代数幾何学シンポジウム記録
1992年度 pp.72-76



# On false hyperelliptic surfaces
# (擬超楕円曲面について)

奈良女子大学理学部　武田 好史 (Yoshifumi TAKEDA)

## 1 Multiple fibres of elliptic fibrations

Let $f : V \to B$ be an mininal elliptic fibration from a nonsingular projective surface $V$ onto a nonsingular projective curve $B$ over an algebraically closed field $k$. Let us consider multiple fibres of $f$. In the case of characteristic zero, multiple fibres are classified and it is well-known. For example, the following is known:

**Fact 1** (char$=0$) *If $mF$ is a multiple fibre, the $F$ is either an elliptic curve, a nodal rational curve, or a loop of rational curves.*

In the case of positive characteristic, however, the complete classification of multiple fibres is not known as far as the author knows. The following fact are, nevertheless, known:

**Fact 2** (char$=p>0$) *If $mF$ is a multiple fibre and if $F$ is neither an elliptic curve, a nodal rational curve, nor a loop of rational curves, then $m$ is a power of $p$.*

Furthermore, some results and many examples of multiple fibre in positive characteristic due to Katsura and Ueno are known. Yet, the following problem rests open:

**Problem 1 (Néron)** *Does there exist an elliptic fibration which has multiple fibres of supersingular elliptic curve, of multiplicity $p^\nu$ with $\nu > 1$ ?*

Here the definition of supersingular elliptic curve is as follows:

**Definition 1** *Let $E$ be an elliptic curve. We say that $E$ is supersingular if there exists a rational function $h$ such that the exterior differentiation* d*$h$ globally generates the sheaf of differentials*

We shall give an example of multiple fibre $pE$, where $E$ is supersingular.

**Example 1** Let $E$ be a supersingular elliptic curve. Consider the product of $E$ and the projective line, the projection onto $\mathbf{P}^1$, and the Frobenius $k$-morphisms. Take a convenient covering $\sigma : X \to E \times \mathbf{P}^1$ of degree $p$, which factors the Frobenius $k$-morphism of $E \times \mathbf{P}^1$.

$$\begin{tikzcd}
E \times \mathbf{P}^1 \arrow[rr, "\text{Frobenius } k\text{-morphism}"] \arrow[dr, "\deg p"'] \arrow[dd, "\mathrm{pr}"'] & & E \times \mathbf{P}^1 \arrow[dd, "\mathrm{pr}"] \\
& X \arrow[ur, "\sigma"', "\deg p"] \arrow[dr, "f"'] & \\
\mathbf{P}^1 \arrow[rr, "\text{Frobenius } k\text{-morphism}"'] & & \mathbf{P}^1
\end{tikzcd}$$

Then the composite $f = \mathrm{pr} \circ \sigma$ is an elliptic fibration which has a multiple fibre $pE$.

We note that it is difficult to extend this example into the case of $p^\nu E$. So, let us consider another method.

## 2 False hyperelliptic surfaces

Let $X$ be a nonsingular relatively minimal projective surface.

**Definition 2 (H1)** $\kappa(X) = 0$, $\chi(\mathcal{O}_X) = 0$, $(K_X^2) = 0$, $\dim \mathrm{Alb}(X) = 1$.

**(H2)** *Let $\varphi : X \to E$ be the Albanese mapping. Then all fibres of $\varphi$ are elliptic curves.*

*If $X$ satisfies these conditions, we say that $X$ is a hyperelliptic surface.*

Hyperelliptic surfaces have the following properties:

1. Every hyperelliptic surface is obtained as the quotient of the product of two elliptic curves by a finite group scheme.
2. Every hyperelliptic surface has another elliptic fibration onto $\mathbf{P}^1$.

There is another class of surfaces which have the same numerical invariants.

**Definition 3** **(Q1)** $\kappa(X) = 0$, $\chi(\mathcal{O}_X) = 0$, $(K_X^2) = 0$, $\dim \mathrm{Alb}(X) = 1$.

**(Q2)** *Let $\varphi : X \to E$ be the Albanese mapping. Then all fibres of $\varphi$ are rational curves with one ordinary cusp.*

*If $X$ satisfies these conditions, we say that $X$ is a quasi-hyperelliptic surface.*

Quasi-hyperelliptic surfaces have the following properties:

1. Every quasi-hyperelliptic surface is obtained as the quotient of the product of an elliptic curve and a rational curve with one ordinary cusp, by a finite group scheme.
2. Every quasi-hyperelliptic surface has an elliptic fibration onto $\mathbf{P}^1$.
3. Quasi-hyperelliptic surfaces occur in characteristic 2 or 3.

Let us generalize quasi-hyperelliptic surfaces in every positive characteristic. Assume that the characteristic $p$ is positive.

**Definition 4** **(F1)** $\chi(\mathcal{O}_X) = 0$, $(K_X^2) = 0$, $\dim \mathrm{Alb}(X) = 1$.

**(F2)** *Let $\varphi : X \to E$ be the Albanese mapping. Then all fibres of $\varphi$ are rational curves with one cusp of type $x^{p^\nu} + y^n = 0$, where $(n, p) = 1$.*

*If $X$ satisfies these conditions, we say that $X$ is a false hyperelliptic surface.*

Here we note that $\kappa(X) = 0$ is not assumed. The following problems arise.

