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ON ABELIAN CONFORMAL FIELD THEORY

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Abelian conformal field theory is usually discussed from the view point of the universal Grassmann manifold and Krichever maps ([1]). Here, we consider it from the view point of non-abelian conformal field theory developed in [2]. We take the Heisenberg algebra as a gauge group. In the following we shall show that the main ideas of the paper [2] can be applied to our situation.

We thank A. Tuchiya for pointing out a gap of our original proof of the main theorem and showing us an idea of a proof of Lemma 2.3 below.

§1. Main Theorem

For a positive even integer M we let H_M be a Heisenberg algebra generated by operators $a(n)$, $n \in \mathbf{Z}$ with commutation relation

$$[a(n), a(m)] = Mn\delta_{n+m,0} \cdot id. \tag{1.1}$$

The Heisenberg algebra is a universal enveloping algebra of an affine Lie algebra $\{a(n)\}$ associated with a one-dimensional abelian Lie algebra $\mathbf{C}$ with commutation relation (1.1). For each $p \in \mathbf{C}$, by $\mathcal{F}(p)$ we denote an irreducible highest weight module of H_M determined by

$$\begin{aligned} a(0)|p\rangle &= p|p\rangle \\ a(n)|p\rangle &= 0, \quad \text{if} \quad n \geq 1, \end{aligned}$$

where $|p\rangle$ is a highest weight vector. Let $t_0, t_1, t_2, \ldots$ be independent variables. Put

$$\begin{aligned} a(m) &= \frac{\partial}{\partial t_m}, \quad m = 0, 1, 2, \ldots \\ a(-n) &= nMt_n, \quad n = 1, 2, 3, \ldots \end{aligned}$$

Then, the Heisenberg algebra H_M and its irreducible module $\mathcal{F}(p)$ are realized as

$$\begin{aligned} H_M &= \mathbf{C}[t_1, t_2, \dots t_n, \dots, \frac{\partial}{\partial t_0}, \frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_m}, \dots, \frac{\partial}{\partial t_m}, \dots] \\ \mathcal{F}(p) &= \mathbf{C}[t_1, t_2, \dots, t_n, \dots e^{pt_0}, e^{-pt_0}], \end{aligned}$$

where the highest weight vector $|p\rangle$ corresponds to e^{pt_0}. Using there realization, let us introduce an operator $\widehat{q}$ as

$$\widehat{q} = Mt_0.$$

Put

$$\begin{aligned} \phi(z) &= \widehat{q} + a(0)\log z - \sum_{n\neq 0} \frac{a(n)}{n} z^{-n} \\ a(z) &= \sum_{n\in\mathbf{Z}} a(n) z^{-n-1} \end{aligned}$$

Then we have

$$d\phi(z) = a(z)dz$$

For each integer k, the Vertex operator $V_{kM}(z)$ is defined as

$$V_{kM}(z) = {}^{\circ}_{\circ}\, e^{k\phi(z)}\, {}^{\circ}_{\circ}$$

where ${}^{\circ}_{\circ}\quad{}^{\circ}_{\circ}$ is a normal ordering defined by putting $a(n)$, $n \geq 0$ the right hand side and $\widehat{q}$, $a(-n)$,$n \geqslant 1$ the left hand side. Hence, we have

$$V_{kM}(z) = e^{k\sum_{n=1}^{\infty} \frac{a(-n)}{n} z^n} e^{k\widehat{q}} e^{ka(0)\log z} e^{-k\sum_{n=1}^{\infty} \frac{a(n)}{n} z^{-n}}$$

The Vertex operator $V_{kM}(z)$ is an intertwiner between the representations $\mathcal{F}(p)$ and $\mathcal{F}(kM + p)$. Note that in conformal field theory $a(z)$ behaves as a one-form and $V_{kM}(z)$ behaves as a $\frac{k^2}{2}M$-form. The energy-momentum tensor $T(z)$ is defined as

$$T(z) = \frac{1}{2M}\, {}^{\circ}_{\circ}\, a(z)a(z)\, {}^{\circ}_{\circ}$$

There is a formal expansion

$$T(z) = \sum_{n\in\mathbf{Z}} L_n z^{-n-2},$$

and $\{L_n\}$ is a Virasoro algebra. In the following we only consider irreducible highest weight representations of H_M with highest weight vectors $|p\rangle$ where p's are *integers.*

