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RIGID GEOMETRY AND IT'S APPLICATIONS

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§1. Introduction

Analytic Geometry over p-adic fields has been studied over years. Rigid analytic geometry is one of such approaches, defined by J. Tate in 60's [T]. (a different approach was proposed by Berkovich. Cf. [Be])

A naive approach would be the following: $\mathbf{Q}_p$ naturally has a topology, which makes it locally compact, so for an algebraic variety X over $\mathbf{Q}_p$ $X(\mathbf{Q}_p)$ is naturally endowed with the topology. We should be able to do analytic geometry with it.

But this idea is too naive. First of all we do not have a good notion of "analytic function " or " analytic continuation" using such topology. Secondly, the topology is totally disconnected, so it does not give the "expected topological invariants" for analytic varieties, even for π_0.

Tate's idea was to introduce family of "rings of analytic functions", called Tate algebras, and use a Grothendieck topology to patch them together, which fits into the pattern established by Grothendieck for commutative rings. Tate needed such a theory to study the degeneration of elliptic curves. The idea was developed by many people, especially by German school [K], [BGR].

In mid 70's , Raynaud published his very beautiful idea in [Ray 2]. (He claimed it at Nice congress too, so the idea emerged in late 60's.) The idea is to view the category of rigid-analytic spaces as a quotient category of formal schemes, at least for quasi-compact and quasi-separated spaces. By this idea, many basic facts in rigid geometry are reduced to the knowledge of formal geometry, where EGA III is at our disposal. Moreover the construction carries over any noetherian formal schemes, not only over valuation rings. So the rigid geometry is base space free, but the absolute rigid geometry remains almost unexplored. Unfortunately, many basic facts from Raynaud's viewpoint are still unpublished, except recent papers by [BL] and a few references. The author believes that rigid-geometric ideas are effective in algebraic geometry. From the viewpoint of algebraic geometry, the role of formal geometry is rather small, and rigid geometry is a systematic study of " limit of blowing ups along a subvariety", so the analogue of infinite repetition method.

Here we give some relations with Zariski's theory of abstract Riemann spaces, and applications to etale cohomology theory. The author apologizes to those whose contributions he does not properly acknowledge due to his incomplete understanding.

Basic properties

In the following we consider coherent (= quasi-compact and quasi-separated) formal schemes which subject to one of the following conditions:
type n) X is a Noetherian formal scheme.
type v) X is finitely generated over a complete valuation ring V with a-adic topology for some $a \in V$.

By $\mathcal{C}$ we denote the category of coherent formal schemes (the morphisms between them are coherent (quasi-compact and quasi-separated)). The most basic example is $A = V\{\{X\}\}$, the ring of a-adic convergent power series($V = \mathbf{Z}_p, a = p$ or $V = \mathbf{C}[[t]]$, $a = t$). Put K= the fraction field of V, $\mathbf{C}_K$ = the completion of the algebraic closure of $\bar{K}$. For any element $f(X)$ in $A_{\mathbf{C}_K} = A\hat{\otimes}_V\mathbf{C}_K$ (the tensor product is the topological one) we can make substitution $X \mapsto \alpha$ with $\alpha \in D(\mathbf{C}_K) = \{\beta \in \mathbf{C}$, $|\beta| \leq 1\}$. So we want to attach $A_{\mathbf{C}_K}$ as the ring of analytic functions to the closed unit disk $D(\mathbf{C}_K)$. Since the ring $A_{\mathbf{C}_K}$ is integral, the unit disk should be connected, but for the natural topology of $D(\mathbf{C}_K)$ this is false. Tate defined a class of finite coverings, and considered only coverings which are refined by this class. If we calculate π_0 with this topology we get the expected answer. The class of coverings considered by Tate is obtained from an open covering of a formal scheme which is a proper modification of Spf A. We define the class of proper modification, called admissible blowing ups, as follows:
Let $\mathcal{I}$ be an ideal of definition. When X = Spf A is affine, $\mathcal{I} = I \cdot \mathcal{O}_X$, the blow up X' of X along $\mathcal{I}$ is just the formal completion of the blowing up of Spec A along I . In general X' is defined by patching. When X is the p-adic completion of some p-adic scheme Y, admissible blowing up means the (formal completion of) blowing up with a center whose support is concentrated in $p = 0$. So the following definition, due to Raynaud, will be suited for our purpose:

Definition (Raynaud)[Ray 2].

