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A numerical criterion for admissibility of semi-simple elements

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Abstract

In this article, we shall generalize a theorem of Cattani and Kaplan on horizontal representations of SL(2). Their theorem plays an important role in the construction of their partial compactifications of the classifying spaces D modulo an arithmatic subgroup of Hodge structures of weight 2.

Introduction

A horizontal SL_2-representation is a generalization of the notion of "(H_1)-homomorphism" of SL_2 in the case of the classical theory of Hermitian symmetric domains (cf., e.g., [Sa.2, III]). More precisely, let $G = G_{\mathbf{R}} := \mathrm{Aut}(H_{\mathbf{R}}, S)$ be the automorphism group of the classifying space D of Hodge structures of weight w (see §1). A representation $\rho : \mathrm{SL}_2(\mathbf{R}) \to G$ is said to be *horizontal* at $r \in D$ if the morphism $\rho_* : \mathfrak{sl}_2(\mathbf{R}) \to \mathfrak{g}$ of the Lie algebras is a morphism of Hodge structures of type $(0,0)$ with respect to the Hodge structures on $\mathfrak{sl}_2(\mathbf{C})$ and $\mathfrak{g}_{\mathbf{C}}$ induced by $i \in U :=$ (upper-half plane) and $r \in D$ respectively (see Definition (2.1)). In this case, the pair (ρ, r) is uniquely determined by the pair $(Y, r) \in \mathfrak{g} \times D$ with

(0.1)
$$Y := \rho_* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

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Conversely, a pair $(Y, r) \in \mathfrak{g} \times D$ is said to be *admissible* if there exists a representation $\rho : \mathrm{SL}_2(\mathbf{R}) \to G$ horizontal at r and satisfying (0.1). The main result in the present article is a numerical criterion for admissibility of a pair (Y, r) in the case of general weight.

Given a pair (ρ, r) as above, one can refine the Hodge decomposition $H_{\mathbf{C}} = \oplus H_r^{p,q}$, corresponding to $r \in D$, under the horizontal action of $\mathfrak{sl}_2(\mathbf{C})$ at r, called a *Hodge-*$(Z, X_\pm)$ *decomposition* (see (2.7)). Our proof of the main result is based on an elementary but useful observation (Corollary (2.11), see also Remark (2.12)), which says that the transformation of the Hodge-$(Z, X_\pm)$ decomposition by the inverse c^{-1} of the Cayley element

$$c := \rho\left(\exp\frac{\pi i}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$$

yields a split mixed Hodge structure, called a *mixed Hodge-*$(Y, N_\pm)$ *decomposition*, which is nothing but the limiting mixed Hodge structure of the associated SL_2-orbit $\widetilde{\rho} : U \to D$ defined by $\widetilde{\rho}(gi) := \rho(g)r$ for $g \in \mathrm{SL}_2(\mathbf{R})$ (cf. [Sc, Theorem (6.16)] and its proof). By virtue of this observation, we can view the relationship between the pairs (ρ, r) and (Y, r) from a better perspective, and generalize a numerical criterion [CK, Theorem (2.22)] for admissibility of (Y, r) in the case of weight 2 to the case of general weight.

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§1. Preliminaries

We recall first the definition of a (polarized) Hodge structure of weight w.

Fix a free $\mathbf{Z}$-module $H_{\mathbf{Z}}$ of finite rank. Set $H_{\mathbf{Q}} := \mathbf{Q} \otimes H_{\mathbf{Z}}$, $H = H_{\mathbf{R}} := \mathbf{R} \otimes H_{\mathbf{Z}}$ and $H_{\mathbf{C}} := \mathbf{C} \otimes H_{\mathbf{Z}}$, whose complex conjugation is denoted by σ. Let w be an integer. A *Hodge structure of weight* w on $H_{\mathbf{C}}$ is a decomposition

$$H_{\mathbf{C}} = \bigoplus_{p+q=w} H^{p,q} \quad \text{with} \quad \sigma H^{p,q} = H^{q,p}. \tag{1.1}$$

The integers

$$h^{p,q} := \dim H^{p,q} \tag{1.2}$$

are called the Hodge numbers.

A polarization S for a Hodge structure (1.1) of weight w is a non-degenerate bilinear form on $H_{\mathbf{Q}}$, symmetric if w is even and skew-symmetric if w is odd, such that its $\mathbf{C}$-bilinear extension, denoted also by S, satisfies

$$\begin{aligned} &S(H^{p,q}, \sigma H^{p',q'}) = 0 \quad \text{unless} \quad (p,q) = (p',q'), \\ &i^{p-q} S(v, \sigma v) > 0 \quad \text{for all} \quad 0 \neq v \in H^{p,q}. \end{aligned} \tag{1.3}$$

Remark (1.4) In the geometric case, i.e., the Hodge structure on the w-th cohomology group $H^w(X, \mathbf{Q})$ of a smooth projective variety $X \subset \mathbf{P}^N$ of dimension d over $\mathbf{C}$, we take as a polarization

$$S(u,v) := (-1)^{w(w-1)/2} \int_X u \wedge v \wedge \eta^{d-w}$$

for primitive classes $u, v \in H^w_{\text{prim}}(X, \mathbf{C}) \simeq H^w_{\text{prim}}(X, \Omega^{\cdot}_X)$ where $\eta \in H^1(X, \Omega^1_X)$ is the cohomology class of a hyperplane section of X.

For fixed S and $\{h^{p,q}\}$, the classifying space D for Hodge structures and its "compact dual" $\check{D}$ are defined by

$$\begin{aligned} \check{D} &:= \{\{H^{p,q}\} \mid \text{Hodge structure on } H_{\mathbf{C}} \text{ with } \dim H^{p,q} = h^{p,q}, \\ &\qquad\qquad \text{satisfying the first condition in (1.3)}\}, \\ D &:= \{\{H^{p,q}\} \in \check{D} \mid \text{satisfying also the second condition in (1.3)}\}. \end{aligned} \tag{1.5}$$

These are homogeneous spaces under the natural actions of the groups

$$G_{\mathbf{C}} := \operatorname{Aut}(H_{\mathbf{C}}, S), \quad G = G_{\mathbf{R}} := \{g \in G_{\mathbf{C}} \mid gH_{\mathbf{R}} = H_{\mathbf{R}}\}, \tag{1.6}$$

respectively. Taking a reference point $r \in D$, one obtains identifications

$$\check{D} \simeq G_{\mathbf{C}}/B_{\mathbf{C}}, \quad D \simeq G/V, \tag{1.7}$$

where $B_{\mathbf{C}}$ and V are the isotropy subgroups of $G_{\mathbf{C}}$ and of G at $r \in D$, respectively. It is a direct consequence of the definition that

$$G \simeq \begin{cases} O(2h,k), \\ \mathrm{Sp}(2h,\mathbf{R}), \end{cases} \quad V \simeq \begin{cases} U(h^{w,0}) \times \cdots \times U(h^{t+1,t-1}) \times O(h^{t,t}) & \text{if } w = 2t, \\ U(h^{w,0}) \times \cdots \times U(h^{t+1,t}) & \text{if } w = 2t+1, \end{cases} \tag{1.8}$$

where $k := \sum_{|j| \le [t/2]} h^{t+2j,t-2j}$ and $h := (\dim H - k)/2$ if $w = 2t$, and $h := \dim H/2$ if $w = 2t+1$. It is an important observation that V is compact, but not maximal compact in general. Hence D is a symmetric domain of Hermitian type if and only if

$$h^{p,q} = 0 \quad \text{unless } (p,q) = \begin{cases} (t+1,t-1), (t,t) \text{ or } (t-1,t+1), \\ \qquad \text{and } h^{t+1,t-1} = 1 \quad \text{if } w = 2t, \\ (t+1,t) \text{ or } (t,t+1) \quad \text{if } w = 2t+1. \end{cases} \tag{1.9}$$

A reference Hodge structure $r = \{H_r^{p,q}\} \in D$ induces a Hodge structure of weight 0 on the Lie algebra $\mathfrak{g}_{\mathbf{C}} := \operatorname{Lie} G_{\mathbf{C}}$ by

$$\mathfrak{g}_{\mathbf{C}}^{s,-s} := \{X \in \mathfrak{g}_{\mathbf{C}} \mid XH_r^{p,q} \subset H_r^{p+s,q-s} \text{ for all } p,q\}. \tag{1.10}$$

One can define the associated Cartan involution θ_r on $\mathfrak{g}_{\mathbf{C}}$ by

$$\theta_r(X) := \sum_s (-1)^s X^{s,-s} \quad \text{for} \quad X = \sum_s X^{s,-s} \in \mathfrak{g}_{\mathbf{C}} = \bigoplus_s \mathfrak{g}_{\mathbf{C}}^{s,-s}. \tag{1.11}$$

This can be interpreted in the following way: Set

$$(1.12)\qquad \begin{aligned} H_r^+ &:= H_r^{w,0} \oplus H_r^{w-2,2} \oplus H_r^{w-4,4} \oplus \cdots, \\ H_r^- &:= H_r^{w-1,1} \oplus H_r^{w-3,3} \oplus H_r^{w-5,5} \oplus \cdots. \end{aligned}$$