Let $\Lambda = \{\overline{0}, \overline{1}, \dots, \overline{M-1}\}$ be representatives of the module $\mathbf{Z}/M\mathbf{Z}$. For each $\overline{p} \in \{\overline{0}, \overline{1}, \dots, \overline{M-1}\}$, put

$$\mathcal{H}(\overline{p}) := \bigoplus_{p \equiv \overline{p} \mod M} \mathcal{F}(p).$$

Let $\mathfrak{X} = (C; Q_1, \dots, Q_N; \xi_1, \dots, \xi_N)$ be an N-pointed stable curve of genus g with formal neighbourhoods. To each point Q_j we associate an element $\overline{p_j} \in \Lambda$ and put

$$\vec{p} = (\overline{p}_1, \overline{p}_2, \dots, \overline{p}_N),$$
$$\mathcal{H}(\vec{p}) = \mathcal{H}(\overline{p}_1) \otimes \mathcal{H}(\overline{p}_2) \otimes \cdots \otimes \mathcal{H}(\overline{p}_N)$$

Put also

$$\mathcal{H}^\dagger(\vec{p}) = \mathrm{Hom}_{\mathbf{C}}(\mathcal{H}(\vec{p}), \mathbf{C}).$$

We have a natural pairing

$$\begin{aligned} \mathcal{H}^\dagger(\vec{p}) \times \mathcal{H}(\vec{p}) &\to \mathbf{C} \\ (\langle\psi|, |\phi\rangle) &\mapsto \langle\psi|\phi\rangle \end{aligned}$$

where $\langle\psi|\phi\rangle$ means $\psi(|\phi\rangle)$.

Definition 1.1. The space of vacua $\mathcal{V}^\dagger_{\vec{p}}(\mathfrak{X})$ attached to the N-pointed stable curve with formal neighbourhoods $\mathfrak{X}$ is a subspace of $\mathcal{H}^\dagger(\vec{p})$ consisting of vectors $\langle\psi|$ satisfying the following conditions.

(1) For each $|\phi\rangle \in \mathcal{H}(\vec{p})$, the data $\langle\psi|\rho_j(a(\xi_j))|\phi\rangle d\xi_j$, $j = 1, 2, \dots, N$ are the Laurent expansions of an element $\omega \in H^0(C, \omega_C(*\sum Q_j))$ at Q_j's with respect to the formal coordinates ξ_j's,

(2) For each $|\phi\rangle \in \mathcal{H}(\vec{p})$, the data $\langle\psi|\rho_j(V_{\pm M}(\xi_j)|\phi\rangle(d\xi_j)^{\frac{M}{2}}$, $j = 1, 2, \dots, N$, are the Laurent expansions of an element $\tau \in H^0(C, \omega_C^{\otimes\frac{M}{2}}(*\sum Q_j))$ at Q_j's with respect to the formal coordinates ξ_j.

Main Theorem. *We have*

$$\dim_{\mathbf{C}} \mathcal{V}^\dagger_{\vec{p}}(\mathfrak{X}) = \begin{cases} M^g, & \textit{if } \overline{p}_1 + \cdots + \overline{p}_N = \overline{0} \\ 0, & \textit{otherwise} \end{cases}$$

where g is the genus of the stable curve C.

§2. Outline of a proof of Main Theorem.

First we shall rewrite the conditions (1), (2) in Definition 1.1. Note that the condition (1) is equivalent to the condition

$$(1^*)\qquad \sum_{j=1}^{n} \operatorname*{Res}_{\xi_j=0}(\langle\psi|\rho_j(a(\xi_j))|\phi\rangle g(\xi_j)d\xi_j) = 0$$

for every $g \in H^0(C, \mathcal{O}_C(*\sum Q_j))$, where $g(\xi_j)$ is the Laurent expansion of g at Q_j. The condition (2) is equivalent to the condition

$$(2^*)\qquad \sum_{j=1}^{N} \operatorname*{Res}_{\xi_j=0}(\langle\psi|\rho_j(V_{\pm M}(\xi_j))|\phi\rangle h(\xi_j)d\xi_j) = 0$$

for every $h \in H^0(C, \omega_C^{\otimes(1-\frac{M}{2})}(*\sum Q_j))$, where $h(\xi_j)(d\xi_j)^{\frac{M}{2}}$ is the Laurent expansion of h at Q_j. In the following we choose integers p_j such that $p_j \equiv \overline{p}_j \mod M$. Put

$$|p_1, p_2, \dots, p_N\rangle = |p_1\rangle \otimes |p_2\rangle \otimes \cdots \otimes |p_N\rangle.$$

