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ON ELLIPTIC FIBRATIONS AND HYPER-KÄHLER STRUCTURES

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§0. Introduction

In this paper we shall discuss the class of Ricci-flat manifolds called hyper-Kähler manifolds. A hyper-Kähler manifold is, by definition, a Riemannian manifold equipped with three complex structures I, J, K satisfying the quaternionic relations, with respect to all of which the metric is Kählerian. The standard example of hyper-Kähler manifold is the vector space over the quaternion. The well known compact hyper-Kähler manifolds are $K3$ surfaces and even dimensional complex tori. A class of hyper-Kähler manifolds arises as moduli spaces of certain geometric structures, such as the moduli spaces of instantons on S^4, monopoles on $\mathbb{R}^3$ or Higgs bundles on Riemannian surfaces.

Eguchi-Hanson firstly discovered an interesting example of noncompact complete hyper-Kähler manifold which is diffeomorphic to the holomorphic cotangent bundle of $\mathbb{C}P^1$. Its $4m$ dimensional generalization was obtained by Calabi. He showed that the holomorphic cotangent bundle of $\mathbb{C}P^m$ has a hyper-Kähler structure. Gibbons-Hawking constructed hyper-Kähler structures on all minimal resolutions of rational double points of type A_k. They are called hyper-Kähler 4 manifolds of type A_k. Kronheimer-Nakajima constructed $4m$ dimensional noncompact complete hyper-Kähler manifolds which generalize hyper-Kähler 4 manifolds of type A_k. In [G-1], the author has studied geometrical and topological properties of these hyper-Kähler $4m$ manifolds of type A_k.

The first purpose of this paper is to construct new families of noncompact complte hyper-Kähler 4 manifolds. In [G-2], we shall give the $4m$ dimensional generalization. The following table (i) will show the significance of our new manifolds.

Hyper-Kähler manifolds of type A

	4 dim	4m dim ($m > 1$)
A_1	$T^*\mathbb{C}P^1$ (by Eguchi-Hanson)	$T^*\mathbb{C}P^m$ (by Calabi)
A_k	Hyper-Kähler 4 manifolds of type A_k (by Gibbons-Hawking)	4m dimensional hyper-Kähler manifolds of type A_k (in [KN], [G-1])
A_∞	Hyper-Kähler manifolds of type A_∞ (constructed in this paper)	4m dimensional hyper-Kähler manifolds of type A_∞ (constructed in [G-2])
A_∞^+	Hyper-Kähler manifolds of type A_∞^+ (in [AKL])	4m dimensional hyper-Kähler manifolds of type A_∞^+ (constructed in [G-2])

Table (i)

As shown in Table (i), we obtain two families of type A_∞ and of type A_∞^+. In 4 dimensional case, hyper-Kähler manifolds of type A_∞^+ were constructed by Anderson-Kronheimer-Lebrun [AKL]. But our construction is different from their construction. Our construction of hyper-Kähler 4-manifolds of type A_∞ is well understood by the comparison with Kronheimer's construction of hyper-Kähler 4-manifolds of type A_k. This comparison will be well explained by the following table (ii).

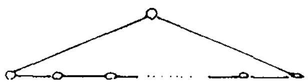
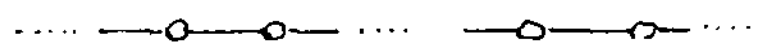
Construction of hyper-Kähler 4 manifolds of type A_k by Kronheimer	Construction of hyper-Kähler 4 manifolds of type A_∞ in this paper
cyclic group Γ_k of order $k+1$ in $SU(2)$	maximal torusS^1 in SU(2)
extended Dynkin diagram of type A_k 	extended Dynkin diagram of type A_∞ 
regular representation R of Γ_k	regular representation $L^2(S^1)$ of S^1
module over $\mathbb{H}$ $M = (\mathrm{End}(R) \otimes_{\mathbb{C}} \mathbb{H})^{\Gamma_k}$	Hilbert manifold $\hat{M} \subset (\mathrm{Hom}(L^2(S^1)) \otimes_{\mathbb{C}} \mathbb{H})^{S^1}$
Lie group $G = U(R)^{\Gamma_k}$	Hilbert Lie group $G \subset U(L^2(S^1))^{S^1}$
hyper-Kähler moment map μ	map μ given by (0-1)
The hyper-Kähler quotient $\mu^{-1}(\zeta)/G$	Our new quotient $\mu^{-1}(\zeta)/G$

Table (ii)

