# On Moishezon Threefolds
# Homeomorphic to a Cubic Hypersurface in $\mathbf{P}^4$

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## § 0. Introduction.

The three dimensional projective space $\mathbf{P}^3$ and a smooth quadric hypersurface $\mathbf{Q}^3$ in the four dimensional projective space $\mathbf{P}^4$ have the unique complex structure as Moishezon manifolds ([Ko2, 5.3.5] [N1][N2]).[1] See also [P1][P2] [Ko2, 5.3.13]. The major purpose of the present paper is to report recent progress on the similar problems for Fano 3-folds of index two or more specifically cubic hypersurfaces in $\mathbf{P}^4$.

There are, besides smooth ones, various normal singular cubic hypersurfaces admitting small (smooth) resolutions. *A cubic 3-fold* is by definition a smooth Moishezon 3-fold which is a small resolution

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[1]   There is a rumor that it has been proved that the six dimensional sphere $S^6$ has a complex structure, a fortiori, $\mathbf{P}^3$ has an exotic complex structure with nonzero irregularity. It may probably true that any compact complex 3-fold homeomorphic to $\mathbf{P}^3$ (resp. $\mathbf{Q}^3$) is isomorphic to $\mathbf{P}^3$ (resp. $\mathbf{Q}^3$) if the irregularity vanishes.

of a normal cubic hypersurface in $\mathbf{P}^4$. Any cubic 3-fold with $b_2 = 1$ is a simply connected closed 6-manifold with the second integral homology group infinite cyclic, whose first Chern class is divisible by two. Some of cubic 3-folds are shown to have torsion free integral homology groups. Since the second homology group is generated by the dual of a hyperplane section, the cubic form on the second homology group is the same as that of a smooth cubic hypersurface. Therefore if any integral homology group of it is torsion free, (though we do not know whether this is true for arbitrary cubic 3-folds) then the topology of a cubic 3-fold with $b_2 = 1$ is by [W] uniquely determined by its third Betti number $b_3$. By an inequality which will be proved in (4.2) we see that $b_3$ is an even integer with $0 \le b_3 \le 10$. An arbitrary even integer between 0 and 10 is realized as the third Betti number of some cubic 3-fold with $b_2 = 1$, where a cubic 3-fold with $b_3 = 10$ is a smooth cubic hypersurface. We prove

**Theorem 0.1.** *Let $X$ be a Moishezon 3-fold with $c_1^3$ positive, $b_2 = 1$ and $2 \le b_3 \le 10$. Then $X$ is homeomorphic to a cubic 3-fold if and only if it is isomorphic to either a cubic 3-fold $(2 \le b_3 \le 10)$ or a certain blowing down of a small resolution of a blowing-up of $\mathbf{Q}^3$ $(2 \le b_3 \le 4)$. In particular, any Moishezon 3-fold with $c_1^3$ positive which is homeomorphic to a smooth cubic hypersurface in $\mathbf{P}^4$ is isomorphic to a smooth cubic hypersurface in $\mathbf{P}^4$.*

**Theorem 0.2.** *Let $X$ be a Moishezon 3-fold with $b_2 = 1$, $b_3 = 0$. Then $X$ is homeomorphic to a cubic 3-fold if and only if it is*

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*isomorphic to either a cubic 3-fold or a certain blowing down of a*
*small resolution of a blowing-up of* $\mathbf{Q}^3$.

The blowing down of a blowing-up of a smooth quadric hypersurface $\mathbf{Q}^3$ mentioned above, which we refer to as *a fake cubic 3-fold*, has $b_2 = 1$ and $b_3 \leq 4$. Since any fake cubic 3-fold is simply connected and has torsion free integral homology groups isomorphic to those of one of cubic 3-folds, it is diffeomorphic to some cubic 3-fold by the same reason as before. However it seems that no fake cubic 3-folds are global deformations of cubic 3-folds.

Here we would like to remark that for quadric hypersurfaces in $\mathbf{P}^4$ with Hessian rank four it seems very hard to give their characterization similar to the above because $b_2 \geq 2$. In fact, any normal quadric hypersurface in $\mathbf{P}^4$ with Hessian rank four has a small resolution, which is a $\mathbf{P}^2$-bundle over $\mathbf{P}^1$, and has infinitely many distinct complex structures as $\mathbf{P}^2$-bundles over $\mathbf{P}^1$.

Our proof of (part of) (0.1) roughly goes as follows. Let $X$ be a Moishezon 3-fold with $c_1^3$ positive which is homeomorphic to a smooth cubic hypersurface. Then $b_2 = 1$ and $b_3 = 10$, while the canonical line bundle of $X$ is divisible by two. Let $L$ be a (positive) generator of $H^2(X, \mathbf{Z})$ with $L^3 = 3$. Since $c_1^3$ is positive by the assumption, we have $K_X = -2L$ and $h^0(X, L) = 5$. If the base locus $B := \mathrm{Bs}\,|L|$ is empty, then the associated rational map $\rho_L$ is a birational morphism of $X$ onto a possibly singular normal cubic hypersurface in $\mathbf{P}^4$ with at worst isolated singularities. Thus $X$ is a cubic 3-fold. Then it follows from the inequality proved in section seven that $X$ is isomorphic to

a smooth cubic hypersurface.