The category $\mathcal{R}$ of coherent rigid-analytic spaces is the quotient category of $\mathcal{C}$ by making all admissible blowing ups into isomorphisms, i.e.

$$\mathrm{Hom}_{\mathcal{R}}(X,Y) = \varinjlim_{X' \in \mathcal{B}_X} \mathrm{Hom}(X',\ Y).$$

For $X \in \mathcal{C}$, X viewed as an object of $\mathcal{R}$ is denoted by X^{rig} or X^{an}. X is called a formal model of X^{an}.

Note that we can fix a base if necessary. For example, in case of type v), it might be natural to work over the valuation ring V. Though this definition seems to be a global one, i.e. there are no a priori patching properties, but it indeed does. The equivalence with the classical Tate rigid-spaces is shown in [BL].

Riemann space associated with a rigid space.

Let $\mathcal{X} = X^{an}$ is a coherent rigid space. Then the projective limit

$$< \mathcal{X} >= \varprojlim_{X' \in \mathcal{B}_X} X'$$

in the category of local ringed spaces exists. The topological space is quasi-compact. We call it the Zariski-Riemann space associated to $\mathcal{X}$. The projection $< \mathcal{X} > \to X$

is called the specialization map and written as $\mathrm{sp} = \mathrm{sp}_{\mathcal{X}}$. *The structural sheaf* $\tilde{\mathcal{O}}_{\mathcal{X}}$ *yields*

$$\mathcal{O}_{\mathcal{X}} = \varinjlim_{n} \mathrm{Hom}(\mathcal{I}^n \tilde{\mathcal{O}}_{\mathcal{X}},\ \tilde{\mathcal{O}}_{\mathcal{X}})$$

which is local ringed. This $\mathcal{O}_{\mathcal{X}}$ *is the structural sheaf in rigid geometry ((classical) rigid geometry is a* $\mathbf{Q}$*-theory, i.e. invert* $\mathcal{I}$*).* $\tilde{\mathcal{O}}_{\mathcal{X}}$ *is the (canonical) model of* $\mathcal{O}_{\mathcal{X}}$.

In the followings we sometimes call the topology, or rather the Grothendieck topology associated to the topological space, admissible, to make it compatible with the classical terminology. The category $\mathcal{R}$, with the admissible topology, is called large admissible site.

Note that the model sheaf $\tilde{\mathcal{O}}_{\mathcal{X}}$ itself gives a local ringed space structure. I wonder why people had not used this Zariski-Riemann space until now. (Berthelot told me that the approach was written in a letter of Deligne to Berkovich. It is quite likely that such an idea came from Gabber, since he studied such a limit , for example, see [V] p.194 .) The author was lead to the idea by the necessity to define a fixed point set in rigid geometry.

As in the Zariski case, each point x of $< \mathcal{X} >$ corresponds to a valuation ring V_x which is henselian along $I =$ the inverse image of $\mathcal{I}$, i.e. x is considered as the image of the closed point of Spf $\hat{V}_x$. The local ring $A = \tilde{\mathcal{O}}_{\mathcal{X},x}$ has the following property: $B = \mathcal{O}_{\mathcal{X},x} = A[1/\{I \backslash \{0\}\}]$ is a noetherian henselian local ring, whose residue field K_x is the quotient field of V_x, $A =$ the inverse image of V_x by the reduction map $B \rightarrow K_x$.

Conversely, any morphism Spf $V \rightarrow X$ (V is a valuation ring) lifts uniquely to any admissible blowing ups by the valuative criterion, so the image of the closed point of V define a point x.