It is clear by definition that the isotropy subgroup of the decomposition $H_{\mathbf{C}} = H_r^+ \oplus H_r^-$ induces the maximal compact subgroup

$$(1.13)\qquad K \simeq \begin{cases} U(h) \times O(k) & \text{if } w = 2t, \\ U(h) & \text{if } w = 2t+1, \end{cases}$$

of G which contains V, and the Cartan involution θ_r in (1.11) is the one associated to K. Define a $\mathbf{C}$-linear automorphism

$$(1.14)\qquad E_r : H_{\mathbf{C}} \to H_{\mathbf{C}} \quad \text{by} \quad E_r := \begin{cases} 1 & \text{on } H_r^+, \\ -1 & \text{on } H_r^-. \end{cases}$$

Then the Cartan involution θ_r in (1.11) can also be written as

$$(1.15)\qquad \theta_r X = (\mathrm{Int}\, E_r) X \quad \text{for} \quad X \in \mathfrak{g}_{\mathbf{C}}.$$

We recall now well-known results on SL_2-representations. Let ξ, η be two variables, and write

$$(1.16)\qquad \begin{pmatrix} \xi \\ \eta \end{pmatrix}^{(m)} := \begin{pmatrix} \xi^m \\ \xi^{m-1}\eta \\ \vdots \\ \eta^m \end{pmatrix} \qquad (m = 0, 1, 2, \cdots).$$

A representation

$$(1.17)\qquad \rho_m : \mathrm{SL}_2(\mathbf{R}) \to \mathrm{SL}_{m+1}(\mathbf{R}) \text{ defined by } \rho_m(g) \begin{pmatrix} \xi \\ \eta \end{pmatrix}^{(m)} := \left(g \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right)^{(m)}$$

is called a symmetric tensor representation of dimension $m+1$. It is known that the ρ_m $(m = 0, 1, 2, \cdots)$ are absolutely irreducible and constitute a full set of representatives for the equivalence classes of finite dimensional irreducible representations of $\mathrm{SL}_2(\mathbf{R})$.

We take the standard generators for the Lie algebras $\mathfrak{sl}_2(\mathbf{R})$ and $\mathfrak{su}(1,1)$ which are related by the Cayley transformation $\mathrm{Int}\, c_1$, where

$$(1.18) \qquad c_1 := \exp \frac{\pi i}{4} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix},$$

as follows:

$$(1.19) \qquad \begin{array}{lllll} \mathfrak{sl}_2(\mathbf{R}) & \ni y := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & n_+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & n_- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ \mathrm{Int}\, c_1 \downarrow & \quad \downarrow & \quad \downarrow & \quad \downarrow \\ \mathfrak{su}(1,1) & \ni z := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, & x_+ := \frac{1}{2} \begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}, & x_- := \frac{1}{2} \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}. \end{array}$$

The following lemma can be verified directly by using the monomial basis (1.16) and the definition (1.19), and so we omit the proof.

Lemma (1.20). **(i)** *In the above notation,* $Y_m := \rho_{m*}(y)$ *and* $N_{m\pm} := \rho_{m*}(n_\pm)$ *satisfy*

$$Y_m(\xi^{m-j}\eta^j) = (m-2j)\xi^{m-j}\eta^j,$$

$$N_{m+}(\xi^{m-j}\eta^j) = (m-j)\xi^{m-j-1}\eta^{j+1},$$

$$N_{m-}(\xi^{m-j}\eta^j) = j\xi^{m-j+1}\eta^{j-1}.$$

(ii) *For the Cayley element* $c_m := \rho_{m*}(c_1) \in \mathrm{SL}_{m+1}(\mathbf{C})$,

$\sigma c_m = c_m^{-1}\sigma$, *where* σ *is the complex conjugation.*

$$c_m^{\pm 2}(\xi^{m-j}\eta^j) = (\pm i)^m \eta^{m-j}\xi^j,$$

$$c_m^4(\xi^{m-j}\eta^j) = (-1)^m \xi^{m-j}\eta^j.$$

Remark (1.21). The Hodge structure on $\mathfrak{g}_{1\mathbf{C}} := \mathfrak{sl}_2(\mathbf{C})$ induced by $i \in U :=$ (upper-half plane) $\simeq \mathrm{SL}_2(\mathbf{R})/U(1)$ coincides with the canonical decomposition by the standard "H-element" $(n_+ - n_-)/2$ (cf., e.g., [Sa.2, II. §7]):

$$\mathfrak{g}_{1\mathbf{C}} = \mathfrak{g}_{1\mathbf{C}}^{1,-1} + \mathfrak{g}_{1\mathbf{C}}^{0,0} + \mathfrak{g}_{1\mathbf{C}}^{-1,1} = \mathfrak{p}_- + \mathfrak{k}_\mathbf{C} + \mathfrak{p}_+ = \{x_-\}_\mathbf{C} + \{z\}_\mathbf{C} + \{x_+\}_\mathbf{C}.$$

§2. Horizontal SL_2-representations

From now on, we assume that $w > 0$ and all Hodge structures of weight w satisfy $H^{p,q} = 0$ unless $p, q \geq 0$.

Definition (2.1) (cf. [Sc, p.258]). *An* SL_2*-representation* $\rho : \mathrm{SL}_2(\mathbf{R}) \to G$ *is said to be horizontal at* $r = \{H_r^{p,q}\} \in D$ *if* $\rho_*(x_+) \in \mathfrak{g}_\mathbf{C}^{-1,1} := \{X \in \mathfrak{g}_\mathbf{C} \mid XH_r^{p,q} \subset H_r^{p-1,q+1}$ *for all* $p, q\}$.

Remark (2.2). It is clear that an SL_2-representation ρ is horizontal if and only if $\rho_* : \mathfrak{sl}_2(\mathbf{R}) \to \mathfrak{g}$ is a morphism of Hodge structures of type $(0,0)$ with respect to the Hodge structures induced by $i \in U$ and $r \in D$, respectively. A horizontal SL_2-representation ρ induces an equivariant horizontal map $\tilde{\rho} : \mathbf{P}^1 \to \check{D}$ with $\tilde{\rho}(i) = r$:

$$\begin{array}{ccc} \mathrm{SL}_2(\mathbf{C}) & \xrightarrow{\rho} & G_\mathbf{C} \\ \downarrow & & \downarrow \\ \mathbf{P}^1 & \xrightarrow{\tilde{\rho}} & \check{D} \end{array}$$

This is a generalization to the present context of the notion of '(H_1)-homomorphism' in the case of symmetric domains of Hermitian type (cf., e.g., [Sa.2, II. (8.5), III. §1]).

Let $\rho : \mathrm{SL}_2(\mathbf{R}) \to G$ be a representation horizontal at $r = \{H_r^{p,q}\} \in D$, and set

(2.3) $\qquad Y := \rho_*(y), \quad N_\pm := \rho_*(n_\pm); \quad Z := \rho_*(z), \quad X_\pm := \rho_*(x_\pm).$

Notice that by (1.19) these are related under the Cayley transformation:

$$(2.4) \qquad Z = (\operatorname{Int} c)Y, \quad X_\pm = (\operatorname{Int} c)N_\pm, \quad c := \rho(c_1).$$

$(Y, N_\pm)$ and $(Z, X_\pm)$ define direct sum decompositions of H and $H_{\mathbf{C}}$ whose summands are

$$(2.5) \qquad P_\lambda^{(\lambda+2k)} := N_-^k(H(Y; \lambda + 2k) \cap \operatorname{Ker} N_+),$$

$$(2.6) \qquad Q_\lambda^{(\lambda+2k)} := X_-^k(H_{\mathbf{C}}(Z; \lambda + 2k) \cap \operatorname{Ker} X_+),$$

for all eigenvalues $\lambda \in \{0, \pm 1, \pm 2, \cdots \pm w\}$ of Y and Z and for $k \geq \max\{-\lambda, 0\}$, respectively. Here we denote by $H(Y; \lambda+2k)$ etc. the eigenspace of an endomorphism Y of H with eigenvalue $\lambda+2k$. Since ρ is horizontal at $r = \{H_r^{p,q}\}$, (2.6) is compatible with this Hodge structure and we set

$$(2.7) \qquad Q_\lambda^{(\lambda+2k)\, a+k, b+\lambda+k} := Q_\lambda^{(\lambda+2k)} \cap H_r^{a+k, b+\lambda+k} \quad (a, b \geq 0).$$

These form a refined direct sum decomposition which we call the *Hodge-*$(Z, X_\pm)$ *decomposition of* (ρ, r) (cf. Remark (2.12) below). Transforming this by the inverse c^{-1} of the Cayley element, we define

$$(2.8) \qquad P_\lambda^{(\lambda+2k)\, a+k, b+k} := c^{-1} Q_\lambda^{(\lambda+2k)\, a+k, b+\lambda+k}.$$

Lemma (2.9). (i) $\sigma Q_\lambda^{(\lambda+2k)\, a+k, b+\lambda+k} = Q_{-\lambda}^{(-\lambda+2(\lambda+k))\, b+\lambda+k, a+k}$.

(ii) $c Q_\lambda^{(\lambda+2k)\, a+k, b+\lambda+k} = c^2 P_\lambda^{(\lambda+2k)\, a+k, b+k} = P_{-\lambda}^{(-\lambda+2(\lambda+k))\, a+\lambda+k, b+\lambda+k}$.