Next we assume in general that $X$ is a Moishezon 3-fold with $L^3 = 3$, $K_X = -2L$, $h^0(X, L) = 5$ and that the base locus $B :=$ Bs $|L|$ of $|L|$ is nonempty. Two distinct general members $D$ and $D'$ of $|L|$ have no irreducible components in common so that the complete intersection $\ell = D \cap D'$ is pure one dimensional, which turns out to be a cycle of two smooth rational curves. Thereby $B$ turns out to be a single reduced smooth rational curve. Thus the base locus of $\rho_L$ can be eliminated by blowing up $X$ only once with $B$ center so that we have a morphism from the blowing up $\hat{X}$ of $X$ onto $\mathbf{Q}^3$. By studying the morphism we prove that $\hat{X}$ is a small resolution of a blowing-up of $\mathbf{Q}^3$, and that $X$ is therefore a fake cubic 3-fold. However, no fake cubic 3-fold is homeomorphic to a smooth cubic hypersurface because their third Betti numbers are different. This proves (0.1) when $X$ is homeomorphic to a smooth cubic hypersurface.

In the last we would like to mention that the positivity of $b_3$ is the obstacle for removing the assumption $c_1^3 > 0$ from (0.1). Since $b_3 \geq 2$ in these cases, $X$ may have nontrivial holomorphic three forms so that $K_X = 0$ or $K_X = 2L$ can happen. We were unable to exclude these possibilities without the assumption on $c_1^3$. However we need no extra assumption in (0.2) by the vanishing of $b_3$. It is an interesting question whether there exists a Calabi-Yau 3-fold homeomorphic to some cubic 3-fold with $b_2 = 1$, hence having Euler number $-6$ or $\pm 4$, or $\pm 2$.

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## § 1. Cubic 3-folds

(1.1) Let $W$ be an irreducible cubic hypersurface in $\mathbf{P}^4$. If $W$ is smooth, then $X := W$ is a Fano 3-fold of index two with Pic $X \simeq \mathbf{Z}$ whose integral homology groups are all torsion free and $b_2 = 1$, $b_3 = 10$. Now we consider also normal cubic hypersurfaces in $\mathbf{P}^4$, which admit small (smooth) resolutions. We call those smooth 3-folds *cubic threefolds*. The purpose of this section is to show that besides smooth cubic hypersurfaces there are smooth cubic 3-folds $X$ with Pic $X \simeq \mathbf{Z}L$, $L^3 = 3$, $K_X = -2L$ and $h^0(X, L) = 5$ whose integral homology groups are all torsion free. By [W] it is easy to determine (classify) the topology of those 3-folds.

Let $W$ be a singular normal cubic hypersurface in $\mathbf{P}^4$, $p_0$ a singular point of $W$. Taking homogeneous coordinates $x_0, \cdots, x_4$ such that $p_0 = [0, 0, 0, 0, 1]$, we write the defining equation $F$ of $W$ as

$$F = f_2(x)x_4 + f_3(x)$$

where $f_k(x)$ is a homogeneous polynomial of degree $k$ in $x_0, \cdots x_3$.

**Lemma 1.2.** *$f_2$ is not identically zero if $W$ has a small resolution.*

**Lemma 1.3.** *Assume* $f_2(x) = x_0 x_1 - x_2 x_3$. *Let* $\Delta$ *be a curve on* $\mathbf{P}^3$ *defined by* $f_2 = f_3 = 0$. *If* $W$ *has at worst isolated singularities, then* $\Delta$ *is reduced. If moreover* $W$ *has a small resolution* $smr : X \to W$, *then any local irreducible component of* $\Delta$ *is smooth and* $b_2(X) = b_2(\Delta)$. *The small resolution* $X$ *is simply connected.*

*Proof.* Let $G(x) = f_2(x)$ and $H(x) = f_3(x)$. Assume $x_k \neq 0$ for some $0 \leq k \leq 3$. Hence we have $G_k \neq 0$ for some $k$. Therefore

$$F_k := \partial F / \partial x_k = 0 \ \text{ for any } \ k$$

$$\Updownarrow$$

$$G = H = 0, \quad x_4 G_k + H_k = 0 \ \text{ for any } \ k$$

$$\Updownarrow$$

$$x_4 + H_k / G_k = 0 \ \text{ for some } \ k, \quad G = H = 0,$$

$$\text{rank} \begin{pmatrix} G_0 & G_1 & G_2 & G_3 \\ H_0 & H_1 & H_2 & H_3 \end{pmatrix} \leq 1$$

where $G_k := \partial G / \partial x_k$ and $H_k := \partial H / \partial x_k$. Since $G_k \neq 0$ for some $k$, $x_4$ at the singular point of $W$ is uniquely determined by $x_4 + H_k / G_k = 0$. It follows that the singularities of $W$ except at $p_0$ are in one to one correspondence with the singularities of $\Delta$. If $\Delta$ is nonreduced along an irreducible component, then $\Delta$ is *singular* along the component so that $W$ has nonisolated singularities. Therefore if $W$ has at worst isolated singularities, then $\Delta$ is reduced.

