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The Fundamental Group of the Smooth part of a Log Del Pezzo Surface

By

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Introduction

A normal projective surface $S/\mathbf{C}$ is called *a log del Pezzo surface* if S has at worst quotient singularities and $-K_S$ is ample, where K_S denotes the canonical divisor of S.

Recall that the divisor class group of a quotient singularity is always finite. Hence for any Weil divisor D on a log del Pezzo surface S, nD is a Cartier divisor for some integer $n \geq 1$.

The principal result of this paper is the following :

Main Theorem. *The fundamental group of the space of smooth points of a log del Pezzo surface is finite.*

In the case of a Gorenstein log del Pezzo surface, this result was proved in [6] by first classifying such surfaces. In this paper, we also give a very easy proof of the result in the case of Gorenstein log del Pezzo surfaces. So far, there are not many results about general log del Pezzo surfaces. Recently, V.A. Alekseev and V.V. Nikulin have classified all log del Pezzo surfaces of index ≤ 2 (i.e., where $2K_S$ is Cartier).

The index of a log del Pezzo surface S is defined to be the smallest positive integer n such that nK_S is a Cartier divisor. Nikulin has proved that the rank of the Picard group of a minimal resolution of S is bounded by a universal function of the index of S. From this also one can deduce that $\pi_1^{alg}(S^o)$ is finite.

There are easy examples of normal projective rational surface $/\mathbf{C}$ with quotient singularities (even double points) and with numerically effective anti-canonical divisor, such that the fundamental group of the space of smooth points is infinite. This shows that the condition about the ampleness of $-K_S$ in the main theorem cannot be dropped.

From the main theorem, we see easily that any log del Pezzo surface S is a quotient of a log del Pezzo surface T modulo a finite group such that the space of smooth points of T is simply-connected (the group acting freely outside a finite set of points of T).

§1. Some easy results

In this section we fix the following notations and terminology which will be used throughout the paper.

Let S be a log del Pezzo surface as defined in the introduction. Denote by $S^o := S-$ (Sing S) the smooth part of S. Let $f : \widetilde{S} \to S$ be a minimal resolution of singularities and denote by $D := f^{-1}$(Sing S) the exceptional divisor.

Write $D = \sum_{i=1}^{n} D_i$ where D_i is irreducible. The first part of Lemma 1.1 below follows from the definition of a quotient singularity. The second part is trivial and the third part follows from the ampleness of $-K_S$.

Lemma 1.1. (1) *There exists a* $\mathbf{Q}$*-coefficient divisor* $D^* = \sum_{i=1}^{n} \alpha_i D_i$ *such that* $0 \le \alpha_i < 1$ *and*

$$f^*(K_S) \equiv K_{\widetilde{S}} + D^*.$$

Moreover, $\alpha_j = 0$ *if and only if the connected component of* D *containing* D_j *is contracted to a rational double point on* S.

(2) *Let* p *be the smallest positive integer such that* pD^* *is an integral divisor. Then* pK_S *is a Cartier divisor and*

$$f^*(pK_S) \sim p(K_{\widetilde{S}} + D^*).$$

(3) $-(K_{\widetilde{S}} + D^*)$ *is a nef and big divisor. Moreover,* $-(K_{\widetilde{S}} + D^*) \cdot B = 0$ *if and only if the support of* B *is contained in* D.

(4) *Suppose that* B *is an irreducible curve on* $\widetilde{S}$ *with negative self intersection. Then either* B *is a* (-1)*-curve or* $B \le D$.

Proof. (4) Suppose that B is not contained in D. Then $B.K_{\widetilde{S}} < 0$ by (3). Now it follows from the genus formula that B is a (-1)-curve.

The following two results are easy but useful.

Lemma 1.2. *Let* T *be a norma projective surface with a finite morphism* $\varphi : T \to S$ *which is unramified over* S^o. *Then* T *is a log del Pezzo surface.*

Lemma 1.3. *A log del Pezzo surface is rational.*
A proof of the following result is available in [6].

Lemma 1.4. *$H_1(S^o, \mathbf{Z})$ is finite.*

The following one is the first step towards the proof of our Main Theorem.