$c^{-1} P_\lambda^{(\lambda+2k)\, a+k, b+k} = c^{-2} Q_\lambda^{(\lambda+2k)\, a+k, b+\lambda+k} = Q_{-\lambda}^{(-\lambda+2(\lambda+k))\, a+\lambda+k, b+k}$.

Proof. It is easy to see, by definition, that $cP_\lambda^{(\lambda+2k)} = Q_\lambda^{(\lambda+2k)}$. Hence, by the first equality in (1.20.ii), we have

$$\sigma Q_\lambda^{(\lambda+2k)} = \sigma c P_\lambda^{(\lambda+2k)} = c^{-1} \sigma P_\lambda^{(\lambda+2k)} = c^{-1} P_\lambda^{(\lambda+2k)} = c^{-2} Q_\lambda^{(\lambda+2k)}.$$

On the other hand, by the second equality in (1.20.ii), the third and the second equalities in (1.20.i), we see that on $P_\lambda^{(\lambda+2k)}$

$$c^{-2} = \begin{cases} i^{\lambda+2k}\frac{k!}{(\lambda+k)!}N_-^\lambda & \text{if } \lambda \geq 0, \\ i^{\lambda+2k}\frac{(\lambda+k)!}{k!}N_+^\lambda & \text{if } \lambda < 0. \end{cases}$$

Taking their Cayley transforms, we see that on $Q_\lambda^{(\lambda+2k)}$

$$\textbf{(2.10)} \qquad c^{-2} = \begin{cases} i^{\lambda+2k}\frac{k!}{(\lambda+k)!}X_-^\lambda & \text{if } \lambda \geq 0, \\ i^{\lambda+2k}\frac{(\lambda+k)!}{k!}X_+^\lambda & \text{if } \lambda < 0. \end{cases}$$

Thus, by the definition of the $Q_\lambda^{(\lambda+2k)}$, we have in both cases that

$$\sigma Q_\lambda^{(\lambda+2k)} = c^{-2}Q_\lambda^{(\lambda+2k)} = X_\mp^{\pm\lambda}Q_\lambda^{(\lambda+2k)} = Q_{-\lambda}^{(\lambda+2k)} = Q_{-\lambda}^{(-\lambda+2(\lambda+k))}.$$

This together with $\sigma H_r^{a+k,b+\lambda+k} = H_r^{b+\lambda+k,a+k}$ yields the assertion (i).

By horizontality, $X_- \in \mathfrak{g}_{\mathbf{C}}^{1,-1}$ and $X_+ \in \mathfrak{g}_{\mathbf{C}}^{-1,1}$, hence $X_\mp^{\pm\lambda} \in \mathfrak{g}_{\mathbf{C}}^{\lambda,-\lambda}$. This together with (2.10) shows that

$$c^{-2}Q_\lambda^{(\lambda+2k)a+k,b+\lambda+k} = X_\mp^{\pm\lambda}Q_\lambda^{(\lambda+2k)a+k,b+\lambda+k} = Q_{-\lambda}^{(-\lambda+2(\lambda+k))a+\lambda+k,b+k}.$$

Thus we obtain the second equality in (ii). The first equality in (ii) follows from the second. ∎

Corollary (2.11). *Let (ρ, r) be as above. For each eigenvalue λ of Y and for $k \geq \max\{-\lambda, 0\}$, we see that*

$$\mathbf{C} \otimes P_\lambda^{(\lambda+2k)} = \bigoplus_{\substack{a+b+2k=w-\lambda \\ a,b\geq 0}} P_\lambda^{(\lambda+2k)\,a+k,b+k}$$

is a Hodge structure of weight $w - \lambda$. Moreover, in the case $\lambda = k = 0$, this is S-polarized.

Proof. We should observe the behavior under the complex conjugation σ:

$$\begin{aligned}\sigma P_{\lambda}^{(\lambda+2k)\,a+k,b+k} &= \sigma c^{-1} Q_{\lambda}^{(\lambda+2k)\,a+k,b+\lambda+k} = c\, Q_{-\lambda}^{(-\lambda+2(\lambda+k))\,b+\lambda+k,a+k} \\ &= c^2 P_{-\lambda}^{(-\lambda+2(\lambda+k))\,b+\lambda+k,a+\lambda+k} = P_{\lambda}^{(\lambda+2k)\,b+k,a+k}.\end{aligned}$$

This shows the first assertion.

The representation ρ is trivial on $Q_0^{(0)}$, hence $P_0^{(0)\,a,b} = c^{-1} Q_0^{(0)\,a,b} = Q_0^{(0)\,a,b}$, and so the second assertion trivially holds. ∎

We call a direct sum decomposition in (2.11) the *mixed Hodge-$(Y, N_{\pm})$ decomposition of (ρ, r).*

Remark (2.12). We remark here some observations which are verified easily by (1.20.i), their Cayley transforms and horizontality of ρ at r. A Hodge-$(Z, X_{\pm})$ decomposition and a mixed Hodge-$(Y, N_{\pm})$ decomposition form "nests of diamonds", respectively. For example, in the case of weight $w = 3$, these nests of diamonds are illustrated respectively as in Figures 1 and 2.

(Figure 1)

$$\begin{array}{ccccccccccc}
 & & & Q_{-3}^{(3)3,0} & & & & & & & \\
 & & Q_{-2}^{(2)3,0} & & Q_{-2}^{(2)2,1} & & & & & & \\
 & Q_{-1}^{(1)3,0} & & Q_{-1}^{(3)2,1} & & Q_{-1}^{(1)1,2} & & & & Q_{-1}^{(1)2,1} & \\
Q_0^{(0)3,0} & & Q_0^{(2)2,1} & & Q_0^{(2)1,2} & & Q_0^{(0)0,3} & , & Q_0^{(0)2,1} & & Q_0^{(0)1,2}, \\
 & Q_1^{(1)2,1} & & Q_1^{(3)1,2} & & Q_1^{(1)0,3} & & & & Q_1^{(1)1,2} & \\
 & & Q_2^{(2)1,2} & & Q_2^{(2)0,3} & & & & & & \\
 & & & Q_3^{(3)0,3} & & & & & & &
\end{array}$$

(Figure 2)

$$
\begin{array}{ccccccccccc}
 & & & P_{-3}^{(3)3,3} & & & & & & & \\
 & & P_{-2}^{(2)3,2} & & P_{-2}^{(2)2,3} & & & & & & \\
 & P_{-1}^{(1)3,1} & & P_{-1}^{(3)2,2} & & P_{-1}^{(1)1,3} & & & & P_{-1}^{(1)2,2} & \\
P_{0}^{(0)3,0} & & P_{0}^{(2)2,1} & & P_{0}^{(2)1,2} & & P_{0}^{(0)0,3} & , & P_{0}^{(0)2,1} & & P_{0}^{(0)1,2}. \\
 & P_{1}^{(1)2,0} & & P_{1}^{(3)1,1} & & P_{1}^{(1)0,2} & & & & P_{1}^{(1)1,1} & \\
 & & P_{2}^{(2)1,0} & & P_{2}^{(2)0,1} & & & & & & \\
 & & & P_{3}^{(3)0,0} & & & & & & &
\end{array}
$$

On these nests of diamonds, the complex conjugation by σ sends respectively a summand $Q_{\lambda}^{(\lambda+2k)a+k,b+\lambda+k}$ to a summand $Q_{-\lambda}^{(-\lambda+2(\lambda+k))b+\lambda+k,a+k}$ which are symmetric with respect to the origin of the diamonds, and a summand $P_{\lambda}^{(\lambda+2k)a+k,b+k}$ to a summand $P_{\lambda}^{(\lambda+2k)b+k,a+k}$ which are symmetric with respect to the vertical axis. The operator X_+ (resp. X_-) sends a summand $Q_{\lambda}^{(\lambda+2k)a+k,b+\lambda+k}$ one step down (resp. up) to a summand $Q_{\lambda+2}^{(\lambda+2+2(k-1))a+k-1,b+\lambda+2+k-1}$ (resp. $Q_{\lambda-2}^{(\lambda-2+2(k+1))a+k+1,b+\lambda-2+k+1}$), and $X_\pm$ are inverse to each other up to non-zero constant between these summands whenever both summands actually appear in the nest of diamonds. Similarly, the operator N_+ (resp. N_-) sends a summand $P_{\lambda}^{(\lambda+2k)a+k,b+k}$ one step down (resp. up) to a summand $P_{\lambda+2}^{(\lambda+2+2(k-1))a+k-1,b+k-1}$ (resp. $P_{\lambda-2}^{(\lambda-2+2(k+1))a+k+1,b+k+1}$), and $N_\pm$ are inverse to each other up to non-zero constant between these summands whenever both summands actually appear in the nest of diamonds. The Cayley element c transforms the second nest of diamonds together with the action of the operators $Y, N_\pm$ to the first nest of diamonds together with the action of the operators $Z, X_\pm$: $cP_{\lambda}^{(\lambda+2k)a+k,b+k} = Q_{\lambda}^{(\lambda+2k)a+k,b+\lambda+k}$.