We assume that $\Delta$ is reduced. Let $smr : X \to W$ be a small resolution. The point $p_0$ is an ordinary double point of $W$. On the other

hand, any singularity of $X$ except $p_0$ corresponds to a singularity of $\Delta$ as observed above. The singularity of $X$ except $p_0$ is defined by an equation of the form $xy = f(z, w)$ where $f(z, w) = 0$ is a local equation of $\Delta$ at the corresponding singular point of $\Delta$. Hence by (1.2), any local irreducible component of $\Delta$ is smooth.

Since $p_0$ is an ordinary double point of $W$, $C := smr^{-1}(p_0)$ is a smooth rational curve. Let $\phi_C : \hat{X} \to X$ be the blowing-up of $X$ with $C$ center. Let $f : W \to \mathbf{P}^3$ be a rational map defined by $f([x_0, \cdots, x_4]) = [x_0, \cdots, x_3]$, which is an isomorphism over $\mathbf{P}^3 \setminus Q$ where $Q$ is a smooth quadric surface defined by $f_2 := x_0 x_1 - x_2 x_3 = 0$ in $\mathbf{P}^3$. Then the induced morphism $\pi := f \cdot smr \cdot \phi_C : \hat{X} \to \mathbf{P}^3$ is birational, which induces an isomorphism of the exceptional set $E := (smr \cdot \phi_C)^{-1}(p_0)$ onto $Q$ ($\subset \mathbf{P}^3$).

Let $S := \pi^{-1}(\Delta)$. Then $\Delta = \pi(S)$, and $smr \cdot \phi_C(S)$ is a cone over $\Delta$ in $\mathbf{P}^4$. The divisor $S$ consists of $b_2(\Delta)$ irreducible components. It is clear that $\hat{X} \setminus S \cup E \simeq X \setminus \phi_C(S) \cup C \simeq \mathbf{P}^3 \setminus Q$ and $S \cap E \simeq \Delta$.

Now we compute $b_2(X)$. Let $T := S \cup E$. Since $\mathbf{P}^3 \setminus Q \simeq \hat{X} \setminus T$, we have $\pi_1(\hat{X} \setminus T) \simeq H_1(\hat{X} \setminus T) \simeq \mathbf{Z}/2\mathbf{Z}$. It follows easily from $\pi_1(\hat{X} \setminus T) \simeq \mathbf{Z}/2\mathbf{Z}$ that $\hat{X}$ is simply connected. Similarly $H_2(\hat{X} \setminus T) \simeq H_2(\mathbf{P}^3 \setminus Q) = 0$. By the realtive homology exact sequence for the pair $(\hat{X}, \hat{X} \setminus T)$, we have $H_2(\hat{X}, \hat{X} \setminus T) \simeq H^4(T) \simeq H^4(S) \oplus H^4(E)$, so that $b_2(X) = b_2(\hat{X}) - 1 = \mathrm{rank}\, H_2(\hat{X}, \hat{X} \setminus T) - 1 = b_4(T) - 1 = b_4(S) = b_2(\Delta)$. This completes the proof.            q.e.d.

**Lemma 1.4.** *Assume* $f_2(x) = x_0 x_1 - x_2 x_3$. *If* $W$ *admits a small*

*resolution* $smr : X \to W$ *with* $b_2(X) = 1$, *then the integral homology groups of* $X$ *are torsion free and we have*

$$b_q = 1 \ (q : \text{even}), b_1 = b_5 = 0, b_3 = 8 - 2r$$

*where* $r = \frac{1}{2}(\deg \omega_\Delta - \deg \omega_{\tilde{\Delta}})$ *for the normalization* $\tilde{\Delta}$ *of* $\Delta$.

Now we recall a theorem of C.T.C. Wall [W, Theorem 5].

**Theorem 1.5.** (Wall) *Diffeomorphism classes of oriented closed 6-manifolds with torsion free integral homology groups and* $w_2 = 0$ *corresponds bijectively to isomorphism classes of systems of invariants :*

*free abelian groups* $H := H^2(M)$, $G := H^3(M)$,

*a symmetric trilinear form (cup product)* $\mu : H \times H \times H \to \mathbf{Z}$, *a homomorphism (Pontrjagin class)* $p_1 : H \to \mathbf{Z}$ *subject to*

$$\mu(x,x,y) = \mu(x,y,y) \bmod 2, \quad p_1(x) = 4\mu(x,x,x) \bmod 24.$$

In our case $p_1$ is given by $p_1 = c_1^2 - 2c_2$. It is easy to check that the conditions on $p_1$ and $\mu$ in (1.5) are automatically true in our case. Thus we see

**Lemma 1.6.** *Let* $X$ *and* $Y$ *be cubic 3-folds with* $b_2 = 1$. *Assume that the integral homology groups of* $X$ *and* $Y$ *are torsion free. Then the following are equivalent.*

*(1.6.1)* $X$ *and* $Y$ *are homeomorphic.*

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*(1.6.2)  X and Y are diffeomorphic.*

*(1.6.3)  $b_3(X) = b_3(Y)$.*

*Proof.* It is easy to show that any cubic 3-fold is simply connected. Therefore the assertion of (1.6) follows from (1.5).          q.e.d.