Proposition 1.5. *The algebraic fundamental group of S^o is finite.*

Now we can make an easy proof for the following result. In [7], it is proved further that $\pi_1(S^o)$ has order ≤ 9.

Proposition 1.6. *Let S be a Gorenstein log del Pezzo surface. Then $\pi_1(S^o)$ is abelian and finite.*

Proof. By Lemma 1.5, it is enough to prove that $\pi_1(S^o)$ is abelian. By [2, Theorem 1, p.39], there is a nonsingular elliptic curve $A \in |-K_S|$. Since $-K_S$ is ample and $K_{\widetilde{S}} = f^*(K_S)$, the Iitaka D-dimension $\kappa(\widetilde{S}, -K_{\widetilde{S}}) = 2$. Also A is disjoint from D as S has only rational double points. Now by [4], we have a surjective map $\pi_1(A) \to \pi_1(\widetilde{S} - D)$. Thus $\pi_1(S^o)$ is abelian.

Remark 1.7. The proof shows that if $|-K_S|$ contains a member A which is a rational cuspidal curve then $\pi_1(S^o) = (1)$ because $\pi_1(A) = (1)$.

§2. Reduction to the rank one case

In this section, using Kawamata's contraction theorem, we will show that it is enough to prove the main theorem when Pic $S \cong \mathbf{Z}$. (Note that since S is simply-connected, Pic S is isomorphic to $\mathbf{Z}$ if the rank of Pic S is one.)

Suppose rank Pic $S \geq 2$. Since K_S is not nef, there is a contraction $\varphi : S \to Y$ of an extremal ray by Kawamata's Cone Theorem. (Note that a two-dimensional quotient singularity is nothing but a log-terminal singularity.) We have two cases :

Case (1). Y is a surface.
Then we can prove that Y is a log del Pezzo surface with $\rho(Y) < \rho(S)$ and with a surjection

$$\pi_1(Y^o) \to \pi_1(S^o).$$

Case (2). Y is a smooth projective curve.
We can prove that φ is a $\mathbf{P}^1$-fibration and $Y \cong \mathbf{P}^1$. By Fenchel's conjecture solved by S. Bundgaard, J. Nielsen, R.H. Fox and T.C. Chau, one can take an unramified covering

of S^o and reduce to the case where the restriction morphism $\varphi_1 := \varphi_{|S^o} : S^o \to Y$ is multiple-fiber free. Then a similar arguments as in Nori [4] shows that we have an exact sequence :

$$(1) = \pi_1(F) \to \pi_1(S^o) \to \pi(Y) = (1)$$

for a general fiber F of φ_1. Hence $\pi_1(S^o) = (1)$ in the present case.

Combining the arguments in Cases (1) and (2), by a repeated application of contractions of extremal rays we reduce the proof of the main theorem to the case when Pic $S \cong \mathbf{Z}$.

§3. The Proof of the Main Theorem

Let S be a log del Pezzo surface of rank one. We use the notation introduced in the beginning of §1. Let p be the smallest positive integer such that pD^* is an integral divisor. Then for every curve B on $\widetilde{S}$ not contained in D, $-(K_{\widetilde{S}} + D^*) \cdot B \in \frac{1}{p}\mathrm{N} = \{n/p | n \in \mathrm{N}\}$ (cf. Lemma 1.1). From this we obtain the following :

Lemma 3.1. *There exists an irreducible curve C on $\widetilde{S}$ such that $-(K_{\widetilde{S}} + D^*) \cdot C$ attains the smallest positive value. Such a curve satisfies $C^2 \geq -1$ (cf. Lemma 1.1,(4)).*

For the time being, we fix the curve C of Lemma 3.1. We shall treat the two cases $|K_{\widetilde{S}} + C + D| \neq \phi, = \phi$ separately.

§3.1. The case $|K_{\widetilde{S}} + C + D| \neq \phi$.

In this subsection, we always assume $|K_{\widetilde{S}} + C + D| \neq \phi$.