By using these operators, we can explain why the summands outside the nests of diamonds vanish in the following way. We claim first that $Q_{\lambda}^{(\lambda+2k)a+k,b+\lambda+k} = 0$ for $\lambda > 0$ and $b < 0$. Indeed, $X_-^{\lambda+k}$ is injective on this summand by the Cayley transform of the third equality in (1.20.i). On the other hand, looking at the Hodge type, we see that $X_-^{\lambda+k}Q_\lambda^{(\lambda+2k)a+k,b+\lambda+k} \subset Q_{-\lambda-2k}^{(-\lambda-2k+2(\lambda+2k))a+\lambda+2k,b} = 0$ by horizontality. Thus we get our claim. It follows by symmetry under the complex conjugation σ that $Q_\lambda^{(\lambda+2k)a+k,b+\lambda+k} = 0$ for $\lambda < 0$ and $a < 0$. Finally, by the inverse of the Cayley transformation, we have $P_\lambda^{(\lambda+2k)a+k,b+k} = 0$ for $\lambda > 0$ and $b < 0$, and for $\lambda < 0$ and $a < 0$.

We call the length of the side of the biggest diamond in a nest the *size* of the nest of diamonds.

Another remark is that a mixed Hodge-$(Y, N\pm)$ decomposition is nothing but the limiting split mixed Hodge structure of the associated SL_2-orbit $\widetilde{\rho} : U \to D$, $\widetilde{\rho}(gi) := \rho(g)r$ $(g \in SL_2(\mathbf{R}))$, and the monodromy weight filtration L is described as $L_i = \bigoplus_{\lambda\leq i}\bigoplus_k P_{-\lambda}^{(\lambda+2k)}$ (cf. [Sc, (6.16)] and its proof, [CK, pp. 13–14]).

In the above notation, for all λ, a and b, put

$$(2.13)\qquad \begin{aligned} n_\lambda &:= \dim_{\mathbf{R}} H(Y;\lambda) = \dim_{\mathbf{C}} H_{\mathbf{C}}(Z;\lambda), \\ p_\lambda^{a,b} &:= \dim_{\mathbf{C}} P_{\lambda-2k}^{(\lambda)\,a+k,b+k} = \dim_{\mathbf{C}} Q_{\lambda-2k}^{(\lambda)\,a+k,b+\lambda-k}. \end{aligned}$$

Notice that, by construction, the middle terms and the terms on the extreme right hand side of the second equality in (2.13) are independent of k (cf. Remark (2.12)).

Lemma (2.14). *For (ρ, r) as above, the following hold:*

(i) $\displaystyle\sum_{a+b=w-\lambda} p_\lambda^{a,b} = n_\lambda - n_{\lambda+2}$ *for all* $0 \leq \lambda \leq w$.

(ii) $p_\lambda^{b,a} = p_\lambda^{a,b}$ *for all* λ, a, b *with* $0 \leq \lambda \leq w, a \geq 0, b \geq 0$ *and* $a + b = w - \lambda$.

(iii) $$h^{a,b} = h^{a+1,b-1} - (p_0^{a+1,b-1} + p_1^{a+1,b-2} + \cdots + p_{b-1}^{a+1,0}) + (p_0^{a,b} + p_1^{a-1,b} + \cdots + p_a^{0,b})$$

for all a, b *with* $a \geq 0, b \geq 0$ *and* $a + b = w$.

Proof. We first observe that there is an exact sequence

$$0 \to P_\lambda^{(\lambda)} \longrightarrow H(Y;\lambda) \xrightarrow{N_+} H(Y;\lambda+2) \to 0$$

for every $\lambda \geq 0$ (and N_- yields a right splitting). (i) and (ii) follow from this and (2.11).

In order to show (iii), we look at the morphism $X_+ : H^{a+1,b-1} \to H^{a,b}$ and its kernel and cokernel:

$$\begin{aligned}
\mathrm{Ker} &= Q_0^{(0)\,a+1,b-1} \oplus Q_1^{(1)\,a+1,b-1} \oplus \cdots \oplus Q_{b-1}^{(b-1)\,a+1,b-1} \\
&\overset{c}{\xleftarrow{\sim}} P_0^{(0)\,a+1,b-1} \oplus P_1^{(1)\,a+1,b-2} \oplus \cdots \oplus P_{b-1}^{(b-1)\,a+1,0}, \\
\mathrm{Coker} &\simeq Q_0^{(0)\,a,b} \oplus Q_{-1}^{(1)\,a,b} \oplus \cdots \oplus Q_{-a}^{(a)\,a,b} \\
&\simeq Q_0^{(0)\,a,b} \oplus Q_1^{(1)\,a-1,b+1} \oplus \cdots \oplus Q_a^{(a)\,0,b+a} \\
&\overset{c}{\xleftarrow{\sim}} P_0^{(0)\,a,b} \oplus P_1^{(1)\,a-1,b} \oplus \cdots \oplus P_a^{(a)\,0,b}.
\end{aligned}$$

Looking at the dimension, we get (iii). ∎

Definition (2.15). *We call a set of integers $\{p_\lambda^{a,b}\}$, which satisfies the conditions (i), (ii) and (iii) of (2.14), a set of primitive Hodge numbers belonging to* $\{h^{p,q}, n_\lambda\}$.

§3. Admissible R-semi-simple elements

We continue to use the notation in the previous sections.

Proposition (3.1). *Given a pair $(Y, r) \in \mathfrak{g} \times D$, there exists at most one representation $\rho : \mathrm{SL}_2(\mathbf{R}) \to G$ which is horizontal at r and $\rho_* y = Y$.*

Proof. Since y and z generate $\mathfrak{sl}_2(\mathbf{C})$, it is enough to show that if such a representation ρ exists then the eigenspaces of Z, and hence Z itself, are determined by the pair (Y, r). Actually, we shall show by induction on the size w of the nest of diamonds of the Hodge-$(Z, X_\pm)$ decomposition (2.7) (cf. Remark (2.12)) that this nest of diamonds is completely determined by (Y, r).

First notice that

$$Y = i(X_+ - X_-). \tag{3.2}$$

For a subspace M of $H_{\mathbf{C}}$, we put, throughout this proof,

$$M^{\perp} := \{v \in H_{\mathbf{C}} \mid S(v, \sigma u) = 0 \text{ for all } u \in M\},$$

$$\text{projection}\{M \to H_r^{p,q}\} := \operatorname{Im}\{M \subset H_{\mathbf{C}} = \bigoplus_{p'+q'=w} H_r^{p',q'} \to H_r^{p,q}\}.$$

Then we see that

$$Q_w^{(w)\,0,w} = \text{projection}\left\{Y^w H_r^{w,0} \to H_r^{0,w}\right\},$$

$$Q_{w-2k}^{(w)\,k,w-k} = \text{projection}\left\{Y^k Q_w^{(w)\,0,w} \to H_r^{k,w-k}\right\} \quad (0 \le k \le w),$$

$$\bigoplus_{0\le\lambda\le w-1} Q_{-\lambda}^{(\lambda)\,w,0} = H_r^{w,0} \cap \left(Q_{-w}^{(w)\,w,0}\right)^{\perp},$$

$$Q_{w-1}^{(w-1)\,1,w-1} = \text{projection}\left\{Y^{w-1}\Big(\bigoplus_{0\le\lambda\le w-1} Q_{-\lambda}^{(\lambda)\,w,0}\Big) \to H_r^{1,w-1}\right\},$$

$$Q_{w-1-2k}^{(w-1)\,1+k,w-1-k} = \text{projection}\left\{Y^k Q_{w-1}^{(w-1)\,1,w-1} \to H_r^{1+k,w-1-k}\right\} \quad (0 \le k \le w-1),$$

$$\bigoplus_{0\le\lambda\le w-2} Q_{-\lambda}^{(\lambda)\,w,0} = H_r^{w,0} \cap \Big(\bigoplus_{w-1\le\lambda\le w} Q_{-\lambda}^{(\lambda)\,w,0}\Big)^{\perp},$$

$$Q_{w-2}^{(w-2)\,2,w-2} = \text{projection}\left\{Y^{w-2}\Big(\bigoplus_{0\le\lambda\le w-2} Q_{-\lambda}^{(\lambda)\,w,0}\Big) \to H_r^{2,w-2}\right\},$$

$$Q_{w-2-2k}^{(w-2)\,2+k,w-2-k} = \text{projection}\left\{Y^k Q_{w-2}^{(w-2)\,2,w-2} \to H_r^{2+k,w-2-k}\right\} \quad (0 \le k \le w-2),$$

$$\cdots\cdots.$$

Thus $Q_{\lambda-2k}^{(\lambda)\,w-\lambda+k,\lambda-k}$ $(0 \le \lambda \le w,\ 0 \le k \le \lambda)$ are determined. Taking the complex conjugation by σ of these, we get $Q_{-\lambda+2k}^{(\lambda)\,\lambda-k,w-\lambda+k} = \sigma Q_{\lambda-2k}^{(\lambda)\,w-\lambda+k,\lambda-k}$ $(0 \le \lambda \le w,\ 0 \le k \le \lambda)$. Applying the induction hypothesis to the nest of diamonds of size $\le w-2$ in

$$\Big(\bigoplus_{\substack{0\le\lambda\le w\\ 0\le k\le\lambda}} \left(Q_{\lambda-2k}^{(\lambda)\,w-\lambda+k,\lambda-k} \oplus Q_{-\lambda+2k}^{(\lambda)\,\lambda-k,w-\lambda+k}\right)\Big)^{\perp}$$

(cf. Remark (2.12)), we get our assertion. ∎

Definition (3.3). *A pair $(Y, r) \in \mathfrak{g} \times D$ is admissible if there exists a representation $\rho : \mathrm{SL}_2(\mathbf{R}) \to G$ which is horizontal at r and $\rho_*(y) = Y$.*

The set of primitive Hodge numbers $\{p_\lambda^{a,b}\}$ belonging to $\{h^{p,q}, n_\lambda\}$ is called the type of an admissible pair (Y, r).