**Remark 1.7.** By computations we see that there are some cubic 3-folds with torsion free integral homology groups also in the case where Hessian rank $f_2 \leq 3$. Therefore in view of (1.6) some cubic threefolds can be diffeomorphic to each other even when they arise from quadratic polynomials $f_2$ with different Hessian ranks. We know so far no cubic 3-folds whose integral homology groups have torsions.

**Example 1.8.** Let $f_2 = x_0 x_1 - x_2 x_3$. Let $S$ be a surface in $\mathbf{P}^3$ defined by $f_2 = 0$. Then $S$ has two rulings $f$ and $g$. Let $\Delta := \{f_2 = f_3 = 0\}$ be a reduced curve of 6 components, the union of 3 fibers of $f$ and 3 fibers of $g$. Then $\Delta$ has 9 double points, whence the cubic hypersurface $W : f_2 x_4 + f_3 = 0$ has 10 ordinary double points. In this case $W$ has a small resolution $X$ with $b_2(X) = 6$ and $b_3(X) = 0$ by (1.3) and (4.1). This gives the maximum of the number of ordinary double points on cubic hypersurfaces. This is shown as follows. We choose one of the ordinary double points on a given cubic hypersurface $W$ as $p : [0,0,0,0,1]$. Then we can choose the equation defining $W$ and the projective coordinates $x_i$ as in (1.1) so that $f_2(x) = x_0 x_1 - x_2 x_3$. Hence by the proof of (1.3) any ordinary double point of $W$ corresponds to that of a curve $\Delta$, which is a member of $|3e_1 + 3e_2|$. Therefore it is easy to see that the above

example gives the maximum.

## § 2. Fake cubic 3-folds and compactifications of $\mathbf{C}^3$

(2.1)   In this section we construct some Moishezon threefolds with $K_X = -2L$ whose rational image by $\rho_L$ is $\mathbf{Q}^3$. This example has been mentioned in [Ko2, 5.3.14].

Let $\Delta$ be an irreducible Gorenstein curve of $\mathbf{Q}^3$ ($\subset \mathbf{P}^4$) with $\deg \Delta = 5$ and $\deg \omega_\Delta = 2$ whose any local irreducible components are smooth. We assume that $\Delta$ lies on a smooth hyperplane section of $\mathbf{Q}^3$, which we denote by $Q$ ($\simeq \mathbf{F}_0 \simeq \mathbf{P}^1 \times \mathbf{P}^1$). Let $e_1$ and $e_2$ be fibers of two natural rulings of $Q$. Then we may assume that $\Delta$ belongs to $|2e_1 + 3e_2|$.

Let $\hat{f} : \hat{X} \to \mathbf{Q}^3$ be a small resolution of a blowing-up of $\mathbf{Q}^3$ with $\Delta$ center, $\hat{\Delta} := \hat{f}^{-1}(\Delta)$ and $\hat{Q}$ the proper transform of $Q$. It is easy to see that $\hat{Q} \simeq Q$ and $N_{\hat{Q}/\hat{X}} \simeq -e_1 - 2e_2$ even if $\Delta$ is singular. Therefore we have a contraction morphism $\phi : \hat{X} \to X$ such that $B := \phi(\hat{Q}) \simeq \mathbf{P}^1$. Let $\hat{L} := \hat{f}^*(O_{\mathbf{Q}^3}(1))$ and $L := \phi_*(\hat{L} + \hat{Q})$. We easily check that

$$\text{Pic } X \simeq \mathbf{Z}L, \quad L^3 = 3, \quad \kappa(L) = 3, \quad K_X = -2L.$$

Let $\Lambda := \phi_* \hat{f}^* |O_{\mathbf{Q}^3}(1)|$. Let $H$ be a general smooth hyperplane section of $\mathbf{Q}^3$, $\hat{D} := \hat{f}^{-1}(H)$ and $D := \phi(\hat{D})$. Then $\hat{D}$ is the blowing-up of $H$ with center $H \cap \Delta$, which consists of 6 distinct points. Hence $\hat{D}$ is smooth, while so is $D$. We easily check that $B = \text{Bs } \Lambda \simeq \mathbf{P}^1$.

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Moreover for a pair of general members $D$ and $D'$ of $\Lambda$, we see that $\ell := D \cap D' = B + C$ is a cycle of smooth rational curves. Moreover

$$LC = 4, \quad LB = -1, \quad \Lambda = |L| \simeq |\hat{L}| \simeq |O_{\mathbf{Q}^3}(1)|.$$

We denote $X$ by $X(\Delta, \mathbf{Q}^3)$ and call it a fake cubic 3-fold, which is uniquely determined by $(\Delta, O_\Delta(1))$. This construction gives no Moishezon 3-folds with $K_X = -2L$ and $b_2(X) = 1$ when $\Delta$ lies on a singular quadric surface by (3.2).