For the rigid-analytic structural sheaf, it can be proved that any coherent $\mathcal{O}_{\mathcal{X}}$ module has a formal model. But the model sheaf itself has a meaning, which has been treated in algebraic geometry. Let Z be a quasi-excellent normal scheme, Y a closed subscheme containing the singular points of Z. Then put $X = \hat{Z}|_Y$, the completion along Y, and $\mathcal{X} = X^{an}$. Then

$$R^q(\mathrm{sp}_{\mathcal{X}})_* \tilde{\mathcal{O}}_{\mathcal{X}} = \widehat{R^q\pi_*\, \mathcal{O}_{\tilde{Z}}}$$

holds if we accept all kinds of Hironaka resolutions. Here $\pi : \tilde{Z} \rightarrow Z$ is a resolution of Z. Without resolution, it seems hard to show the finite generation of this cohomology group. The proof is easy, so omitted here. To define more general rigid spaces, which is inevitable if one treats GAGA-functor, the following lemma is necessary:

Lemma.

For a coherent rigid space $\mathcal{X}$, *the presheaf* $\mathcal{Y} \rightarrow Hom_{\mathcal{R}}(\mathcal{Y}, \mathcal{X})$ *on the large admissible site* $\mathcal{R}$, *is a sheaf.*

Since it exhibits the local nature of our rigid geometry, I give the outline of the proof. Take a formal model X of $\mathcal{X}$. Assume we are given a covering $\{\mathcal{U}_i\}_{i \in I}$ of $\mathcal{Y}$, and $f_i : \mathcal{U}_i \rightarrow \mathcal{X}$ with $(*) : f_i|_{\mathcal{U}_i \cap \mathcal{U}_j} = f_j|_{\mathcal{U}_i \cap \mathcal{U}_j}$. By the quasi-compactness of $\mathcal{Y}$, we may assume that $\mathcal{U}_i$ are coherent and the covering is finite. For a suitable formal model Y of $\mathcal{Y}$, we may assume that $\mathcal{U}_i$ is obtained as $\mathrm{sp}^{-1}(U_i)$, $U_i \subset Y$ is quasi-compact open subformal scheme. Here we have used that any admissible blowing up of an open subformal scheme extends to the whole. Using this property again, we may assume that f_i ($i \in I$) are defined over Y, i.e. come from formal morphisms $F_i : U_i \to X$. We want to patch these local formal morphisms. By (*), for each $(i,j) \in I \times I$, we have an admissible blowing up $\pi_{ij} : \tilde{U}_{ij} \to U_{ij}$ of $U_{ij} = U_i \cap U_j$ defined by finitely generated ideal $\mathcal{I}_{ij}$ such that $F_i|_{U_{ij}} \cdot \pi_{ij} = F_j|_{U_{ij}} \cdot \pi_{ij}$. We extend these $\mathcal{I}_{ij}$ on U_{ij} to Y, say $\tilde{\mathcal{I}}_{ij}$, and blow up the product $\prod_{(i,j)\in I\times I} \tilde{\mathcal{I}}_{ij}$, $\tilde{Y} \to Y$. Then, on each U_{ij}, $\tilde{Y}$ dominates $\tilde{U}_{ij}$ and hence F_i 's patch together on $\tilde{Y}$. By the construction, the rigid morphism defined by the glued formal morphism is the one we wanted.

Definition.

A sheaf $\mathcal{F}$ on the big admissible site $\mathcal{R}$ is called a rigid space if the following conditions are satisfied:

a) There is a morphism $\mathcal{Y} = \coprod_{i\in I} Y_i \to \mathcal{F}$ (Y_i are coherent representable sheaves) which is surjective.

b) Both projections $\mathrm{pr}_i : \mathcal{Y} \times_{\mathcal{F}} \mathcal{Y} \to \mathcal{Y}$ ($i = 1, 2$) are represented by open immersions.

c) $\mathcal{F}$ is quasi-compact if one can take quasi-compact $\mathcal{Y}$ in b).

d) $\mathcal{F}$ is quasi-separated if the diagonal $\mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is quasi-compact.

We can show that if a rigid space in the above sense is quasi-compact and quasi-separated, then it is a representable sheaf, so the terminology " coherent rigid space " is compatible. Assume $\mathcal{F}$ is a quasi-separated rigid space. Then it is written as $\mathcal{F} = \varinjlim_{j\in J} \mathcal{X}_j$ where $\mathcal{X}_j$ is coherent , J is directed and all transition maps $\mathcal{X}_j \to \mathcal{X}_{j'}$ are open immersions. The definition has been used for a long time. For the construction of GAGA-functor for non-separated schemes quasi-separated spaces are not sufficient.