$Y \in \mathfrak{g}$ is said to be admissible if (Y, r) is an admissible pair for some $r \in D$.

Now we prove the following numerical criterion for admissibility:

Theorem (3.4). *$Y \in \mathfrak{g}$ is admissible if and only if Y is semi-simple over $\mathbf{R}$ whose eigenvalues are contained in $\{0, \pm 1, \pm 2, \cdots, \pm w\}$ and there exists a set of primitive Hodge numbers $\{p_\lambda^{a,b}\}$ belonging to $\{h^{p,q}, n_\lambda\}$, where $n_\lambda := \dim H(Y; \lambda)$* (cf. Definition (2.15)).

Proof. Since Y is semi-simple over $\mathbf{R}$, the eigenspaces $H(Y; \lambda)$ are defined over $\mathbf{R}$ and $H(Y; \lambda)$ and $H(Y; \mu)$ are S-orthogonal unless $\lambda + \mu = 0$. Therefore $H(Y; \lambda)$ and $H(Y; -\lambda)$ are S-dual.

Since $n_{\lambda'} - n_{\lambda'+2} \geq 0$ for $\lambda' \geq 0$ by the condition (2.14.i), we can take a direct sum decomposition

$$H(Y; \lambda) = P_\lambda^{(\lambda)} \oplus P_\lambda^{(\lambda+2)} \oplus P_\lambda^{(\lambda+4)} \oplus \cdots \quad \text{for } \lambda \geq 0 \tag{3.5}$$

with $\dim P_\lambda^{(\lambda+2k)} = n_{\lambda+2k} - n_{\lambda+2k+2}$. Moreover, in the case $\lambda = 0$, the decomposition (3.5) can be taken to be S-orthogonal. We denote the S-dual decomposition by

$$H(Y; -\lambda) = P_{-\lambda}^{(\lambda)} \oplus P_{-\lambda}^{(\lambda+2)} \oplus P_{-\lambda}^{(\lambda+4)} \oplus \cdots \quad (\lambda \geq 0), \tag{3.6}$$

i.e, $P_{-\lambda}^{(\lambda+2k)}$ and $P_\lambda^{(\lambda+2m)}$ are S-orthogonal unless $k = m$.

By the conditions (i) and (ii) of (2.14), we can choose a Hodge decomposition

$$\mathbf{C} \otimes P_\lambda^{(\lambda+2k)} = \bigoplus_{\substack{a+b+2k = w-\lambda \\ a,b \geq 0}} P_\lambda^{(\lambda+2k)\,a+k,b+k} \quad \text{for } \lambda \geq 0, k \geq 0, \tag{3.7}$$

with $\dim P_{\lambda}^{(\lambda+2k)\,a+k,b+k} = p_{\lambda+2k}^{a,b}$. Moreover, in the case $\lambda = k = 0$, the Hodge structure (3.7) can be chosen to be S-polarized. We denote the $S(\cdot\,,\sigma\cdot)$-orthogonal decomposition by

$$\text{(3.8)}\quad \mathbf{C}\otimes P_{-\lambda}^{(-\lambda+2(\lambda+k))} = \bigoplus_{\substack{a+b+2\lambda+2k=w+\lambda\\ a,b\geq 0}} P_{-\lambda}^{(-\lambda+2(\lambda+k))\,a+\lambda+k,b+\lambda+k} \quad (\lambda\geq 0, k\geq 0),$$

i.e., $S(P_{-\lambda}^{(-\lambda+2(\lambda+k))\,a+\lambda+k,b+\lambda+k}, \sigma P_{\lambda}^{(\lambda+2k)\,a'+k,b'+k}) = 0$ unless $(a,b)=(a',b')$. Notice that $P_{-\lambda}^{(-\lambda+2(\lambda+k))\,a+\lambda+k,b+\lambda+k} = P_{-\lambda}^{(\lambda+2k)\,a+\lambda+k,b+\lambda+k}$.

Now we consider the cases $\lambda\geq 0$ and $\lambda<0$ altogether. For $k\geq\max\{-\lambda,0\}$ and $a\geq b$, let

$$\text{(3.9)}\qquad \{v_{\lambda,j}^{(\lambda+2k)\,a+k,b+k} \mid 1\leq j\leq p_{\lambda+2k}^{a,b}\}$$

be a $\mathbf{C}$-basis of $P_{\lambda}^{(\lambda+2k)\,a+k,b+k}$ such that

$$\text{(3.10)}\quad S(v_{-\lambda,j}^{(-\lambda+2(\lambda+k))\,a+\lambda+k,b+\lambda+k}, \sigma v_{\lambda,j'}^{(\lambda+2k)\,a+k,b+k}) = \delta_{jj'}(-1)^a i^{w-\lambda}\Big/\binom{\lambda+2k}{k}.$$

In the case $a=b$, we can moreover take the above basis (3.9) to consist of real elements. Put

$$\text{(3.11)}\qquad v_{\lambda,j}^{(\lambda+2k)\,b+k,a+k} = \sigma v_{\lambda,j}^{(\lambda+2k)\,a+k,b+k} \quad (a\geq b).$$

Define now $\mathbf{C}$-linear endomorphisms $N_{\pm}$ of $H_{\mathbf{C}}$ by

$$\text{(3.12)}\quad \begin{aligned} N_{+}v_{\lambda,j}^{(\lambda+2k)\,a+k,b+k} &:= k v_{\lambda+2,j}^{((\lambda+2)+2(k-1))\,a+k-1,b+k-1},\\ N_{-}v_{\lambda,j}^{(\lambda+2k)\,a+k,b+k} &:= (\lambda+k) v_{\lambda-2,j}^{((\lambda-2)+2(k+1))\,a+k+1,b+k+1}, \end{aligned}$$

for all λ, non-negative a, b and $k \geq \max\{-\lambda, 0\}$. By construction, it is easy to see that $N_\pm$ commute with the complex conjugation σ and satisfy the commutation relations: $[N_+, N_-] = Y$, and $[Y, N_\pm] = \pm 2N_\pm$, respectively. It is also easy to verify that $S(N_\pm \cdot, \cdot) + S(\cdot, N_\pm \cdot) = 0$, respectively. Indeed, for example, one can compute as

$$\begin{aligned} &S(N_+ v_{-\lambda, j}^{(-\lambda+2(\lambda+k))\, a+\lambda+k, b+\lambda+k}, \sigma v_{\lambda-2, j'}^{((\lambda-2)+2(k+1))\, a+k+1, b+k+1}) \\ &\quad + S(v_{-\lambda, j}^{(-\lambda+2(\lambda+k))\, a+\lambda+k, b+\lambda+k}, N_+ \sigma v_{\lambda-2, j'}^{((\lambda-2)+2(k+1))\, a+k+1, b+k+1}) \\ =&\delta_{jj'}(-1)^a i^{w-\lambda+2} \frac{(\lambda+k)(\lambda+k-1)!(k+1)!}{(\lambda+2k)!} + \delta_{jj'}(-1)^a i^{w-\lambda} \frac{(k+1)k!(\lambda+k)!}{(\lambda+2k)!} = 0. \end{aligned}$$

Thus we see that $N_\pm \in \mathfrak{g}$ and hence there exists a unique representation

(3.13) $\quad \rho : \mathrm{SL}_2(\mathbf{R}) \to G$ such that $\rho_* y = Y$ and $\rho_* n_\pm = N_\pm$, respectively.

By using the Cayley element $c := \rho(c_1) \in G_{\mathbf{C}}$, we define

(3.14) $$Q_\lambda^{(\lambda+2k)\, a+k, b+\lambda+k} := c P_\lambda^{(\lambda+2k)\, a+k, b+k}, \quad H^{p,q} := \bigoplus_{\substack{a+k=p \\ b+\lambda+k=q}} Q_\lambda^{(\lambda+2k)\, a+k, b+\lambda+k},$$

where, on the right hand side of the second equality, the summation is taken over all the eigenvalues λ of Y, all integers $k \geq \max\{-\lambda, 0\}$ and all non-negative integers a, b with $a + b + \lambda + 2k = w$. This defines a Hodge structure. Indeed, by using (1.20.ii), one sees that

$$\begin{aligned} &\sigma Q_\lambda^{(\lambda+2k)\, a+k, b+\lambda+k} = \sigma c P_\lambda^{(\lambda+2k)\, a+k, b+k} = c^{-1} \sigma P_\lambda^{(\lambda+2k)\, a+k, b+k} \\ =&c^{-1} P_\lambda^{(\lambda+2k)\, b+k, a+k} = c P_{-\lambda}^{(-\lambda+2(\lambda+k))\, b+\lambda+k, a+\lambda+k} = Q_{-\lambda}^{(-\lambda+2(\lambda+k))\, b+\lambda+k, a+k}, \end{aligned}$$

and hence $\sigma H^{p,q} = H^{q,p}$. One can moreover verify that (3.14) is S-polarized. Indeed, the direct sum in (3.14) is S-orthogonal by construction and, for