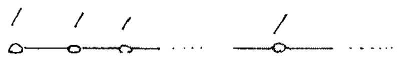
Let us explain Kronheimer's construction briefly. His construction has relied on the hyper-Kähler quotient construction. He has used the regular representation R of Γ_k to construct the $\mathbb{H}$ module $M = (\mathrm{Hom}(R) \otimes_{\mathbb{C}} \mathbb{H})^{\Gamma_k}$ and the Lie group $G = U(R)^{\Gamma_k}$, where M is a set of Γ_k-invariant elements of $\mathrm{Hom}(R) \otimes_{\mathbb{C}} \mathbb{H}$ and G is the set of Γ_k-invariant unitary map of R. The Lie group G acts on the $\mathbb{H}$-module M preserving its hyper-Kähler structure. Then we have the hyper-Kähler moment map $\mu\colon M \to \mathrm{Im}\mathbb{H} \otimes_{\mathbb{R}} \mathfrak{g}^*$, where $\mathrm{Im}\mathbb{H}$ is the imaginary part of the quaternion and $\mathfrak{g}^*$ the dual space of the Lie algebra of G. For generic $\zeta \in \mathrm{Im}\mathbb{H} \otimes \mathfrak{g}^*$, we obtain a hyper-Kähler manifold $\mu^{-1}(\zeta)/G$. Then Kronheimer has showed that this hyper-Kähler manifold $\mu^{-1}(\zeta)/G$ coincides with hyper-Kähler manifold of type A_k. Moreover he has showed that this construction can be described in terms of the extended Dynkin diagram. (Note that Kronheimer has constructed hyper-Kähler metrics on all minimal resolutions of rational double points.)

In our case of hyper-Kähler manifolds of type A_∞, we shall use the regular representation $L^2(S^1)$ of the maximal torus S^1 of $SU(2)$ to construct the HIlbert manifold $\hat{M}$ and the Hilbert Lie group $\hat{G}$. The HIlbert manifold $\hat{M}$ is a subset of $(\mathrm{Hom}(L^2(S^1)) \otimes_{\mathbb{C}} \mathbb{H})^{S^1}$ of S^1-invariant elements of $\mathrm{Hom}(L^2(S^1)) \otimes_{\mathbb{C}} \mathbb{H}$, where

$\mathrm{Hom}(L^2(S^1))$ is the set of operators of $L^2(S^1)$ whose domains are dense. The HIlbert Lie group $\hat{G}$ is a subgroup of S^1-invariant unitary operators $U(L^2(S^1))^{S^1}$. An element of $\hat{M}$ can be regarded as a $\mathbb{H}$ valued operator $\alpha + \beta j$, where α, β are operators of $L^2(S^1)$. We shall define the map $\hat{\mu}$ from $\hat{M}$ to $\mathrm{Im}\mathbb{H} \otimes \mathfrak{g}^*$ by

$$\text{(0-1)} \qquad \begin{cases} \mu_I(\alpha + \beta j) &= ([\alpha, \quad \alpha^*] + [\beta, \quad \beta^*]), \\ \mu_{\mathbb{C}}(\alpha + \beta j) &= -2\sqrt{-1} \quad [\alpha, \quad \beta]. \end{cases}$$

where $\mu = \mu_I + \mu_{\mathbb{C}} j$. Then a hyper-Kähler manifold of type A_∞ will be constructed as a quotient space $\mu^{-1}(\zeta)/G$ for generic $\zeta \in \mathrm{Im}\mathbb{H} \otimes \mathfrak{g}^*$. Moreover we shall show that our construction can be described in terms of the extended Dynkin diagram of type A_∞ in table (**ii**). For the construction of hyper-Kähler manifolds of type A^+_∞, we shall use the representation space $L^{2+}(S^1)$ of L^2 functions on the circle S^1 whose negative Fourier coefficients vanish. Then we follow the same procedure as in type A_∞. This construction of type A^+_∞ corresponds to the following extended Dynkin diagram of type A^+_∞.



where each vertex has the weight number 1.

The second purpose of this paper is to discuss the relation between hyper-Kähler manifolds of type A_∞ and elliptic fibrations. Let X_ζ be the hyper-Käher manifold of type A_∞. When we consider a certain subfamily of hyper-Kähler manifolds of type A_∞, we can choose a special complex structure I and a holomorphic symplectic form $\omega_{\mathbb{C}}$ on each X_ζ in this subfamily. Then there exist three kind of actions on X_ζ. At first $\mathbb{C}^*$ acts on X_ζ preserving I and $\omega_{\mathbb{C}}$. We denote by Φ the holomorphic moment map on X_ζ for the action of $\mathbb{C}^*$. (We shall describe this moment map Φ explicitely.) Secondly there exits the holomorphic action of an additive group $\mathbb{Z}$ on X_ζ. Since the moment map Φ is invariant under the action of $\mathbb{Z}$, we have a map $\tilde{\Phi}$ from the quotient space $X_\zeta/b\mathbb{Z}$ to $\mathbb{C}$ for any positive integer b. Denote by $\triangle$ the disk $\{ \quad t \in \mathbb{C} \quad | \quad |t| < 1 \quad \}$. When we restrict the map $\tilde{\Phi}$ to the inverse image $\tilde{\Phi}^{-1}(\triangle)$, we have the map $\hat{\Phi} : \tilde{\Phi}^{-1}(\triangle) \to \triangle$. Then we obtain the following Theorem.

Main theorem. $\hat{\Phi} : \tilde{\Phi}^{-1}(\triangle) \to \triangle$ *is biholomorphic to the fibre space of elliptic curves of type* I_b.

$$\begin{array}{ccccc} X_\zeta & \longrightarrow & X_\zeta/b\mathbb{Z} & \hookleftarrow & \tilde{\Phi}^{-1}(\triangle) \\ & \searrow & \downarrow \tilde{\Phi} & & \downarrow \hat{\Phi} \\ & & \mathbb{C} & \hookleftarrow & \triangle \end{array}$$

Remark. All fibre space of elliptic curves over the disk were classified by a celebrated theorem of Kodira. Our notation is the same as in his paper [KK].