Lemma 3.2 (cf. [8, Lemma 2.1]). *Let C be as in Lemma 3.2. Suppose $|C+D+K_{\widetilde{S}}| \neq \phi$. Then there exists a unique decomposition $D = D' + D''$ such that :*
(1) *$K_{\widetilde{S}} + C + D'' \sim 0$,*
(2) *D' is disjoint from $C \cup D''$ and consists of (-2)-curves; hence D' is contracted to rational double points on S.*

Now we can outline the proof for the Main Theorem in the present case. First, we can divide the case $|C + D + K_{\widetilde{S}}| \neq \phi$ into the following four subcases :

Case (I-1) $D'' = 0$. Then S is a log del Pezzo surface with only rational double points. By Proposition 1.6, $\pi_1(S^o)$ is finite abelian.

In the following three subcases, assume that $D'' \neq 0$. Now from $K_{\widetilde{S}} + C + D'' \sim 0$ we see that each irreducible component of $C \cup D''$ is isomorphic to $\mathbf{P}^1$ (cf. Lemma 3.4 below), e.g., $K_{\widetilde{S}} + C \sim -D''$ implies that $|K_{\widetilde{S}} + C| = \phi$, etc.

Case (I-2) $D" \neq 0$ and $C+D$ is a divisor with only simple normal crossings. Then $C+D"$ is a simple loop of nonsingular rational curves.

Let U be an open neighbourhood of $C+D$ with D removed. Applying Wagreich's version of Van Kampen Theorem for the case where $U = U_1 \cup U_2$ and $U_1 \cap U_2$ comprises two connected components and using the fact that $\pi_1^{alg}(S^o)$ is finite and that $\pi_1(S^o)$ is a surjection image of $\pi_1(U)$ (cf. [4]), one can prove that $\pi_1(S^o)$ is finite.

Case (I-3) $D" \neq 0$ and $(C^2) \geq 0$. This case can be reduced to the case(I-2) above by replacing C with a new irreducible curve linearly equivalent to C.

Case (I-4) $D" \neq 0, (C^2) \leq -1$ and $C+D$ is not a divisor with only simple normal crossings. Then C is a (-1)-curve by Lemma 3.1. Furthermore, one of the following two subcases occurs.

Case (I-4a) $D"$ is an irreducible curve such that $C \cdot D" = 2$ and $C \cap D"$ is a single point. Since the intersection matrix of $C + D"$ has one positive eigenvalue one has $(D")^2 = -2$ or -3.

Case (I-4b) $D"$ consists of two irreducible components $D_1", D_2"$ such that $C \cdot D_1" = C \cdot D_2" = 1$ and $C \cap D_1" \cap D_2"$ consists of a single point. By the same reasoning as in Case (I-4a), we have $((D_1")^2, (D_2")^2) = (-2,-2), (-2,-3), (-2,-4)$ after interchanging the subscripts of $D_i"$, if necessary.

With the help of Mumford's presentation theorem, the same arguments as in Case(I-2) can show that $\pi_1(S^o)$ is finite.

§3.2. The Case $|K_{\bar{S}} + C + D| = \phi$

In this section we always assume that $|K_{\bar{S}} + C + D| = \phi$. So, we have the following (cf. Miyanishi [5]) :

Lemma 3.4. *$C+D$ has only simple normal crossings, consists of nonsingular rational curves and has a disjoint union of trees as the dual graph.*

We need the following :

Proposition 3.5 (cf. the proof of [8, Lemma 2.2]). *Let C be as in* Lemma 3.2. *Suppose $|C + D + K_{\bar{S}}| = \phi$. Then either C is a (-1)-curve or S is the Hirzebruch surface with the minimal section contracted. In the latter case, S^o is simply connected.*

From now on till the end of the present section, we assume always that C is a (-1)-curve.

Lemma 3.6 (cf. [8, Lemma 4.1]). *Let $D_1, \cdots, D_r$ exhaust all irreducible components of D with $(C, D_i) > 0$. Suppose $(D_1^2) \geq \cdots \geq (D_r^2)$. Then $\{-(D_1^2), \cdots, (D_r^2)\}$ is one of the* *following :*

$$\{2^a, n\}(n \geq 2), \{2^a, 3, 3\}, \{2^a, 3, 4\}, \{2^a, 3, 5\}$$

where 2^a signifies that 2 is repeated a-times.