$$c v_{\lambda, j}^{(\lambda+2k)\, a+k, b+k}, \; c v_{\lambda, j'}^{(\lambda+2k)\, a+k, b+k} \in Q_\lambda^{(\lambda+2k)\, a+k, b+\lambda+k} \subset H^{p,q},$$

one can compute as

$$
\begin{aligned}
&i^{p-q}S(cv_{\lambda,j}^{(\lambda+2k)\,a+k,b+k},\sigma cv_{\lambda,j'}^{(\lambda+2k)\,a+k,b+k})\\
=&i^{a-b-\lambda}S(cv_{\lambda,j}^{(\lambda+2k)\,a+k,b+k},c^{-1}\sigma v_{\lambda,j'}^{(\lambda+2k)\,a+k,b+k})\\
=&i^{a-b-\lambda}S(c^2v_{\lambda,j}^{(\lambda+2k)\,a+k,b+k},\sigma v_{\lambda,j'}^{(\lambda+2k)\,a+k,b+k})\\
=&i^{a-b-\lambda+\lambda+2k}S(v_{-\lambda,j}^{(-\lambda+2(\lambda+k))\,a+\lambda+k,b+\lambda+k},\sigma v_{\lambda,j'}^{(\lambda+2k)\,a+k,b+k})\\
=&\delta_{jj'}i^{a-b+2k+2a+w-\lambda}\Big/\binom{\lambda+2k}{k}=\delta_{jj'}\Big/\binom{\lambda+2k}{k}.
\end{aligned}
$$

Thus we have $\{H^{p,q}\}\in D$.

Finally, we claim that the representation ρ in (3.13) is horizontal at $\{H^{p,q}\}\in D$. Indeed, since $Z=(\operatorname{Int}c)Y$, $X_\pm=(\operatorname{Int}c)N_\pm$, one can compute, by (1.20), as

$$
\begin{aligned}
&ZQ_\lambda^{(\lambda+2k)\,a+k,b+\lambda+k}=cYP_\lambda^{(\lambda+2k)\,a+k,b+k}=Q_\lambda^{(\lambda+2k)\,a+k,b+\lambda+k},\\
&X_\pm Q_\lambda^{(\lambda+2k)\,a+k,b+\lambda+k}=cN_\pm P_\lambda^{(\lambda+2k)\,a+k,b+k}\\
&\quad=cP_{\lambda\pm2}^{((\lambda\pm2)+2(k\mp1))\,a+k\mp1,b+k\mp1}=Q_{\lambda\pm2}^{((\lambda\pm2)+2(k\mp1))\,a+k\mp1,b+\lambda+k\pm1}.
\end{aligned}
$$

This completes the proof of the theorem. ∎

We remark that the condition on $\{n_\lambda\}$ in Theorem (3.4) coincides with the one in [CK, (2.20)] in the case of weight 2.

Fix identifications $D\simeq G/V$ and $R\simeq G/K$, where K is a maximal compact subgroup of G containing V and R is the associated Riemannian symmetric domain, and let θ_K be the associated Cartan involution. We denote the projection by

(3.15) $$\pi: D\simeq G/V\to G/K\simeq R.$$

Proposition (3.16). *We use the notation in Theorem (3.4). Let $Y\in\mathfrak{g}$ be an admissible element.*

(i) *If* $r \in D$ *forms an admissible pair* (Y,r)*, then* $\theta_r Y = -Y$*, where* θ_r *is the Cartan involution on* $\mathfrak{g}$ *induced from (1.11).*

(ii) *If* $\theta_K Y = -Y$*, then there exists* $r \in \pi^{-1}([K])$ *such that* (Y,r) *is an admissible pair*

(iii) *For each set of primitive Hodge numbers* $\{p_\lambda^{a,b}\}$ *belonging to* $\{h^{p,q}, n_\lambda\}$*,* $G_Y := \{g \in G \mid (\mathrm{Ad}\, g)Y = Y\}$ *acts transitively on the set* $\{r \in D \mid (Y,r)$ *is an admissible pair of type* $\{p_\lambda^{a,b}\}\}$.

Proof. (i) follows from (3.2) and (1.11).

(ii): Assume $\theta_K Y = -Y$. Take a point $r' \in D$ at which Y is admissible and let K' be the maximal compact subgroup of G associated to the Cartan involution $\theta_{r'}$. By the result in (i) for (Y,r') and the assumption, Y can be viewed as a tangent vector to R at $[K']$ as well as at $[K]$: $Y \in T_R([K'])$, $Y \in T_R([K])$. By the transitivity of tangent spaces of a Riemannian symmetric domain, there exists $g \in G$ such that $(\mathrm{Int}\, g)K' = K$ and $(\mathrm{Ad}\, g)Y = Y$. Hence the admissibility of (Y,r') implies that of $((\mathrm{Ad}\, g)Y, gr') = (Y, gr')$, where $gr' \in \pi^{-1}([K])$.

(iii): Suppose that $r, r' \in D$ are points at which Y is admissible of the same type $\{p_\lambda^{a,b}\}$. Let $\rho, \rho' : \mathrm{SL}_2(\mathbf{R}) \to G$ be the corresponding representations. It is enough to show that there exists $g \in G$ such that $\rho' = (\mathrm{Int}\, g)\rho$. Indeed, if this is the case, then $(\mathrm{Ad}\, g)Y = (\mathrm{Ad}\, g)(\rho_*(y)) = \rho'_*(y) = Y$ and $gr = g\widetilde{\rho}(i) = \widetilde{\rho}'(i) = r'$.

Since (Y,r) and (Y,r') have the same type $\{p_\lambda^{a,b}\}$, the types of the irreducible decompositions of ρ and ρ' coincide. Now we use the well-known fact that any finite-dimensional irreducible representation of SL_2 is isomorphic to a suitable symmetric tensor power representation (cf. §1). Thus we get our assertion. ∎

Appendix

In this appendix, we shallgeneralize the results in [CK] and construct a partial compactification of the classifying space $\Gamma\backslash D$, $\Gamma = G_{\mathbf{Z}} := \{g \in G | gH_{\mathbf{Z}} = H_{\mathbf{Z}}\}$, of Hodge structures in general weight, adding those points which correspond one-parameter degenerations of type II. Since the arguments are analogous to those in [CK], we shall only indicate the outline of the construction. We use the notation in previous sections.

An admissible semi-simple element $Y \in \mathfrak{g}$ is of *type II* if its eigenvalues are 0 or ± 1.

A horizontal SL_2-representation ρ is of *type II* if so is $Y = \rho(y)$.

A period map $\varphi : \Delta^* \to \Gamma\backslash D$ from the punctured disc, i.e., a holomorphic map with horizontal local liftings, is of *type II* if its monodromy logarithm $N := (1/m)\log \gamma^m$ satisfies $N^2 = 0$, where γ is the monodromy of φ and m is the least positive integer such that the eigenvalues of γ^m are all unity.

Throughout this appendix, we shall consider only those Y, ρ and φ of type II.

(A.1) Let H be an $\mathbf{R}$-vector space underlying Hodge structures of weight w, and S the polarizing bilinear form on H (see §1). For an isotropic subspace W_1, of H, we have a filtration $0 =: W_2 \subset W_1 \subset W_0 := W_1^{\perp} \subset W_{-1} := H$, where $W_1^{\perp}$ means the subspace of H perpendicular to W_1 with respect to S. Set $n_\lambda := \dim W_\lambda / W_{\lambda+1}$. We assume that there exists a set of primitive Hodge numbers $\{p_\lambda^{a,b}\}$ belonging to $\{h^{p,q}, n_\lambda\}$ (see Definition (2.15)). We denote by $\tilde{S}$ the non-degenerate bilinear form on W_0/W_1 induced by S. Let ϕ be a polarizing bilinear form of a Hodge structure on $W_{1\mathbf{C}}$ of type $\{p_\lambda^{a,b}\}$. Two such forms are considered to be equivalent if they are different only up to a positive multiplicative constant.

Definition(A.1.1). *Given* $W_1, p := \{p_\lambda^{a,b}\}$ *and* ϕ *as above . The associated* *boundary component* $B = B(W_1, p, \phi) = B^w(W_1, p) \times B^{w-1}(W_1, p, \phi)$ *is defined by*

$B^w(W_1, p)$: *classifying space for* $\tilde{S}$*-polarized Hodge structures on* $(W_0/W_1)_{\mathbf{C}}$ *of type* $\{p_0^{a,b}\}$.

$B^{w-1}(W_1, p, \phi)$: *classifying space for* ϕ*-polarized Hodge structures on* $W_{1\mathbf{C}}$ *of type* $\{p_1^{a,b}\}$.

The boundary bundle $\mathcal{B} = \mathcal{B}(W_1, p) = \mathcal{B}^w(W_1, p) \times \mathcal{B}^{w-1}(W_1, p)$ *is defined as the disjoint union of all boundary components* $B(W_1, p, \phi)$ *where* ϕ *runs over all equivalence classes of polarizing forms on* W_1 *of type* $\{p_1^{a,b}\}$.