Colollary. *The hyper-Kähler 4 manifold of typeA_∞ is the universal cover of the fibre space of elliptic curves of type I_b.*

Finally we shall show that there exists the holomorphic involution σ on X_ζ. By the action of σ, we can obtain the fibre space of elliptic curves of type I_b^*. In the final theorem of this paper, we shall diccuss hyper-Kähler manifolds of type D_∞. We shall show that hyper-Kähler manifolds of type D_∞ can be constructed by the regular representation of the normalizer of S^1 in $Sp(1)$ as in type A_∞ [G-4].

§1. PRELIMINARY RESULTS

In this section, we shall give a brief review of the hyper-Kähler quotient construction and Kronheimer's result.

Definition 1-1. *A hyper-Kähler structure* on a Riemannian manifold (X, g) consists of three almost complex structures (I, J, K) which satisfy following conditions

(1)
$$g(u,v) = g(Iu, Iv) = g(Ju, Jv) = g(Ku, Kv).$$

(2)
$$I^2 = J^2 = K^2 = -1, \quad IJ = -JI = K \in \mathrm{End}(TX).$$

(3) Let ∇ be a Levi-Civita connection of (X, g). Then
$$\nabla I = \nabla J = \nabla K = 0.$$

A hyper-Kähler manifold is a Riemannian manifold with a hyper-Kähler structure. Especially the module over the quaternion is a standard example of a hyper-Kähler manifold. Next we define a hyper-Kähler moment map. Let (X, g, I, J, K) be a hyper-Kähler manifold. We assume that a Lie group G acts on X so as to preserve the hyper-Kähler structure of X. Each element $\xi \in \mathfrak{g}$ of the Lie algebra of G defines a vector field $\hat{\xi}$ on X by the action of G.

Definition 1-2. *A hyper-Kähler moment map* for the action of G on M is a map $\mu = i\mu_I + j\mu_J + k\mu_K \colon M \longrightarrow \mathrm{Im}\mathbb{H}\otimes\, \mathfrak{g}^*$ which satisfies

$$\mu_{I_\alpha}(gx) = \mathrm{Ad}_g^*(\mu_{I_\alpha})(x), \qquad x \in M, g \in G, \alpha = 1,2,3$$
$$\langle \xi, d\mu_{I_\alpha}\rangle = i(\hat{\xi})\omega_{I_\alpha}, \qquad \xi \in \mathfrak{g}, \alpha = 1,2,3$$

where $(I_1, I_2, I_3) = (I, J, K)$, $\mathfrak{g}^*$ the dual space of $\mathfrak{g}$, $\mathrm{Ad}_g^* \colon \mathfrak{g}^* \to \mathfrak{g}^*$ the coadjoint map, $\langle\ ,\ \rangle$ the dual pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$, and $i(\hat{\xi})$ the interior product.

Under the assumption that M is simply connected, a hyper-Kähler moment map always exists and is unique up to addition of a constant $\zeta \in \mathrm{Im}\mathbb{H} \otimes Z \subset \mathrm{Im}\mathbb{H} \otimes \mathfrak{g}^*$, where Z is the set of G-invariant elements of $\mathfrak{g}^*$. The set $\mu^{-1}(\zeta) \in M$ is invariant under G-action for any $\zeta \in \mathrm{Im}\mathbb{H} \otimes Z$. After choosing $\zeta \in \mathrm{Im}\mathbb{H} \otimes \mathfrak{g}^*$, one defines *the hyper-Kähler quotient* as
$$X_\zeta = \mu^{-1}(\zeta)/G.$$
We are now ready to state the hyper-Kähler quotient construction.

Fact 1-3[H-K-L-R]. *Suppose that G acts freely on $\mu^{-1}(\zeta)$, then the hyper-Kähler quotient $\mu^{-1}(\zeta)/G = X_\zeta$ is a hyper-Kähler manifold. Moreover if G is compact and M is complete, then X_ζ is a complete hyper-Kähler manifold.*

Kronheimer constructed the family of hyper-Kähler structures on minimal resolutions of rational double points. Let Γ be a finite subgroup of $SU(2)$. There is the natural action of Γ on the quaternion $\mathbb{H}$ by the identification $SU(2) \cong Sp(1)$. Denote by R the regular representation with the invariant Hermitian metric. Then $\mathrm{End}(R) \otimes_{\mathbb{C}} \mathbb{H}$ is regarded as a module over the quaternion on which Γ acts preserving its hyper-Kähler structure. Let $M := (\mathrm{End}(R) \otimes_{\mathbb{C}} \mathbb{H})^{\Gamma}$ be the set of invariant elements of $\mathrm{End}(R) \otimes_{\mathbb{C}} \mathbb{H}$ under the action of Γ. Denote by G the group of unitary maps of R which are invariant under the adjoint action of Γ. Then M is the module over the quaternion on which the compact Lie group G acts so as to preserve the hyper-Kähler structure.

When we apply hyper-Kähler quotient construction on M and G, we obtain the following Theorem by Kronheimer.