Apply the condition (1^*) to an element $\langle\psi| \in \mathcal{V}_{\overline{p}}^{\dagger}(\mathfrak{X})$ and $1 \in H^0(C, \mathcal{O}_C(*\sum Q_j))$. Since we have

$$\operatorname*{Res}_{\xi_j=0}\{(a(\xi_j)|p_j\rangle d\xi_j\} = a(0)|p_j\rangle = p_j|p_j\rangle,$$

the condition (1^*) implies that

$$(\sum_{j=1}^{N} p_j)\langle\psi|p_1, p_2, \dots, p_N\rangle = 0.$$

Hence, if $\langle\psi|p_1, p_2, \dots, p_N\rangle \neq 0$, then $\sum_{j=1}^{N} p_j = 0$.

First let us consider an N-pointed projective line $(\mathbf{P}^1(\mathbf{C}); a_1, a_2, \dots, a_N)$ with $a_1 = 0, a_2 = 1, a_N = \infty$. Let z (resp.w) be a coordinate of an affine line in $\mathbf{P}^1(\mathbf{C})$ containing 0 (resp.∞) with $z \cdot w = 1$. Put

$$(2.1)\qquad \xi_j = \begin{cases} z - a_j, & j = 1, 2, \cdots, N-1 \\ w, & j = N, \end{cases}$$

and

$$\mathfrak{X} = (\mathbf{P}^1(\mathbf{C}); a_1, a_2, \dots, a_N; \xi_1, \xi_2, \dots, \xi_N).$$

First we shall prove the following proposition.

proposition 2.1.

$$\dim_{\mathbf{C}} \mathcal{V}^{\dagger}_{\vec{p}}(\mathfrak{X}) = \begin{cases} 1, & \textit{if} \ \ \overline{p}_1 + \overline{p}_2 + \cdots + \overline{p}_N = 0 \\ 0, & \textit{otherwise} \end{cases}$$

Let $F_0\mathcal{H}(\overline{p}_j)$ be a subspace of $\mathcal{H}(\vec{p})$ spanned by the highest weight vectors $|lM + p_j\rangle$, $l \in \mathbf{Z}$ over $\mathbf{C}$. Put

$$F_0\mathcal{H}(\vec{p}) = F_0\mathcal{H}(\overline{p}_1) \otimes F_0\mathcal{H}(\overline{p}_2) \otimes \cdots \otimes F_0\mathcal{H}(\overline{p}_N).$$

To prove the above proposition we need the following lemma.

Lemma 2.2. *Under a natural mapping*

$$j : \mathrm{Hom}_{\mathbf{C}}(\mathcal{H}(\vec{p}), \mathbf{C}) \longrightarrow \mathrm{Hom}_{\mathbf{C}}(F_0\mathcal{H}(\vec{p}), \mathbf{C}),$$

the space of vacua $\mathcal{V}^{(\dagger)}_{\vec{p}}(\mathfrak{X})$ of the N-pointed projective line with coordinates (2.1) is mapped injectively.

The lemma and the above consideration imply

$$\mathcal{V}^{\dagger}_{\vec{p}}(\mathfrak{X}) = 0$$

if $\overline{p}_1 + \overline{p}_2 + \ldots + \overline{p}_N \neq 0$. Therefore, assume $\overline{p}_1 + \overline{p}_2 + \ldots + \overline{p}_N = 0$. Choose p_j's in such a way that

$$p_1 + p_2 + \ldots + p_N = 0,$$

and fix them in the following. For an element $\langle\psi| \in \mathcal{V}^{\dagger}_{\vec{p}}(\mathfrak{X})$, put

$$\psi_{l_1, l_2, \ldots, l_N} = \langle\psi|(|l_1 M + p_1\rangle \otimes |l_2 M + p_2\rangle \otimes \cdots \otimes |l_N M + p_N\rangle).$$