Theorem(3.4) shows that for every boundary bundle $\mathcal{B}(W_1, p)$ there exists an admissible element $Y \in \mathfrak{g}$ with the set of primitive Hodge numbers p and $W_1 = H(Y; 1)$. Theorem(3.4) and Proposition(3,16,iii) (and some argument in linear algebra) show that for every boundary component $B(W_1, p, \phi)$ there exists a horizontal SL_2-representation ρ such that $W_1 = H(Y; 1), p_\lambda^{a,b} = \dim P_\lambda^{\{\lambda\}a,b}$ and $\phi = S(N_- \cdot, \cdot)$.

Definition (A.1.2). *Aboundary bundle* $\mathcal{B}(W_1, p)$ *is rational if the isotropic subspace* $W_1 \subset H$ *is defined over* $\mathbf{Q}$. *A boundary component* $B(W_1, p, \phi)$ *is rational if* W_1 *and the form* ϕ *are defined over* $\mathbf{Q}$. *We denote by* $D^{**} \subset D^*$ *the union of all rational boundary components and the union of all rational boundary bundles, respectively.*

(A.2) Let (W_1, p, ϕ) be a polarized isotropic subspace, and $\{P_\lambda^{(\lambda)a,b}\} \in B(W_1, p, \phi)$ a point in the associated boundary component. Then these are transformed by $g \in G$ to the polarized isotropic subspace $(gW_1, p, (g^{-1})^*\phi)$ and the point $\{gP_\lambda^{(\lambda)a,b}\} \in (gW_1, p, (g^{-1})^*\phi)$, respectively. This defines a natural action of G on the union of all boundary components, which restricts to an action of $G_{\mathbf{Q}}$ on D^* and D^{**}.

Notice also that $g \in G$ transforms an admissible pair $(Y, r) \in \mathfrak{g} \times D$ to the admissible pair $(\mathrm{Ad}(g)Y, gr)$, and an SL_2-representation $\rho : \mathrm{SL}_2(\mathbf{R}) \to G$ horizontal at $r \in D$ to the SL_2-representation $\mathrm{Int}(g)\rho$ horizontal at gr.

Definition(A.2.1). *For a boundary bundle $\mathcal{B} = \mathcal{B}(W_1, p)$, we define its normalizer $N(\mathcal{B}) := \{g \in G \,|\, g\mathcal{B} = \mathcal{B}\}$ and its centralizer $Z(\mathcal{B}) := \{g \in G \,|\, g|_{\mathcal{B}} = \mathrm{id}\}$. Let $Y \in \mathfrak{g}$ be an admissible semi-simple element such that $(W_1, p) = (H(Y; 1), p)$. We denote by G_Y the isotropy subgroup of Y in the adjoint action of G, and $G(Y) :=$ $\{g \in G_Y \,|\, \det(g|_{W_1}) = 1\}$.*

In order to express these groups by matrices, we take a basis of $H = H_{-1} \oplus H_0 \oplus H_1$, $H_\lambda := H(Y; \lambda)$, subjected to the decomposition so that the bilinear form S becomes

$$S = \begin{pmatrix} O & J \\ -J & O \end{pmatrix} \text{ if } w \text{ is odd, } \begin{pmatrix} O & O & J \\ O & \pm I & O \\ J & O & O \end{pmatrix} \text{ if } w \text{ is even,}$$

where J is an antidiagonal matrix $J = \mathrm{antidiag}(1, \ldots, 1)$, Then the matrices of $N(\mathcal{B})$ are of the form

$$\begin{pmatrix} g_1 & O & O \\ * & g_0 & O \\ * & * & g_1 \end{pmatrix} \text{ where } g_0 \in \mathrm{Aut}(H_0, S|_{H_0}), g_1 \in \mathrm{GL}(n_1, \mathbf{R}), g_{-1} = J\,{}^t g_1^{-1} J,$$

and

$$Z(\mathcal{B}) = \{g \in N(\mathcal{B}) \,|\, g_0 = \pm I_0, g_1 = aI_1 \,(a \in \mathbf{R}^*)\},$$

$$G_Y = \{g \in N(\mathcal{B}) \,|\, \text{ the } *\text{' s are } 0\},$$

$$G(Y) = \{g \in G_Y \,|\, \det g_1 = 1\}.$$

For these expressions, one can see easily

Proposition(A.2.2). **(i)** $N(\mathcal{B}), Z(\mathcal{B}), G_Y, G(Y)$ *are all independent of the choice of a set of primitive Hodge numbers p.*

(ii) $N(\mathcal{B})$ *is a parabolic subgroup of* G *preserving the filtration* $0 = W_2 \subset W_1 \subset W_0 = W_1^{\perp} \subset W_{-1} = H$.

(iii) $Z(\mathcal{B})$ *is a closed normal subgroup of* $N(\mathcal{B})$.

(iv) $G(Y)$ *is a semi-simple group, acting transitively on* $\mathcal{B}$ *with compact isotropy subgroup.*

(v) $N(\mathcal{B}) = G(Y) \cdot Z(\mathcal{B})$ *is an almost direct product, i.e.,* $G(Y) \cap Z(\mathcal{B})$ *is finite.*

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition with $Y \in \mathfrak{p}$, $\mathfrak{t}$ a maximal abelian subspace, containing Y (i.e., $\mathfrak{t}$ is the intersection of $\mathfrak{g}$ with a maximal $\mathbf{R}$-split Cartan subalgebra of $\mathfrak{g}_{\mathbf{C}}$), $\mathfrak{m}$ the centralizer of $\mathfrak{t}$ in $\mathfrak{k}$, and $\Phi \subset \mathfrak{t}^*$ the system of restricted roots for the adjoint action of $\mathfrak{t}$ on $\mathfrak{g}$. Then we have a root space decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m} \oplus \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha := \{X \in \mathfrak{g} \,|\, \mathrm{Ad}(H)X = \alpha(H)X \ \ (H \in \mathfrak{t})\}$.

If we select a basis of H, compatible with the choice in (A.1), with respect to which the matrices of $\mathfrak{t}$ are of the diagonal form

$$H(\lambda_1, \ldots, \lambda_r) := \begin{cases} \mathrm{diag}(-\lambda_1, \ldots, -\lambda_r) \oplus \mathrm{diag}(\lambda_r, \ldots, \lambda_1), w\text{: odd}, \\ \mathrm{diag}(-\lambda_1, \ldots, -\lambda_r) \oplus \mathrm{diag}(0, \ldots, 0) \oplus \mathrm{diag}(\lambda_r, \ldots, \lambda_1), w\text{: even}, \end{cases}$$

where r is the $\mathbf{R}$-rank of $\mathfrak{g}$ and $\lambda_i \in \mathbf{R}$. Notice that $Y = (1, \ldots, 1, 0, \ldots, 0)$, and that the elements $H_i := H(\lambda_1, \ldots, \lambda_r)$ with $\lambda_j = \delta_{ij}$ form a basis of $\mathfrak{t}$ and define the lexicographic order of the roots in which the system of the positive roots Φ^+ contains $\{\alpha \in \Phi \,|\, \alpha(Y) > 0\}$. Let ξ_i $(1 \leq i \leq r)$ be the basis of $\mathfrak{t}^*$ dual to H_i $(1 \leq i \leq r)$. Then the positive roots are calculated as

Let us denote $\mathfrak{r} := \sum_{\alpha \in \Phi^+} \mathfrak{g}_\alpha, R := \exp \mathfrak{r}, T := \exp \mathfrak{t}$ and by K the maximal compact subgroup of G with $\mathfrak{k} := \mathrm{Lie}\, K$. Then one has the Iwasawa decomposition $G = RTK$. This induces the corresponding decompositions:

Proposition(A.2.3). **(i)** *$N(\mathcal{B}) = RTK_Y$, where $K_Y = K \cap G_Y$.*

(ii) *$Z(\mathcal{B}) = (R \cap Z)(T \cap Z)(K \cap Z)$, where $Z = Z(\mathcal{B})$.*

(iii) *Let $\mathfrak{g}(Y) := \mathrm{Lie}\, G(Y)$. Then $\mathfrak{t} \cap \mathfrak{g}(Y)$ is a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{g}(Y)$, $\Phi^+(Y) := \{\alpha \in \Phi^+ \mid \alpha(Y) = 0\}$ is the system of positive roots for the adjoint action of $\mathfrak{t} \cap \mathfrak{g}(Y)$ on $\mathfrak{g}(Y)$ and $\mathfrak{r} \cap \mathfrak{g}(Y) = \sum_{\alpha \in \Phi^+(Y)} \mathfrak{g}_\alpha$* (because of the compatibility of the orders), *whence one has $G(Y) = (R \cap G(Y))(T \cap G(Y))(K \cap G(Y))$.*

(iv) *Let $r \in D$ be a point with which Y forms an admissible pair and whose isotropy subgroup V_r of G is contained in K (cf. Proposition(3.16.ii)), and let I_b be the isotropy subgroup of $N(\mathcal{B})$ at $b = b(Y, r) := \{P_\lambda^{(\lambda)a,b}\} \in B(Y, r) \subset \mathcal{B}$. Then one has $V_r \cap G(Y) \subset I_b \cap G(Y) \subset K \cap G(Y), I_b \cap RT \subset Z(\mathcal{B}), I_b = (RT \cap Z(\mathcal{B}))(K \cap I_b)$.*

The proof is similar to those for [CK, (3.28), (3.36), (3.40)]. Our present assumption 'type II' will be used in the proof of (iv) of the above proposition.