It is easy to see the following

**Lemma 2.2.** *Let $\Delta$ be an irreducible Gorenstein curve on a smooth quadric surface $Q$ ($\subset \mathbf{P}^3 \subset \mathbf{P}^4$) with $\deg \Delta = 5$ and $\deg \omega_\Delta = 2$, any of whose local irreducible components are smooth. Let $r = \frac{1}{2}(\deg \omega_\Delta - \deg \omega_{\tilde{\Delta}})$ for the normalization $\tilde{\Delta}$ of $\Delta$, $X := X(\Delta, \mathbf{Q}^3)$. Then $0 \le r \le 2$ and any homology group $H_q(X, \mathbf{Z})$ of $X$ is torsion free, $b_q = 1$ ($q = 0, 2, 4, 6$), $b_1 = b_5 = 0$, while $b_3 = 4 - 2r$ ($\le 4$) and the Euler number of $X$ is given by $e(X) = 2r(\le 4)$.*

**Corollary 2.3.** *In the notation in (2.2), $X(\Delta, \mathbf{Q}^3)$ is diffeomorphic to a cubic 3-fold with the same integral homology groups.*

*Proof.* This follows from Lemmas 1.5-1.6 and 2.2.  q.e.d.

**Remark 2.4.** It is possible to construct similar examples when the curve $\Delta$ is on a singular hyperplane section. We also call this a fake cubic 3-fold.

By choosing $\Delta \in |2e_1 + (6 - m)e_2|$ with mild singularities, we can also produce examples of Moishezon 3-folds $X$ with Pic $X = \mathbf{Z}L$, $K_X = -2L$ and $L^3 = m$.

To be more precise, let $Q$ be a smooth hyperpplane section of $\mathbf{Q}^3$, and $\Delta \in |2e_1 + (6 - m)e_2|_Q$ ($-\infty < m \leq 3$) an irreducible curve on $Q$ whose any local irreducible components are smooth. Let $\hat{f} : \hat{X} \to \mathbf{Q}^3$ be a small resolution of the blowing up $B_\Delta(\mathbf{Q}^3)$ of $\mathbf{Q}^3$ with $\Delta$ center, $\hat{\Delta} := \hat{f}^{-1}(\Delta)$, and $\hat{B}$ the proper transform of $Q$. Then we see $N_{\hat{B}/\hat{X}} \simeq O_{\hat{B}}(-e_1 - (5 - m)e_2)$ so that we have a Moishezon 3-fold $X$ and a contraction morphism $\phi : \hat{X} \to X$ such that $B := \phi(\hat{B}) \simeq \mathbf{P}^1$ and $\hat{X} \setminus \hat{B} \simeq X \setminus B$. Letting $\hat{L} = \hat{f}^*(O_{\mathbf{Q}^3}(1))$ and $L := \phi_*(L)$, we have

$$\text{Pix } X = \mathbf{Z}L, \quad K_X = -2L, \quad L^3 = m \ (\leq 3),$$

$$|L| \simeq |\hat{L}| \simeq |O_{\mathbf{Q}^3}(1)|, \quad \text{Bs } |L| = B.$$

We denote $X$ by $X(\Delta, \mathbf{Q}^3)$. We also note that if $m \geq 6$, then there is no irreducible $\Delta$. If $m = 4$ or 5, then Bs $|L|$ is empty and $X$ is either a small resolution of a complete intersection of two quadric hypersurfaces in $\mathbf{P}^5$ ($m = 4$) or a 3-fold hyperplane section of $\text{Gr}(2, 5) \subset \mathbf{P}^9$ ($m = 5$). This is shown directly or by using a theorem of Kollár (3.1).

It is possible to construct similar examples when the curve $\Delta$ is on a singular hyperplane section. These yield compactifications of $\mathbf{C}^3$ by an irreducible divisor.

## § 3. Moishezon 3-folds with $b_2 = 1$ and $K_X = -2L$

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First we recall a theorem of Kollár.

**Theorem 3.1.** (Kollár [Ko2, 5.3.12]) *Let $X$ be a Moishezon 3-fold $X$ with $K_X = -2L$, $s := h^0(X, L) \geq 4$ and $b_2 = 1$. Let $\rho_L$ be the rational map associated with $|L|$. Then $(X, \rho_L)$ is one of the following;*

*(3.1.1)* $s = 4$, $\rho_L$ *is a birational rational map of $X$ onto $\mathbf{P}^3$,*

*(3.1.2)* $s = 4$, $X$ *is a double cover of $\mathbf{P}^3$ ramified along a quartic by the morphism $\rho_L$,*

*(3.1.3)* $s = 5$, $\rho_L$ *is a birational rational map of $X$ onto $\mathbf{Q}^3$,*

*(3.1.4)* $s = 5$, $\rho_L$ *is a birational morphism of $X$ onto a cubic hypersurface in $\mathbf{P}^4$,*

*(3.1.5)* $s = 6$, $\rho_L$ *is a birational morphism of $X$ onto a complete intersection of a pair of quadric hypersurfaces in $\mathbf{P}^5$,*

*(3.1.6)* $s = 7$, $\rho_L$ *is an isomorphism of $X$ onto a 3-fold hyperplane section of the Grassmanian $\mathrm{Gr}\,(2,5) \subset \mathbf{P}^9$, which we call $V_5$.*

We shall nearly settle (3.1.3) by proving (3.2) and (3.3).