$\rho(S) = 1$ implies the following :

Lemma 3.7 (1) *Suppose C meets exactly one irreducible component D_0 of D. Then* $(D_0^2) = -2$.

(2) C meets at least one component of D.

Lemma 3.8 (cf. [8, Lemma 4.4]). *Suppose C meets exactly two irreducible components D_0, D_1 of D. Then $(D_i^2) = -2$ for $i = 0$ or 1.*

Lemma 3.9 (cf. [8, Lemma 4.3]). *Assume that one of the following cases takes place :*

(1) C meets only one irreducible component D_0 of D.

(2) C meets exactly two irreducible components D_0, D_1 of D with $(D_1^2) \leq -3$.

Let $\sigma : \widetilde{S} \to \widetilde{T}$ be the blowing-down of the (-1)-curve C, let $E = \sigma(D_0)$ and let $B = \sigma(D - D_0)$. Then there exists a log del Pezzo surface T of Picard number one and there exists a birational morphism $g : \widetilde{T} \to T$ such that g is a minimal resolution and $B = g^{-1}(SingT)$.

Now we can give some idea to prove the Main Theorem in the present case. Let $D_1, \cdots, D_r$ be all irreducible components of D with $(C, D_i) > 0$ (hence $(C, D_i) = 1$ by Lemma 3.4). Suppose $(D_1^2) \geq \cdots \geq (D_r^2)$. By virtue of Lemmas 3.4, 3.6, 3.7 and 3.8, in the case where C is a (-1)-curve, we can divide into the following four cases :

Case (II-1) $r \geq 2$ and $(D_1^2) = (D_2^2) = -2$.

Case (II-2) $r = 1$ and $(D_1^2) = -2$.

The above Cases (II-1) and (II-2) are described quite precisely in [8]. Arguments similar to those in §2 and §3.1 can be used to prove the finiteness of $\pi_1(S^o)$.

Case (II-3) $r = 3$ and $\{(D_1^2), (D_2^2), (D_3^2)\} = \{-2, -3, -3\}, \{-2, -3, -4\}$ or $\{-2, -3, -5\}$.

Case (II-4) $r = 2$ and $(D_1^2) = -2, (D_2^2) \leq -3$.

The above two Cases (II-3) and (II-4) are the hardest part in proving our Main Theorem. We need first to classify S at certain extent. Then Lemma 3.9 and the arguments in §2 and §3.1 can be used to prove the finiteness of $\pi_1(S^o)$.

References

[1] T.C. Chau, A note concerning Fox's paper on Fenchel's conjecture, Proc. Amer. Math. Soc. 88 (1983), 584-586.

[2] M. Demarzure, Surfaces de del Pezzo, Lecture Notes in Mathematics 777, Berlin-Heidelberg-New York, Springer, 1980.

[3] R.H. Fox, On Fenchel's conjecture about F-groups, Math. Tidsskr. B (1952), 61-65.

[4] M.V. Nori, Zariski's conjecture and related problems, Ann. scient. Ec. Norm. Sup. 16 (1983), 305-344.

[5] M. Miyanishi, Non-complete algebraic surfaces, Lecture Notes in Mathematics 857, Berlin-Heidelberg-New York, Springer, 1981.

[6] M. Miyanishi and D.-Q. Zhang, Gorenstein log del Pezzo surfaces of rank one, J. of Algebra, 118 (1988), 63-84.

[7] M. Miyanishi and D.-Q. Zhang, Gorenstein log del Pezzo surfaces, II, to appear in J. of Algebra.

[8] D.-Q. Zhang, Logarithmic del Pezzo surfaces of rank one with contractible boundaries, Osaka J. Math. 25 (1988), 461-497.

[9] D.-Q. Zhang, Logarithmic del Pezzo surfaces with rational double and triple singular points, Tohoku Math. J. 41 (1989), 399-452.