If $\psi_{l_1, l_2, \ldots, l_N} \neq 0$, then $l_1 + l_2 + \ldots l_N = 0$. The condition (1*) implies that $\psi_{l_1, l_2, \ldots, l_N}$ determines uniquely the values

$$\langle\psi|(a(-n^{(1)}_1)\ldots a(-n^{(1)}_{k_1})|l_1 M + p_1\rangle \otimes a(-n^{(2)}_1)\ldots a(-n^{(2)}_{k_2})|l_2 M + p_2\rangle \otimes \\ \ldots \otimes a(-n^{(N)}_1)\ldots a(-n^{(N)}_{k_N})|l_N M + p_M\rangle),$$

for any positive integers $n^{(i)}_j$. Also,the condition (2*) implies that $\psi_{l_1, l_2, \ldots, l_N}$ can be uniquely determined by the value $\psi_{0,0,\ldots,0}$. Thus, we conclude that

$$\dim_{\mathbf{C}} \mathcal{V}^{\dagger}_{\vec{p}}(\mathfrak{X}) = 1$$

This proves Proposition 2.1.

Let us consider a bigger subspace $\mathcal{V}_{\vec{p}}^{\dagger}(n)$ of $\mathcal{H}^{\dagger}(\vec{p})$. An element $\langle\psi|$ is in $\mathcal{V}_{\vec{p}}^{\dagger}(n)$, if $\langle\psi|$ satisfies the following two conditions (1_n^{**}) and (2_n^{**}).

$$(1_n^{**}) \qquad \sum_{j=1}^{n} \operatorname*{Res}_{\xi_j=0}(\langle\psi|a(\xi_j)|\phi\rangle g_j(\xi_j)d\xi_j) = 0$$

for all

$$(g_j(\xi_j)d\xi_j) \in \mathbf{C}\bigoplus \oplus_{j=1}^{N}(\mathbf{C}[\xi_j]\xi_j^{-n}), \quad \text{and} \quad |\phi\rangle \in \mathcal{H}(\vec{p}),$$

where an element $c \in \mathbf{C}$ in the right hand side can be considered as $(c, c, \ldots, c)$.

$$(2_n^{**}) \qquad \sum_{j=1}^{n} \operatorname*{Res}_{\xi_j=0}(\langle\psi|V_{\pm M}(\xi_j)|\phi\rangle g_j(\xi_j)d\xi_j) = 0$$

for all

$$(h_j(\xi_j)(d\xi_j)^{\frac{M}{2}}) \in \bigoplus_{j=1}^{N}(\mathbf{C}[\xi_j]\xi_j^{-n})(d\xi_j)^{(1-\frac{M}{2})}, \quad \text{and} \quad |\phi\rangle \in \mathcal{H}(\vec{p}).$$

Key Lemma. *Under the above notation we have*

$$\dim \mathcal{V}_{\vec{p}}^{\dagger}(n) < \infty.$$

To prove the Key Lemma we need the following Lemma due to Tuchiya.

Lemma 2.3. *Let $\mathfrak{X}$ be an N-pointed smooth curve of genus g with formal neighbourhoods. Then we have*

$$\dim \mathcal{V}_{\vec{p}}^{\dagger}(\mathfrak{X}) \leq n^g.$$

The idea of the proof is as follows. For each non-zero element $\langle\psi| \in \mathcal{V}_{\vec{p}}^{\dagger}(\mathfrak{X})$ and any element $|v\rangle \in F_0\mathcal{H}(\vec{p})$ we can define a meromorphic form

$$\langle\psi|V_{\pm M}(z_1)V_{\pm M}(z_2)\cdots\ V_{\pm M}(z_m)|v\rangle(dz_1)^{\frac{M}{2}}(dz_2)^{\frac{M}{2}}\cdots(dz_m)^{\frac{M}{2}}$$

on $\underbrace{C \times C \times \cdots \times C}_{m}$. By the operator product expansion of the energy momentum tensor $T(z)$ we know singularities of this form and we can express the form by means of prime forms. This shows Lemma 2.3. To prove Key Lemma we need also the following lemma.

Lemma 2.4. *For positive integers n and N there exist a smooth curve D of genus g and points $Q_1, \cdots, Q_N$ on D with local coordinates $\xi_1, \xi_2, \cdots, \xi_N$ such that*

$$Gr^F_\bullet H^0(D, \mathcal{O}_D(*\sum Q_j)) \subset \mathbf{C} \bigoplus \oplus_{j=1}^N \mathbf{C}[\xi_j^{-1}]\xi_j^{-n}$$
$$Gr^F_\bullet H^0(D, \omega_D^{\otimes(1-\frac{M}{2})}(*\sum Q_j)) \subset \mathbf{C} \bigoplus \oplus_{j=1}^N \mathbf{C}[\xi_j^{-1}]\xi_j^{-n}(d\xi_j)^{1-\frac{M}{2}}$$

where the filtration F can be defined by the order of poles at Q_j.