Theorem 1-4. *Let M, G be as before. Let μ be the hyper-Kähler moment map from M to $\mathrm{Im}\mathbb{H} \otimes \mathfrak{g}^*$. For generic $\zeta \in \mathrm{Im} \otimes Z$, the hyper-Kähler manifold $\mu^{-1}(\zeta)/G$ is diffeomorphic to the minimal resolution of $\mathbb{C}^2/\Gamma$.*

§2. Hyper-Kähler manifolds of type A_∞

Let V_n be a irreducible representation of the circle group S^1 which is generated by the function $e^{in\theta}$ on S^1. By the basis $e^{in\theta}$ each V_n may be regarded as the one dimensional complex vector space $\mathbb{C}$. Consider the following diagrm with all edges doubled up and assigned orientations both ways :

$$\cdots \underset{\beta_{n-2}}{\overset{\alpha_{n-2}}{\rightleftarrows}} V_{n-1} \underset{\beta_{n-1}}{\overset{\alpha_{n-1}}{\rightleftarrows}} V_n \underset{\beta_n}{\overset{\alpha_n}{\rightleftarrows}} V_{n+1} \underset{\beta_{n+1}}{\overset{\alpha_{n+1}}{\rightleftarrows}} \cdots$$

where each arrow implies a homomphism between irreducible representations. Denote by (α_n, β_n) an element of $\mathrm{Hom}(V_n, V_{n+1}) \oplus \mathrm{Hom}(V_{n+1}, V_n)$. Consider the infinite dimensional module

$$H := \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}(V_n, V_{n+1}) \oplus \mathrm{Hom}(V_{n+1}, V_n).$$

We define the Hilbert space M by

$$M := \left\{ \quad (\alpha_n, \beta_n)_{n\in\mathbb{Z}} \in H \quad \Big| \quad \sum_{n\in\mathbb{Z}} |\alpha_n|^2 + |\beta_n|^2 < \infty \quad \right\}$$

Since M is a vector space over $\mathbb{C}$, we have the almost complex structure I on M by the mutiplication of i. An almost complex structure J is defined by

$$J(\alpha_n, \beta_n)_{n\in\mathbb{Z}} := (\beta_n^*, -\alpha_n{}^*)_{n\in\mathbb{Z}}.$$

When we set $K = IJ$, then M has a hyper-Kähler structure. Define an element $\Lambda = (\Lambda_n)_{n\in\mathbb{Z}}$ by

$$\Lambda_n = \begin{cases} (ni, 0) & \text{if } n \geqq 0, \\ (0, ni) & \text{if } n < 0. \end{cases}$$

Definition 2-1. *We define the Hilbert manifold $\hat{M}$ by*

$$\hat{M} := \Lambda + M \subset H$$

An element of $\hat{M}$ may be written as $(\alpha_n, \beta_n)_{n\in\mathbb{Z}}$ where $(\alpha_n, \beta_n) = \Lambda_n + (x_n, y_n)$ for $(x_n, y_n)_{n\in\mathbb{Z}} \in M$. Define the Hilbert space $\mathfrak{g}$ by

$$\mathfrak{g} := \{ \quad (\xi_n)_{n\in\mathbb{Z}} \in \bigoplus_{n\in\mathbb{Z}} u(V_n) \quad | \quad \sum_{n\in\mathbb{Z}} (1+n^2)|\xi_n - \xi_{n+1}| < \infty, \quad \lim_{|n|\to\infty} \xi_n = 0 \quad \}.$$

Definition 2-2. *Define the Hilbert Lie group G by*

$$G := \left\{ \quad (e^{\xi_n})_{n\in\mathbb{Z}} \in \underset{n\in\mathbb{Z}}{\times} U(V_n) \quad \middle| \quad (\xi_n)_{n\in\mathbb{Z}} \in \mathfrak{g} \quad \right\}.$$

The Lie algebra of the Hilbert Lie group G is the Hilbert space $\mathfrak{g}$. Since G is the subgroup of $\underset{n\in\mathbb{Z}}{\times} U(V_n)$, we have the action of G on $H := \underset{n\in\mathbb{Z}}{\oplus} \mathrm{Hom}(V_n, V_{n+1}) \oplus \mathrm{Hom}(V_{n+1}, V_n)$. Then the Hilbert manifold $\hat{M}$ is invariant undrer the action of G. Hence we have the action of G on $\hat{M}$. It is clear that the G acts on $\hat{M}$ preserving the hyper-Kähler structure. Then there exists the hyper-Kähler moment map μ on M for the action of G.