**Theorem 3.2.** *Let $X$ be a Moishezon 3-fold with $b_2(X) = 1$, $L^3 = 3$, $K_X = -2L$ and $h^0(X, L) = 5$. If $\mathrm{Bs}\,|L|$ is nonempty, then $X$ is a fake cubic 3-fold. In particular, $X$ is homeomorphic to no smooth cubic hypersurface.*

**Theorem 3.3.** *Let $X$ be a Moishezon 3-fold with $b_2(X) = 1$, $L^3 = m$, $K_X = -2L$ and $h^0(X, L) = 5$. If $\mathrm{Bs}\,|L|$ is empty, then $m = 3$ and*

$X$ is a cubic 3-fold. If Bs $|L|$ is nonempty, then $-\infty < m \le 3$ and $X$ is isomorphic to $X(\Delta, \mathbf{Q}^3)$ resp. the blowing up-and-down or the blowing down of a $\mathbf{Q}$-factorialization of the blowing up $B_\Delta(\mathbf{Q}^3)$ of $\mathbf{Q}^3$ with center a curve $\Delta$ of $\mathbf{Q}^3$ ($\subset \mathbf{P}^4$) where $\Delta$ is an irreducible reduced curve contained in a smooth (resp. singular) hyperplane section of $\mathbf{Q}^3$ with $\deg \Delta = 8 - m$ and $\deg \omega_\Delta = 8 - 2m$.

See (2.4) (resp. (3.5)) for the detail when $\Delta$ is contained in a smooth (resp. singular) hyperplane section of $\mathbf{Q}^3$.

*Proof of (3.3).* If Bs $|L|$ is empty, then $m = 3$ and $X$ is a cubic 3-fold by (3.1) or [Ko2, 5.3.12]. Now we assume that $B := \mathrm{Bs}\,|L|$ is nonempty. Then we see $B \simeq \mathbf{P}^1$. Moreover we have $h^0(O_B(L)) = 0$, whence $LB \le -1$. This proves that $m = L^3 = LB + LC = LB + 4 \le 3$. Let $\hat{X}$ be the blowing-up of $X$ with $B$ center, $\hat{B}$ the total transform of $B$. The we have a morphism $f : \hat{X} \to \mathbf{Q}^3$. Let $\hat{L} := f^*(O_{\mathbf{Q}^3}(1))$.

There are two cases $\hat{B} \simeq \mathbf{F}_0$ or $\hat{B} \simeq \mathbf{F}_2$. In each case we have

**Lemma 3.4.** *If* $\hat{B} \simeq \mathbf{F}_0$, *then* $\hat{X}$ *is a small resolution of* $B_\Delta(\mathbf{Q}^3)$. *The curve* $\Delta$ *is an irreducible reduced curve contained in a smooth hyperplane section of* $\mathbf{Q}^3$ *with* $\deg \Delta = 8 - m$ *and* $\deg \omega_\Delta = 8 - 2m$, *any of whose local irreducible components is smooth.*

**Lemma 3.5.** *Assume* $\hat{B} \simeq \mathbf{F}_2$. *Let* $e_\infty$ *be a smooth rational curve on* $\mathbf{F}_2$ *with* $e_\infty^2 = -2$, $I_{e_\infty}$ *the defining ideal of* $e_\infty$ *in* $\hat{X}$, $I := I_{e_\infty}^{2-k} + I_{\hat{B}}$ ($m = 2k, 2k+1$) *and* $\phi_I : B_I(\hat{X}) \to \hat{X}$ *the blowing-up of* $\hat{X}$ *with*

the ideal $I$ center. Then we have a natural surjective morphism $g^{\mathrm{norm}} : B_I(\hat{X}) \to B_{\Delta}^{\mathrm{norm}}(\mathbf{Q}^3)$.

(3.5.1)  If $m = 2k$, then $m \leq 1$, and $g^{\mathrm{norm}}$ contracts $E := \phi_I^{-1}(e_{\infty}) \simeq \mathbf{F}_0$ into a smooth rational curve. There exists a contraction morphism $\mathrm{cont}_E : B_I(\hat{X}) \to \mathrm{Cont}_E B_I(\hat{X})$ of $B_I(\hat{X})$ onto a 3-dimensional normal Moishezon space $\mathrm{Cont}_E B_I(\hat{X})$ and a surjective morphism $h^{\mathrm{norm}} : \mathrm{Cont}_E B_I(\hat{X}) \to B_{\Delta}^{\mathrm{norm}}(\mathbf{Q}^3)$ such that $g^{\mathrm{norm}} = h^{\mathrm{norm}} \cdot \mathrm{cont}_E$. We have $\mathrm{cont}_E(E) \simeq \mathbf{P}^1$ and $\mathrm{Cont}_E B_I(\hat{X})$ is a $\mathbf{Q}$-factorialization of $B_{\Delta}^{\mathrm{norm}}(\mathbf{Q}^3)$ by $h^{\mathrm{norm}}$.