The first inclusion can be proved, if the divisor $n(Q_1+Q_2+\cdots+Q_N)$ is not special on a curve D. The second inclusion is trivially true, if we have $(2g-2)(1-\frac{M}{2}) > nN$.

Now introducing the filtration on $\mathcal{H}(\vec{p})$ and $\mathcal{H}^\dagger(\vec{p})$ compatible with the filtration in Lemma 2.4, we can show *finite dimensionality* of $\mathcal{V}_{\vec{p}}(\mathfrak{X})$ for all N-pointed stable curve with formal neighbourhoods.

Now let us consider a semi-stable curve C. For a double point $P \in C$ we let $\pi : \tilde{C} \to C$ be the normalization at the point P. Then, the inverse image $\pi^{-1}(P)$ of the point P consists of two points P_+, P_-. Let η_+, η_- be formal coordinates of P_+ and P_-respectively such that C is defined formally in a neighbourhood of the origin of $\mathbf{C}^2$ by an equation $\eta_+ \cdot \eta_- = 0$. Let $\mathfrak{X} = (C; Q_1, \ldots, Q_N; \xi_1, \ldots, \xi_N)$ be an N-pointed stable curve with formal neighbourhoods whose underling curve is the semi-stable curve C. Put

$$\tilde{\mathfrak{X}} = (\tilde{C}; Q_1, \ldots, Q_N, P_+, P_-; \xi_1, \ldots, \xi_N, \eta_+, \eta_-).$$

Then,we have the following theorem.

Theorem 2.5. *Under the above notation and assumptions, we have a canonical isomorphism.*

$$\bigoplus_{\overline{q} \in \mathbf{Z}/M\mathbf{Z}} \mathcal{V}^\dagger_{\overline{q}, -\overline{q}, \vec{p}}(\tilde{\mathfrak{X}}) \simeq \mathcal{V}^\dagger_{\vec{p}}(\mathfrak{X}).$$

From this theorem and Proposition 2.1 we infer the following lemma.

Lemma 2.6. *Let $\mathfrak{X} = (C; Q_1, \ldots, Q_N; \xi_1, \ldots, \xi_N)$ be an N-pointed stable curve with formal neighbourhoods. Assume that all the irreducible component of the semi-stable curve C are $\mathbf{P}^1(\mathbf{C})$ and the genus of C is g. Then, we have*

$$\dim_{\mathbf{C}} \mathcal{V}^\dagger_{\vec{p}}(\mathfrak{X}) = \begin{cases} M^g, & \text{if } \ \overline{p}_1 + \cdots + \overline{p}_N = 0 \\ 0, & \text{otherwise}. \end{cases}$$

Now we need to show that $\dim_{\mathbf{C}} \mathcal{V}^\dagger_{\vec{p}}(\mathfrak{X})$ depends only on the genus of the underlying curve C. For that purpose we need to consider the family $\mathcal{V}_{\vec{p},N} = \bigcup_{\mathfrak{X}} \mathcal{V}^\dagger_{\vec{p}}(\mathfrak{X})$

over the moduli space $\overline{\mathcal{M}}_{g,N}^{(\infty)}$ of N-pointed curves of genus g with formal neighbourhoods. By a similar method as the one in [2], we can show that $\overline{\mathcal{V}}_{\vec{p},N}$ comes from a sheaf $\mathcal{V}_{\vec{p},N}^{(1)}$ on $\mathcal{M}_{g,N}^{(1)}$, the moduli space of N-pointed curves of genus g with first order neighbourhoods. Then, by Key Lemma we can show that $\mathcal{V}_{\vec{p},N}^{(1)}$ is a coherent $\mathcal{O}_{\overline{\mathcal{M}}_{g,N}^{(1)}}$ -module and it carries a logarithmic projectively flat connection. From these fact we infer that $\mathcal{V}_{\vec{p},N}^{(1)}$ is locally free on the open part of $\overline{\mathcal{M}}_{g,N}^{(1)}$ corresponding to non-singular curves.

Again, using a similar arguments as in [2] we can show that $\mathcal{V}_{\vec{p},N}^{(1)}$ is locally free. By Lemma 2.6 this implies our main theorem.

References

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