$$\mu : M \longrightarrow \mathrm{Im}\mathbb{H} \otimes \mathfrak{g}^*.$$

The map μ is described by

$$\begin{aligned} < \mu^I(q), \xi^I > =& \sum_{n\in\mathbb{Z}} < ({\alpha_n}^*\alpha_n - \beta_n{\beta_n}^* - \alpha_{n-1}{\alpha_{n-1}}^* + {\beta_{n-1}}^*\beta_{n-1}), \quad \xi_n^I > \\ & - < C_n^I(\Lambda), \quad \xi_n^I > \\ < \mu^{\mathbb{C}}(q), \xi^{\mathbb{C}} > =& \sum_{n\in\mathbb{Z}} 2i < \beta_n\alpha_n - \alpha_{n-1}\beta_{n-1}, \quad \xi_n^{\mathbb{C}} > - < C_n^{\mathbb{C}}(\Lambda), \quad \xi_n^{\mathbb{C}} >, \end{aligned}$$

where $q = (\alpha_n, \beta_n)_{n\in\mathbb{C}} \in \hat{M}$, $\xi = (i\xi_n^I + j\xi_n^J + k\xi_n^K)_{n\in\mathbb{Z}} \in \mathrm{Im}\mathbb{H}\otimes\mathfrak{g}$, $\xi_n^{\mathbb{C}} = \xi_n^J + i\xi_n^K$ and $C_n(\Lambda) := C_n^I(\Lambda) + C_n(\Lambda)^{\mathbb{C}} j$ is a constant which does not depend on $q \in \hat{M}$. We consider the following element $\hat{e}^n \in \mathrm{Im}\mathbb{H} \otimes \mathfrak{g}$

$$\hat{e}_m^n = \begin{cases} i(i+j+k) & \text{if } m = n \\ 0 & \text{otherwise} \end{cases}$$

Definition 2-3. *An element $\zeta \in \mathrm{Im}\mathbb{H} \otimes \mathfrak{g}^*$ is said to be nondegenerate if $\sum_{i=n}^{m} \zeta_i \neq 0 \in \mathrm{Im}\mathbb{H}$ for all $n \in \mathbb{Z}, m \in \mathbb{Z}_{\geq 0}$, where $\zeta_i := < \zeta, \quad \hat{e}^n > - < C_n(\Lambda), \quad \hat{e}_n^n >$.*

Form the definition of μ, an inverse image $\mu^{-1}(\zeta)$ is invariant under the action of G. So we have the quotient space $X_\zeta := \mu^{-1}(\zeta)/G$ for $\zeta \in \mathrm{Im}\mathbb{H} \oplus \mathfrak{g}^*$.

Theorem 2-4. *The quotient space $X_\zeta := \mu^{-1}(\zeta)/G$ is a noncompact complete hyper-Kähler 4 manifold for any nondegenerate element $\zeta \in \mathrm{Im}\mathbb{H} \otimes \mathfrak{g}^*$.*

It is natural that X_ζ in Teorem 2-4 is called a hyper-Kähler manifold of type A_∞ by the following Theorem.

Theorem 2-5. *Let X_ζ be a hyper-Kähler manifold of type A_∞. Then there exist submanifolds $L_n \in X_\zeta, n \in \mathbb{Z}$ such that*

(1) *each L_n is homeomorphic to the sphere S^2*
(2) *the inclusion $\underset{n\in\mathbb{Z}}{\cup} L_n \subset X_\zeta$ is a deformation retract.*
(3) *each intersection number is given by*

$$L_{n_1} \cdot L_{n_2} = \begin{cases} -2 & \text{if } n_1 = n_2, \\ 1 & \text{if } |n_1 - n_2| = 1 \\ 0 & \text{otherwise}, \end{cases}$$

The intersection form of $\mathrm{H}(X_\zeta, \mathbb{Z})$ can be interpreted as (-1) times of Cartan matrix of type A_∞.

Sketch of a proof of Theorem 2-4. For any nondegenerate element ζ, we see that $\mu^{-1}(\zeta)$ is a submanifold of $\hat{M}$ by using an implicit functin theorem. A key point of a proof is an existence of a slice on $\mu^{-1}(\zeta)$ for the action of G. Denote by S_q a slice on $q \in \mu^{-1}(\zeta)$. Then we have an othogonal decomposition

$$T_q\mu^{-1}(\zeta) = T_qS_q + T_qG(q),$$

where $T_qG(q)$ is the tangent space of $G-$orbit through q. By the action of G on $\hat{M}$, each element $\xi \in \mathfrak{g}$ defines a vector field V_ξ on $\hat{M}$. So we define the map $d_q : \mathfrak{g} \to T_q\hat{M}$ by

$$d_q(\xi) := V_\xi(q) \in T_q\hat{M}.$$

Consider the following diagram

$$0 \longrightarrow \hat{\mathfrak{g}} \xrightarrow{d_q} T_q\hat{M} \xrightarrow{d\hat{\mu}_q} T_{\hat{\mu}(q)}\hat{N} \longrightarrow 0,$$

where $d\mu_q$ is the differencial of the map μ at q. Let d_q^* be the adjoint operator of d_q. Then the decomposition $T_q\hat{\mu}^{-1}(\zeta) = T_qS_q + T_q\hat{G}_q$ implies that

$$T_qS_q \cong \mathrm{Ker}d_q^* \cap \mathrm{Ker}d\mu_q.$$

We define the map $D_q := d_q^* + d\mu_q : T_q\hat{M} \to \mathfrak{g} \oplus (\mathrm{Im}\mathbb{H} \otimes \mathfrak{g}^*) \cong \mathbb{H} \otimes \mathfrak{g}^*$, where the image of the map d_q is in the real part of $\mathbb{H} \otimes \mathfrak{g}^*$ and $\mathfrak{g}$ is identified with $\mathfrak{g}^*$ by the metric. Then the map D_q satisfies the followings,

(1) $D_q : T_q\hat{M} \to \mathbb{H} \otimes \mathfrak{g}^*$ is a linear operator between Hilbert spaces over $\mathbb{H}$,
(2) D_q is a Fredholm operator whose index is equal to 4 for all $q \in \mu^{-1}(\zeta)$,
(3) $\mathrm{Coker}D_q = \{0\}$.