(A.3) Now we choose as $\mathfrak{t}$ a maximal $\mathbf{Q}$-split Cartan subalgebra of $\mathfrak{g}$ and choose a maximal compact subgroup K of G such that $\mathfrak{t} \subset \mathfrak{p}$ for the associated Cartan decomposition. Let $\mathfrak{t}^+$ be the positive Weyl chamber, and $\overline{\mathfrak{t}^+}$ its closure. We denote by $\mathfrak{S}$ the set of complete representatives of the $G_{\mathbf{Q}}$-equivalent classes of $\mathbf{Q}$-rational admissible element Y of type II in $\overline{\mathfrak{t}^+}$. It is easy to see by definition that $\mathfrak{S}$ is a finite set and that, for any admissible element $Y \in \mathfrak{g}_{\mathbf{Q}}$ of type II, there exists $g \in G_{\mathbf{Q}}$ satisfying $\mathrm{Ad}(g)Y \in \mathfrak{S}$.

Definition(A.3.1). *The boundary bundles $\mathcal{B}(Y, p)$ for $Y \in \mathfrak{S}$ and p being a set of primitive Hodge numbers compatible with Y will be called the standard rational boundary bundles.*

Let $\pi : D \to G/K$ be the canonical projection. By Proposition(3.16.ii) and Remark(2.12), one can choose a reference point $r_{Y,p} \in \pi^{-1}([K]) \subset D$ for each compatible pair (Y,p) with $Y \in \mathfrak{S}$, so that $(Y, r_{Y,p})$ is an admissible pair. Let $B(Y, r_{Y,p})$ be the boundary component contained in the boundary bundle $\mathcal{B}(Y,p)$.

Let $G = RTK$ be the Iwasawa decomposition in the presrnt context.

Definition(A.3.2). *A Siegel set in G is defined as $\sigma = \omega_R T_\lambda K$, where $\omega_R \subset R$ is a compact subset and $T_\lambda := \{t \in T \mid e^\alpha(t) \geq \lambda \ (\alpha \in \Phi^+)\}$ for a positive real number λ.*

The extended Siegel set in D^* is the subset $\sigma^* = \bigcup_{Y\in\mathfrak{S},p} \sigma_Y b_{Y,p} \subset D^*$, where $\sigma_Y := \sigma \cap N(\mathcal{B}(Y,p)), b_{Y,p} = b(Y, r_{Y,p}) := \{P_\lambda^{(\lambda)a,b}\} \in B(Y, r_{Y,p}) \subset \mathcal{B}(Y,p)$, and the union is taken over the finite set of all compatible pairs (Y,p) with $Y \in \mathfrak{S}$.

Notice that the extended Siegal set σ^* in D^* is independent of the choice of a set of complete representatives of the reference points $r_{Y,p} \in \pi^{-1}([K]) \subset D$. It is known that a Siegel set σ in G has the Siegel property: for any $g \in G_{\mathbf{Q}}, \{\gamma \in \Gamma \mid \gamma\sigma \cap g\sigma \neq \emptyset\}$ is a finite set. Moreover, if the subset ω_R and the constant λ are adequately chosen, then there exists a finite subset $C \subset G_{\mathbf{Q}}$ containing 1 such that $G = \Gamma C\sigma$ and $D^* = \Gamma C \sigma^*$.

Let $\pi_{Y,p} : N(\mathcal{B}(Y,p)) \to \mathcal{B}(Y,p)$, sending g to $gb_{Y,p}$, be the natural projection.

Definition(A.3.3). *Let $U_1 \subset \mathcal{B}(Y,p)$ be an open set, U_2 an open neiborhood of $1 \in K$, and λ a positive real number. Then the open set, in D, $V(U_1, U_2, \lambda) := \{g r_{Y,p} \mid g \in \pi_{Y,p}^{-1}(U_1)U_2, \ e^\alpha(g) > 0 \ (\alpha \in \Phi, \alpha(Y) > 0)\}$ will be called a tube over $U_1 \subset \mathcal{B}(Y,p)$.*

Theorem(A.3.4). (i) *The sets $\mathcal{U}(U_1, U_2, \lambda) := (U_1 \cup V(U_1, U_2, \lambda)) \cap \sigma^*$, together with the natural topology on $\sigma' := \sigma r_{Y,p} \subset D$, form a basis of a Hausdorff topology τ^* on the extended Siegel set σ^*.*

(ii) *Let $g \in G$ and $x \in \sigma$. If $g\sigma \in \sigma^*$, then, for any τ^*-neighborhood $\mathcal{U}'$ of $gx \in \sigma^*$, there exists a τ^*-neighborhood $\mathcal{U}$ of $x \in \sigma^*$ such that $g\mathcal{U} \cap \sigma^* \subset \mathcal{U}'$. If*

$gx \notin \sigma^*$, then there exists a τ^*-neighborhoos $\mathcal{U}$ of $x \in \sigma^*$ such that $g\mathcal{U} \cap \sigma^* = \emptyset$.

The proof is analogous to those of [CK, (4.16), (4.25)]. In the proof, the following lemma will play an important role, and a Hodge-$(Z, X_\pm)$ decomposition will also used.

Lemma(A.3.5). *Let $W \subset RT$ and $V \subset K_Y$ be open subsets satisfying*

(i) $W(Z(\mathcal{B}(Y,p)) \cap RT) \subset W$ *and*

(ii) $V(I_{b_{Y,p}} \cap K_Y) \subset V$.

Then there exists an open subset $U \subset \mathcal{B}(Y,p)$ such that $\pi_{Y,p}^{-1}(U) = WV$.

This lemma is proved by using various kind of the Iwasawa decompositions in Proposition(A.2.3).

As in [Sa.1], the results in Theorem(A.3.4) will be transformed to the corresponding assertions on the fundamental domain $\Omega^* := C\sigma^*$ in D^* for the action of Γ, and finally one gets a *Satake topology* τ^Γ on D^* which has the following properties:

Theorem(A.3.6). (i) *The topology τ^Γ on D^* induces the topology τ^* on σ^*.*

(ii) *The operations of Γ are continuous.*

(iii) *If $\Gamma x \cap \Gamma x', x, x' \in D^*$, then there exists τ^Γ-neighborhoods $\mathcal{U}, \mathcal{U}'$ of $x, x' \in D^*$ such that $\Gamma\mathcal{U} \cap \Gamma\mathcal{U}' = \emptyset$.*

(iv) *For each $x \in D^*$, there exists a fundamental system of τ^Γ-neighborhoods $\{\mathcal{U}\}$ of $x \in D^*$ such that $\gamma\mathcal{U} = \mathcal{U}$ for $\gamma \in \Gamma_x$ and $\gamma\mathcal{U} \cap \mathcal{U} = \emptyset$ for $\gamma \notin \Gamma_x$.*

As a corollary, one obtaines

Corollary(A.3.7). *the quotients $\Gamma\backslash D^*, \Gamma\backslash D^{**}$ endowed with the topologies induced from τ^Γ have the following properties:*

(i) $\Gamma\backslash D^*$ *and* $\Gamma\backslash D^{**}$ *are locally compact and Hausdorff.*

(ii) $\Gamma\backslash D \subset \Gamma\backslash D^{**}$ *is open and everywhere dense.*

(iii) $\Gamma\backslash D^{*} = \coprod \Gamma(\mathcal{B}_i)\backslash\mathcal{B}_i$, *where* $\mathcal{B}_i$ *runs over a finite set of complete representatives of* Γ*-equivalence classes of rational boundary bundles, and* $\Gamma(\mathcal{B}_i) := (\Gamma \cap N(\mathcal{B}_i))/(\Gamma \cap Z(\mathcal{B}_i))$ *are arithmetic subgroups of the semi-simple groups* $N(\mathcal{B}_i)/Z(\mathcal{B}_i)$.

(A.4) Let $(Y, r) \in \mathfrak{g} \times D$ be an admissible pair, ρ the corresponding horizontal SL_2-representation, and $\tilde{\rho} : U \to D$ the associated horizontal embedding of the upperhalf plane. Then, as [CK, (6.17)], one obtains

Proposition(A.4.1). *If* ρ *is defined over* $\mathbf{Q}$, *then in the Satake topology*

$$\lim_{t\to\infty} \exp(tY)r = \lim_{\operatorname{Im} z\to\infty} \tilde{\rho}(z) = b(Y, r) \in D^{**}.$$

This is an analogous result to [Sa.2, (8.1) and its proof].

The following theorem will be proved similarly to [CK, (6.1), (6.18)]. A proof is based on the SL_2-orbit theorem in [Sc], the Iwasawa decompositions (A.2.3), the Satake topology (A.3.6), (A.3.7) and Theorem(3.4) and Proposition(3.16).

Theorem(A.4.2). (i) *Let* $\varphi : \Delta^* \to \Gamma\backslash D$ *be a period map of type II. Then* φ *can be extended continuously over the puncture to* $\overline{\varphi} : \Delta \to \Gamma\backslash D^{**}$.

(ii) *Let* $\tilde{b} \in \Gamma\backslash D^{**}$ *be an arbitrary point. Then there exists a period map* $\varphi : \Delta^* \to \Gamma\backslash D$ *of type II such that* $\lim_{t\to 0}\varphi(t) = \tilde{b}$.

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