1. Do false hyperelliptic surfaces exist?
2. Do false hyperelliptic surfaces have elliptic fibrations?

In this article, we consider only the case where the Albanese mapping has a section $S$. We can prove the following theorems.

**Theorem 1** *We have the following cartesian diagram:*

$$
\begin{tikzcd}
E \times \mathbf{P}^1 \arrow[r] \arrow[dd, "pr"'] & Z \arrow[d, "\mu"] & \\
 & \widetilde{X} \arrow[r] \arrow[d] & X \arrow[d, "\varphi"] \\
\widetilde{E} \arrow[r, "\omega"'] & \widetilde{E} \arrow[r, "F"'] & E,
\end{tikzcd}
$$

*where $F$ is the $\nu$-th power of the Frobenius $k$-morphism, $\mu$ is the normalization, $\omega$ is an étale covering, $E = \mathrm{Alb}(X)$, $\mathcal{L} = F^*\mathcal{O}_S(S)$, and $Z = \mathbf{P}(\mathcal{O}_{\widetilde{E}} \oplus \mathcal{L})$.*

We can also prove that $\mathcal{L}^n$ is trivial. So, let $r$ be the order of $\mathcal{L}$. Set $T = \{P \in X \mid \varphi \text{ is not smooth at } P\}$.

**Theorem 2** *There exists an elliptic fibration $\psi$ onto rational curve, induced from the second projection $\widetilde{E} \times \mathbf{P}^1 \to \mathbf{P}^1$ in the previous theorem. Moreover $\psi$ has multiple fibres $rp^\nu S$ and $rT$.*

Here if $r$ is equal to one, then $rT$ is not multiple fibre.

**Theorem 3** *If $\psi$ has at most two multiple fibres $rp^\nu S$ and $rT$, then $E$ is a supersingular elliptic curve.*

## 3 An example

To close this article, we shall give an example of false hyperelliptic surface with section, which gives an affirmative answer about Néron's problem in Section 1 . We begin with

**Lemma 1** *Let $\omega$ be an exact differential form on a smooth projective curve $C$. Suppose the divisor of $\omega$ has the form $p^\nu nH$, where $H$ is an effective divisor, and where $\nu$ and $n$ are positive integers. Then there exists a local coordinate system $\{\eta_i \in \Gamma(U_i, \mathcal{O}_C)\}_{i \in I}$ for an affine open covering $\{U_i\}_{i \in I}$ of $C$, satisfying $\eta_i = {a_{ij}}^{np^\nu} \eta_j + {b_{ij}}^{p^\nu}$, where $\{a_{ij}\}$ are transition functions of $\mathcal{O}_C(H)$ for $\{U_i\}_{i \in I}$, and $b_{ij} \in \Gamma(U_i \cap U_j, \mathcal{O}_C)$.*

This statement is difficult to understand. So, we shall give an example.

**Example 2** Let $C$ be an affine plane curve defined by an equation $y^{np^\nu} - y = x^{np^\nu - 1}$, where $\nu$ and $n$ are positive integers and $n > 1$, $(p, n) = 1$. Since the genus of $C$

is $(np^\nu - 1)(np^\nu - 2)/2$, we obtain that the divisor of an exact differential $dx$ is $np^\nu(np^\nu - 3)P_\infty$, where $P_\infty$ is the point at infinity. Hence $C$ and $dx$ satisfy the hypothesis of the previous lemma. Let $\eta_i$, $a_{ij}$, and $b_{ij}$ be the sections mentioned in the lemma. Let $\mathcal{E}$ be a vector bundle generated by $\{e_i\}$ and $\{f_i\}$, where $e_i = e_j$ and $f_i = a_{ij}{}^{np^\nu} f_j + b_{ij}{}^{p^\nu} e_j$. Let $F^{(\mu)}$ be the $\mu$-th power of the Frobenius $k$-morphism of $C$. Then we have: $F^{(\mu)*}\mathcal{E}$ is stable if $0 \leq \mu < \nu$ ; $F^{(\mu)*}\mathcal{E}$ is unstable if $\mu \geq \nu$.

Now, we shall construct an example of false hyperelliptic surface with section. Let $E$ be a supersingular elliptic curve and take arbitrary positive integers $\nu$ and $n$ with $(p, n) = 1$. By the above lemma, we have $\{U_i\}$, $\{\eta_i\}$, $\{a_{ij}\}$, $\{b_{ij}\}$ as above. (Note that $H$ is zero in this case.) Consider an open surface $X_0$ and a morphism $\phi : X_0 \to E$ defined as

$$\phi^{-1}(U_i) = \operatorname{Spec}\Gamma(U_i, \mathcal{O}_E)[x_i, y_i]/(x_i{}^{p^\nu} - y_i{}^n - \eta_i),$$

where $x_i = a_{ij}{}^n x_j + b_{ij}$, $y_i = a_{ij}{}^{p^\nu} y_j$. Let $X$ be the smooth completion of $X_0$. Then $X$ is a false hyperelliptic surface. Furthermore, the Albanese variety is $E$, the Albanese mapping has a section $S$ which is equal to the complement of $X_0$, and the normal bundle of $S$ is trivial. Moreover, we note that $X$ has an elliptic fibration $\psi : X \to \mathbf{P}^1$ which has a multiple fibre $p^\nu S$, where $S$ is a supersingular elliptic curve.