \in \sigma^*$ such that $\partial U \cap \sigma^* = \emptyset$.

The proof is analogous to those of [CK, (4.16), (4.25)]. In the proof, the following lemma will play an important role, and a Hodge-$(Z, X_{\pm})$ decomposition will also used.

**Lemma (A.3.5).** Let $W \subset RT$ and $V \subset KY$ be open subsets satisfying

(i) $W(Z(B(Y, p)) \cap RT) \subset W$ and

(ii) $V(I_{bY,p} \cap KY) \subset V$.

Then there exists an open subset $U \subset B(Y, p)$ such that $\pi^{-1}_{Y,p}(U) = WV$.

This lemma is proved by using various kind of the Iwasawa decompositions in Proposition (A.2.3).

As in [Sa.1], the results in Theorem (A.3.4) will be transformed to the corresponding assertions on the fundamental domain $\Omega^* := C\sigma^*$ in $D^*$ for the action of $\Gamma$, and finally one gets a Satake topology $\tau^\Gamma$ on $D^*$ which has the following properties:

**Theorem (A.3.6).** (i) The topology $\tau^\Gamma$ on $D^*$ induces the topology $\tau^*$ on $\sigma^*$.

(ii) The operations of $\Gamma$ are continuous.

(iii) If $\Gamma x \cap \Gamma x', x, x' \in D^*$, then there exists $\tau^\Gamma$-neighborhoods $U, U'$ of $x, x' \in D^*$ such that $\Gamma U \cap \Gamma U' = \emptyset$.

(iv) For each $x \in D^*$, there exists a fundamental system of $\tau^\Gamma$-neighborhoods $\{U\}$ of $x \in D^*$ such that $\gamma U = U$ for $\gamma \in \Gamma_x$ and $\gamma U \cap U = \emptyset$ for $\gamma \notin \Gamma_x$.

As a corollary, one obtaines

**Corollary (A.3.7).** the quotients $\Gamma \backslash D^*, \Gamma \backslash D^{**}$ endowed with the topologies induced from $\tau^\Gamma$ have the following properties:

(i) $\Gamma \backslash D^*$ and $\Gamma \backslash D^{**}$ are locally compact and Hausdorff.
(ii) $\Gamma \backslash D \subset \Gamma \backslash D^{**}$ is open and everywhere dense.

(iii) $\Gamma \backslash D^* = \bigsqcup \Gamma(B_i) \backslash B_i$, where $B_i$ runs over a finite set of complete representatives of $\Gamma$-equivalence classes of rational boundary bundles, and $\Gamma(B_i) := (\Gamma \cap N(B_i))/(\Gamma \cap Z(B_i))$ are arithmetic subgroups of the semi-simple groups $N(B_i)/Z(B_i)$.

(A.4) Let $(Y, r) \in g \times D$ be an admissible pair, $\rho$ the corresponding horizontal $SL_2$-representation, and $\tilde{\rho} : U \to D$ the associated horizontal embedding of the upperhalf plane. Then, as [CK, (6.17)], one obtains

**Proposition (A.4.1).** If $\rho$ is defined over $\mathbb{Q}$, then in the Satake topology

$$\lim_{t \to -\infty} \exp(tY)r = \lim_{\text{Im } z \to -\infty} \tilde{\rho}(z) = b(Y, r) \in D^{**}.$$ 

This is an analogous result to [Sa.2, (8.1) and its proof].

The following theorem will be proved similarly to [CK, (6.1), (6.18)]. A proof is based on the $SL_2$-orbit theorem in [Sc], the Iwasawa decompositions (A.2.3), the Satake topology (A.3.6), (A.3.7) and Theorem(3.4) and Proposition(3.16).

**Theorem (A.4.2).** (i) Let $\varphi : \Delta^* \to \Gamma \backslash D$ be a period map of type II. Then $\varphi$ can be extended continuously over the puncture to $\overline{\varphi} : \Delta \to \Gamma \backslash D^{**}$.

(ii) Let $\tilde{b} \in \Gamma \backslash D^{**}$ be an arbitrary point. Then there exists a period map $\varphi : \Delta^* \to \Gamma \backslash D$ of type II such that $\lim_{t \to 0} \varphi(t) = \tilde{b}$.

References


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