(3.5.2)  If $m = 2k + 1$, then $m \leq 1$ and $B_I(\hat{X})$ is a normal $\mathbf{Q}$-factorialization of $B_{\Delta}^{\mathrm{norm}}(\mathbf{Q}^3)$ by $g^{\mathrm{norm}}$.

## § 4.  $V_5$ and cubic 3-folds

**Theorem 4.1.**  (Kollár) Let $X$ be a Moishezon 3-fold homeomorphic to a $V_5$. Then it is isomorphic to a $V_5$.

**Lemma 4.2.**  Let $W$ be a normal cubic hypersurface with isolated singularities which admits a small resolution $f : X \to W$. Let $H$ be a general hyperplane section of $W$, and $L := f^*(H)$. Let $s :=$ the number of singular points of $W$, and $e(X)$ the Euler number of $X$. Then $X$ is a Moishezon 3-fold with $K_X = -2L$, $h^q(X, O_X) = 0$ $(q \geq 1)$. Moreover there exists a coherent sheaf $\mathcal{G}$ on $X$ with $\mathrm{supp}\,\mathcal{G}$ contained in the exceptional set of $f$ such that $\frac{1}{2}e(X) + 3 = h^0(X, \mathcal{G}) \geq s$.

*Proof.*  Since the singularities of $W$ are rational by [E], $h^q(O_X) =$

$h^q(O_W) = 0$ for $q \geq 1$ and $\chi(O_X) = 1$. Let $T_X$ be the holomorphic tangent bundle of $X$. Then we have exact sequences,

$$0 \longrightarrow O_X \longrightarrow O_X(L)^{\oplus 5} \longrightarrow f^*T_{\mathbf{P}^4} \longrightarrow 0$$

$$0 \longrightarrow T_X \longrightarrow f^*T_{\mathbf{P}^4} \longrightarrow O_X(3L)$$

$$0 \longrightarrow T_X \longrightarrow f^*T_{\mathbf{P}^4} \longrightarrow \mathcal{F} \longrightarrow 0$$

$$0 \longrightarrow \mathcal{F} \longrightarrow O_X(3L) \longrightarrow \mathcal{G} \longrightarrow 0$$

where $\mathcal{F}$ and $\mathcal{G}$ are the cokernels of the sequences, while the second sequence is the normal sequence over the smooth locus of $W$. Note that $h^q(X, mL) = h^q(W, mH) = 0$ for any $q \geq 1$ and $m \geq 0$ because $f_*(mL) \simeq mH$ and $R^q f_*(mL) = 0$ for $q \geq 1$. From the above sequences we infer

$$h^q(X, f^*T_{\mathbf{P}^4}) = 0 \ (q \geq 1), \quad h^3(X, T_X) = h^0(X, \Omega^1_X(-2L)) = 0$$
$$h^q(X, \mathcal{G}) = h^{q+1}(X, \mathcal{F}) = h^{q+2}(X, T_X) = 0 \ (q \geq 1)$$

It follows that

$$
\begin{aligned}
h^0(X, \mathcal{G}) &= \chi(X, \mathcal{G}) \\
&= \chi(X, 3L) - \chi(X, f^*T_{\mathbf{P}^4}) + \chi(X, T_X) \\
&= \chi(X, 3L) - 5\chi(X, L) + \chi(X, O_X) + \chi(X, T_X) \\
&= 34 - 25 + 1 + (12c_1^3 - 19c_1 c_2 + 12c_3)/24 \\
&= e(X)/2 + 3
\end{aligned}
$$

where $c_1 c_2 = 24\chi(X, O_X) = 24$ and $c_1 := c_1(X) = f^* c_1(W) = 2c_1(L)$ because the resolution $f : X \to W$ is small. Let $F := F(x_0, \cdots, x_4)$ be the defining equation of $W$, $I$ the Jacobian ideal of $F$ (generated by $\partial F/\partial x_i$). Let $J$ be the ideal of $O_X$ generated by $f^*(\partial F/\partial x_i)$. Then by the definition of the above sequences we have $\mathcal{G} \simeq (O_X/J) \otimes O_X(3L)$. Since the singularities of $W$ are isolated, $(O_X/J) \otimes O_X(3L) \simeq O_X/J$. Therefore $h^0(X, \mathcal{G}) \geq s$, where $s :=$ the number of singular points of $W$. This completes the proof of (4.2).

$$\text{q.e.d.}$$

**Proposition 4.3.** *Let $W$ be a normal cubic hypersurface in $\mathbf{P}^4$ which admits a small resolution $f : X \to W$, and $H$ a general hyperplane section of $W$. Then $H_1(X, \mathbf{Z}) = 0$, and the inclusion homomorphism $i_* : H_2(H, \mathbf{Z}) \to IH_2(W, \mathbf{Z}) \simeq H_2(X, \mathbf{Z})$ is surjective where $IH_2(W, \mathbf{Z})$ is the second intersection homology of $W$ for any perversity. In particular, $b_2(X) \leq 7$ and $-6 \leq e(X) \leq 16$.*