We see that

$$T_qS_q = \mathrm{Ker}D_q.$$

Hence $\dim_{\mathbb{R}} T_qS_q = \mathrm{ind}D_q = 4$. Since D_q is a linear operator over $\mathbb{H}$, $\mathrm{Ker}D_q$ is a vector space over $\mathbb{H}$. This implies that T_qS_q has a hyper-Kähler structure. Let π denote a natural projectin from $\mu^{-1}(\zeta)$ to $X_\zeta := \mu^{-1}(\zeta)/G$. Then each tangent space T_xX_ζ may be considered as T_qS_q for $\pi(q) = x$. Hence X_ζ has an almost hyper-Kähler structure. Finally we can prove that this almost hyper-Kähler structure defines a hyper-Kähler structure.

§3. Holomorphic descriptin of hyper-Kähler manifolds of type A_∞

We use the same notation as in section 2. In this sectin, we choose $\zeta \in i\mathfrak{g} \subset \mathrm{Im}\mathbb{H} \otimes \mathfrak{g}^*$, i.e., $\zeta^J, \zeta^K = 0$.

Proposition 3-1. *Let X_ζ be a hyper-Kähler manifolds of type A_∞ and let L_n be submanifolds of $X_\zeta, n \in \mathbb{Z}$. Then there exists a complex structure I on X_ζ such that each L_n is a complex submanifold with repect to I.*

This proposition implies that X_ζ has an infinite chain of rational curves. Since $\zeta^{\mathbb{C}} := \zeta^J + \zeta^K i = 0$, we have an inclusion $\mu^{-1}(\zeta) \hookrightarrow \mu_{\mathbb{C}}^{-1}(0)$, where $\mu_{\mathbb{C}} := \mu_J + \mu_K i$. By the explicit description of the hyper-Kähler moment map μ, we see that

$$\mu_{\mathbb{C}}^{-1}(0) = \left\{ \quad (\alpha_n, \beta_n)_{n\in\mathbb{Z}} \in \hat{M} \quad | \quad \beta_n\alpha_n = \alpha_{n-1}\beta_{n-1}, \quad \forall n \in \mathbb{Z} \quad \right\}.$$

We consider the following open subset $\mu_{\mathbb{C}}^{-1}(0)_+ \subset \mu_{\mathbb{C}}^{-1}(0)$,

$$\mu_{\mathbb{C}}^{-1}(0)_+ := \{ \quad (\alpha_n, \beta_n)_{n\in\mathbb{Z}} \in \mu_{\mathbb{C}}^{-1}(0) \quad | \quad |\alpha_n|^2 + |\beta_n|^2 \neq 0, \quad \forall n \in \mathbb{Z} \quad \}$$

For simplicity, we assume that $\zeta_i > 0$ for all $i \in \mathbb{Z}$. Then we have a natural inclusion

$$\mu^{-1}(\zeta) \hookrightarrow \mu_{\mathbb{C}}^{-1}(0)_+.$$

Let $G^{\mathbb{C}}$ denote the complexification of G. Then we have the map

$$\iota : X_\zeta \to \mu_{\mathbb{C}}^{-1}(0)_+/G^{\mathbb{C}}.$$

Theorem 3-2. *$\mu_{\mathbb{C}}^{-1}(0)_+/G$ is a complex surface with a holomorphic symplectic form $\tilde{\omega}_{\mathbb{C}}$. The map $\iota : X_\zeta \to \mu_{\mathbb{C}}^{-1}(0)_+/G^{\mathbb{C}}$ is biholomorphic with respect to the complex structure I on X_ζ. Moreover $\iota^*(\tilde{\omega}_{\mathbb{C}}) = \omega_{\mathbb{C}}$.*

§4. Elliptic fibrations and hyper-Kähler manifolds of type A_∞ and D_∞

Let (X_ζ, I) be a pair of a hyper-Kähler manifolds of type A_∞ and a complex structure in Proposition 3-1. Then we have three kind of actions on X_ζ.

(1) the action of $\mathbb{C}^*$,
(2) the action of an additive group $\mathbb{Z}$,
(3) the involution σ.

At first we shall define the action of $\mathbb{C}^*$ on X_ζ. Choose an element $(\alpha_n, \beta_n)_{n\in\mathbb{Z}}$ of $\mu_{\mathbb{C}}^{-1}(0)_+$. Denote by $[\alpha_n, \beta_n]_{n\in\mathbb{Z}}$ the equivalent class of $(\alpha_n, \beta_n)_{n\in\mathbb{Z}}$ in X_ζ. Then the action of $\mathbb{C}^*$ is defined by

$$\phi : \mathbb{C}^* \times X_\zeta \longrightarrow X_\zeta$$

$$\phi(\lambda, [\alpha_n, \beta_n]_{n\in\mathbb{Z}})_n := \begin{cases} (\lambda\alpha_0, \lambda^{-1}\beta_0) & \text{if } n = 0 \\ (\alpha_n, \beta_n) & \text{if } n \neq 0. \end{cases}$$

It is clear that this definition is well defined. Since the action of $\mathbb{C}^*$ on X_ζ is preserving the holomorphic symplectic form $\omega_{\mathbb{C}}$, we have the holomorphic moment map Φ on X_ζ for the action of $\mathbb{C}^*$.

$$\Phi : X_\zeta \longrightarrow \mathbb{C},$$

where $\mathbb{C}$ is considered as the dual space of the Lie algebra of $\mathbb{C}$. The map Φ is explicitly described by

$$\Phi([\alpha_n, \beta]_{n\in\mathbb{Z}}) = \alpha_0\beta_0.$$

Secondly we shall define the action of $\mathbb{Z}$ on X_ζ. Consider the following map

$$f : \mu_{\mathbb{C}}^{-1}(0)_+ \to V$$

$$f((\alpha_n, \beta_n)_{n\in\mathbb{Z}}) := \begin{cases} (-in^{-1}\alpha_n, ni\beta_n) & \text{if } n > 0, \\ (\alpha_0, \beta_0) & \text{if } n = 0, \\ (ni\alpha_n, -in^{-1}\beta_n) & \text{if } n < 0, \end{cases}$$

where $V = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}(V_n, V_{n+1}) \oplus \mathrm{Hom}(V_{n+1}, V_n)$. Let Y denote the image of the map f. Denote by $\hat{X}_\zeta$ the quotient space $Y/G^{\mathbb{C}}$. Then X_ζ may be considered as $\hat{X}_\zeta$.

$$X_\zeta \cong \hat{X}_\zeta.$$

We define the map ψ by

$$\psi : \hat{X}_\zeta \longrightarrow \hat{X}_\zeta,$$

$$\psi([\hat{\alpha}_n, \hat{\beta}_n]_{n\in\mathbb{Z}}) := [\hat{\alpha}_{n-1}, \hat{\beta}_{n-1}]_{n\in\mathbb{Z}}.$$

Notice that $(\hat{\alpha}_{n-1}, \hat{\beta}_{n-1})_{n\in\mathbb{Z}}$ is an element of Y. The map ψ is well explained by the following diagram:

$$\cdots \underset{\beta_{n-2}}{\overset{\alpha_{n-2}}{\rightleftarrows}} V_{n-1} \underset{\beta_{n-1}}{\overset{\alpha_{n-1}}{\rightleftarrows}} V_n \underset{\beta_{n}}{\overset{\alpha_{n}}{\rightleftarrows}} V_{n+1} \underset{\beta_{n+1}}{\overset{\alpha_{n+1}}{\rightleftarrows}} \cdots$$

$$\Downarrow \psi$$

$$\cdots \underset{\beta_{n-3}}{\overset{\alpha_{n-3}}{\rightleftarrows}} V_{n-1} \underset{\beta_{n-2}}{\overset{\alpha_{n-2}}{\rightleftarrows}} V_n \underset{\beta_{n-1}}{\overset{\alpha_{n-1}}{\rightleftarrows}} V_{n+1} \underset{\beta_{n}}{\overset{\alpha_{n}}{\rightleftarrows}} \cdots$$

The action of $\mathbb{Z}$ on $\hat{X}_\zeta$ is defined by

$$n \longrightarrow \psi^n \in \mathrm{Aut}(\hat{X}_\zeta),$$

where $n \in \mathbb{Z}$. By the identificaton $X_\zeta \cong \hat{X}_\zeta$, we have the action of $\mathbb{C}^*$ on X_ζ. Note that $\hat{\beta}_n\hat{\alpha}_n = \hat{\alpha}_{n-1}\hat{\beta}_{n-1}$ for all $n \in \mathbb{Z}$ where $(\hat{\alpha}_n, \hat{\beta}_n)_{n\in\mathbb{Z}} \in Y$ This implies that the holomorphic moment map Φ is invriant under the action of $\mathbb{Z}$. Hence we have the map $\tilde{\Phi} : X_\zeta/b\mathbb{Z} \to \mathbb{C}$ for any positive integer b.

$$\tilde{\Phi} : X_\zeta \longrightarrow X_\zeta/b\mathbb{Z}$$

Set $\triangle := \{ \ t \in \mathbb{C} \ \mid \ |t| < 1 \ \}$. Then we shall show that $\tilde{\Phi}^{-1}(\triangle) \to \triangle$ is biholomorphic to the fibre space of elliptic curves of type I_b.