*Proof.* Let $W$ be a normal cubic hypersurface which admits a small resolution $X$. By [GM, 6.2], $IH_2(W, \mathbf{Z}) \simeq H_2(X, \mathbf{Z})$ by the smoothness of $X$. The first assertion of (4.2) follows from [GM, 7.1]. Similarly $b_2(X) \leq b_2(H) \leq 7$. Because $H$ is a smooth cubic surface so that it is $\mathbf{P}^2$ blown-up at 6 points. We also see $e(X) = 2 + 2b_2 - b_3 \leq 16$, we have $e(X) \geq -6$ by (4.2). $\qquad$ q.e.d.

**Corollary 4.4.** *Let $W$ be a normal cubic hypersurface in $\mathbf{P}^4$ with at worst ordinary double points. Then the number of ordinary double points on $W$ is at most 10.*

*Proof.* We have $s \leq 10$ by (1.9), though we have only $s \leq 11$ by (4.2) and (4.3). q.e.d.

**Theorem 4.5.** *Let $X$ be a Moishezon 3-fold homeomorphic to a cubic 3-fold with $b_2 = 1$ and $6 \leq b_3 \leq 10$. If $c_1^3$ is positive or if $X$ has no holomorphic 3-forms, then $X$ is isomorphic to a cubic 3-fold. In particular if $X$ is homeomorphic to a smooth cubic hypersurface in $\mathbf{P}^4$ and if $c_1^3 > 0$, then $X$ is isomorphic to a smooth cubic hypersurface in $\mathbf{P}^4$.*

*Proof.* Let $X$ be a Moishezon 3-fold homeomorphic to a smooth cubic hypersurface. Then $b_1 = 0$, $b_2 = 1$ and $b_3 = 10$. Hence $h^1(X, O_X) = h^2(X, O_X) = 0$ and $h^3(X, O_X) \leq b_3/2 = 5$ so that Pic $X \simeq H^2(X, \mathbf{Z}) \simeq \mathbf{Z}$. Let $L$ be a generator of Pic $X$ with $L^3 = 3$. Then $K_X = -(2q + 2)L$ for some integer $q$ because $c_1 \mod 2$ is a topological invariant. Then as in [M] we have

$$-4 \leq \chi(X, O_X) = \chi(Z, O_Z(q)) = (q + 1)(q^2 + 2q + 2)/2,$$

where $Z$ is a smooth cubic hypersurface in $\mathbf{P}^4$ and $O_Z(1)$ is the hyperplane bundle. Hence $(q, h^3(X, O_X)) = (0, 0)$, $(-1, 1)$, $(-2, 2)$. When $q$ is negative, we have $c_1^3 = 3(2q + 2)^3 \leq 0$. By either of the assumptions we have $q = 0$, $h^3(X, O_X) = 0$ and $K_X = -2L$. Then by [Ko2, 5.3.12] $h^0(X, L) = 5$. By (3.2), $B := \mathrm{Bs}\,|L|$ is empty. Therefore the half anti-canonical map $\rho_L$ of $X$ is defined everywhere and $X$ is a cubic 3-fold because $b_2 = 1$.

Let $W$ be the cubic hypersurface which is the image of $X$ by $\rho_L$. Then the singularities of $W$ are isolated, hence normal. Since $X$ is a

small resolution of $W$, the singularities of $W$ are terminal so that they are rational [E]. Hence by (4.2), we have $5 - b_3/2 = e(X)/2 + 3 \geq s$, where $s :=$ the number of isolated singular points on $W$. It follows from $b_3 = 10$ that $W$ is smooth and $X \simeq W$.

Next we consider the case where $X$ is homeomorphic to a cubic 3-fold with $6 \leq b_3 \leq 8$. We see in the same manner as above that $K_X = -2L$ and $h^0(X, L) = 5$ for the generator $L$ of Pic $X$ with $L^3 = 3$ by [Ko2]. Let $B := \mathrm{Bs}\,|L|$. If $B$ is empty, then $X$ is a cubic 3-fold by $b_2 = 1$, while if $B$ is nonempty, then $X \simeq X(\Delta, \mathbf{Q}^3)$ by (4.2), which contradicts $b_3(X(\Delta, \mathbf{Q}^3)) \leq 4$ by (2.2).          q.e.d.

**Theorem 4.6.**   *Let $X$ be a Moishezon 3-fold homeomorphic to a cubic 3-fold with $b_2 = 1$, then $X$ is isomorphic to either a cubic 3-fold or a fake cubic 3-fold if one of the following conditions is satisfied.*

*(4.6.1)   $2 \leq b_3 \leq 4$, either $c_1^3$ is positive or  $h^{3,0}(X) = 0$,*

*(4.6.2)   $b_3 = 0$.*

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