Proof of Main theorem. When $t = \hat{\alpha}_0\hat{\beta}_0 \neq 0$, we can define an invariant function X by

$$x := (\prod_{n\geq 0} \hat{\alpha_n})(\prod_{n<0} \hat{\beta}_n)^{-1},$$

where each infinite product converges absolutely. By a simple calculation, we see that (t, x) defines a local coordinate aroung a generic fibre $\Phi^{-1}(t)$ of X_ζ for $t \neq 0$. Moreover any general fibre is written as $\Phi^{-1}(t) \cong \mathbb{C}^* = \{ \ (t, x) \ \mid \ x \in \mathbb{C}^* \ \}$. The action of $\mathbb{Z}$ can be described as the following,

$$\psi((t, x)) = (t, t^{-1}x).$$

This implies that each general fibre $\Phi^{-1}(t)$ is an elliptic curve $\mathbb{C}/(\mathbb{Z}\sqrt{-1} + \mathbb{Z}\log t)$. In order to determine a special fibre $\Phi^{-1}(0)$, we describe the infinite chain of rational curves in Theorem 3-1. Define $\hat{L}_n$ by

$$\hat{L}_n := \left\{ \ [\hat{\alpha}_n, \hat{\beta}_n]_{n\in\mathbb{Z}} \in \hat{X}_\zeta \ \mid \ \hat{\alpha}_i = 0 (i < n), \quad \hat{\beta}_i = 0 (i \geq 0) \ \right\}.$$

Each $\hat{L}_n$ is well explained by the following diagram:

$$\cdots \underset{\beta_{n-2}}{\longleftarrow} V_{n-1} \underset{\beta_{n-1}}{\longleftarrow} V_n \overset{\alpha_n}{\longrightarrow} V_{n+1} \overset{\alpha_{n+1}}{\longrightarrow} \cdots$$

Then we have a holomorphic map $\hat{L}_n \to \mathbb{C}P^1$ by

$$[\hat{\alpha}_n, \hat{\beta}_n]_{n\in\mathbb{Z}} \longrightarrow [\prod_{i\geq n} \hat{\alpha}_i, \quad \prod_{i<n} \hat{\beta}_i] \in \mathbb{C}P^1.$$

We can see that this map is bijective. Hence the infinite chain of rational curves may be considered as $\hat{L}_n, n \in \mathbb{Z}$. By definition of ψ, we have

$$\psi(\hat{L}_n) = \hat{L}_{n+1}.$$

Moreover we see that $\Phi^{-1}(0) = \bigcup_{n\in\mathbf{Z}} \hat{L}_n$. This implies that a special fibre $\tilde{\Phi}^{-1}(0)$ is a circle of rational curves. Hence we can conclude that $\tilde{\Phi}^{-1}(\Delta) \to \Delta$ is biholomorphic to the fibre space of elliptic curves of type I_b. □

Finally we shall define an involution σ on X_ζ. Consider the following map $\tilde{\sigma} : V \to V$ defined by,

$$\tilde{\sigma}((\alpha_n, \beta_n)_{n\in\mathbf{Z}})_n := \begin{cases} (-\beta_{-n-1}, \alpha_{-n-1}) & \text{if } n > 0, \\ (\beta_{-n-1}, -\alpha_{-n-1}) & \text{if } n \leq 0. \end{cases}$$

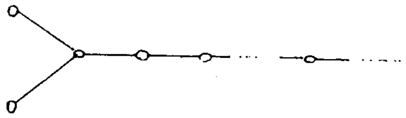
This map $\tilde{\sigma}$ is well understood by the following diagram:

$$\begin{array}{c} \cdots \underset{\beta_{n-2}}{\overset{\alpha_{n-2}}{\rightleftarrows}} V_{-1} \underset{\beta_{n-1}}{\overset{\alpha_{n-1}}{\rightleftarrows}} V_0 \underset{\beta_n}{\overset{\alpha_n}{\rightleftarrows}} V_1 \underset{\beta_{n+1}}{\overset{\alpha_{n+1}}{\rightleftarrows}} \cdots \\ \Downarrow \tilde{\sigma} \\ \cdots \underset{-\alpha_1}{\overset{\beta 1}{\rightleftarrows}} V_{-1} \underset{-\alpha_0}{\overset{\beta_0}{\rightleftarrows}} V_0 \underset{\alpha_{-1}}{\overset{-\beta_{-1}}{\rightleftarrows}} V_1 \underset{\alpha_{-2}}{\overset{-\beta_{-2}}{\rightleftarrows}} \cdots \end{array}$$

It is clear that $\tilde{\sigma}$ defines an involution σ on X_ζ. By using this involution σ, we obtain the fibre space of elliptic curves of type I_b^*. It must be noted that σ is a hyper-Kähler isometry. Hence we have a hyper-Kähler orbifold X_ζ/σ. This orbifold X_ζ/σ has two rational double points of type A_1.

Theorem 4-1. **(hyper-Kähler manifolds of type D_∞).** *Let $\tilde{X}$ be a minimal resolution of X_ζ/σ. Then $\tilde{X}$ has a complete hyper-Kähler structure.*

Remark 4-2. A hyper-Kähker manifold $\tilde{X}$ has a infinite sequence of rational curves. The dual graph of these rational curves coinsides with the following Dynkin diagram of type D_∞:



Remark 4-3. Let D_∞ denote by the normalizer of the maximal torus of $Sp(1)$. When we consider the Hilbert space $L^2(D_\infty)$, we can construct a family of hyper-Kähler manifolds by the hyper-Kähler quotient method. Then we can see that X_ζ/σ and $\tilde{X}$ can be obtained as these hyper-Kähler quotient spaces which correspond to $L^2(D_\infty)$.

Hence it is natural that $\tilde{X}$ of Theorem 4.1 is considered as the hyper-Kähler manifold of type D_∞.

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