As an application of rigid-geometric idea, let me mention the following elementary example:

Flattening theorem of Gruson-Raynaud [GR].

Let $f : X \to S$ be a finitely presented morphism, with S coherent (=quasi-compact and quasi-separated). Assume f is flat over a coherent open set $U \subset S$. Then there is an admissible blow up $S' \to S$ such that the strict transform of f (kill torsions after taking the fiber product) is flat and finitely presented.

There is a principle to prove this kind of statement:

Principle.

Assume we have a canonical global procedure, an element of a cofinal subset $\mathcal{A}_S$ of all admissible blowing ups of S to achieve a property P. Assume the following properties are satisfied:

a) P is of finite presentation.

b) The truth of $P(S')$ for $S' \in \mathcal{A}_S$ implies the truth of $P(S'')$ for all $S'' \in \mathcal{A}_S$ dominating S'.

c) P is satisfied at all stalks $\tilde{\mathcal{O}}_{\mathcal{X},x}$ of the model sheaf.
Then P is satisfied after some blowing up in $\mathcal{A}$.

I explain this in case of flattening. Let $S\backslash U = V(\mathcal{I})$ with $\mathcal{I}$ finitely generated. $\mathcal{A}_S$ is the set of $\mathcal{I}$-admissible blowing ups, for which the total transform of $\mathcal{I}$ is invertible. $P(S')$ is : The strict transform of $X \times_S S'$ is flat and finitely presented over S'.
a) follows from the finite presentation assumption of the strict transform. b) is clear. For c), take a point of the Zariski-Riemann space $< \mathcal{X} >$. Then the local ring $A = \tilde{\mathcal{O}}_{\mathcal{X},x}$ has the property mentioned before. To prove the flattening in this case, using the flatness of $X \times_S \operatorname{Spec} A$ over $(\mathcal{I}\backslash\{0\})^{-1}A$, we are reduced to the valuation ring case. i.e. prove the claim restricted to "curves" passing $V(\mathcal{I})$. In the valuation ring case ("curve case") there is no need to blow up, and strict transform just means that killing torsions. But note that we need to check the finite presentation of the result, i.e.

Lemma.

For a finitely generated ideal I of $V[X]$, the saturation $\tilde{I} = \{f \in V[X];\ af \in I \text{ for some } a \in V\backslash\{0\}\}$ is finitely generated.

The proof of this lemma is not so easy, but I leave it as an exercise.

So the claim is true locally on $< \mathcal{X} >$, since we have the finite presentation property. The quasi-compactness of $< \mathcal{X} >$ implies the existence of a finite covering, which admit models with the desired flattening property. The patching is unnecessary, i.e. it is automatically satisfied since we have a canonical global procedure to achieve the flattening, and once the flattening is achieved, we have it for all admissible blow up in $\mathcal{B}_S$ dominating the model.

Note that our proof applies in case of formal schemes too [Fu 3]. Another proof is found in [BL].

In June 1992, M. Spivakovsky claimed that he proved the canonical resolution of singularities for quasi-excellent schemes. The pattern is similar to the above toy model, but there is no finite presentation property in this resolution case. It is still not clear whether his form of local uniformization is really true or not.

Sometimes we want to use just "usual curves " i.e. Spec of a discrete valuation ring rather than general valuations. Sometimes it is possible. This is plausible, since the general valuation rings does not have any good finiteness conditions. (The value group such as $\mathbf{Z}^n$ with the lexicographic order is good, but even these are not enough sometimes.) Another "toy model " is given by Gabber's extension theorem of locally free sheaves, which played an important role in Vieweg's semipositivity of the direct image of the dualizing sheaves. The structure of locally free module with respect to $\tilde{\mathcal{O}}$ is used: it can be proved that such a module come from some formal model.

Separation

Here we give the explanation of a notion which was unclear in the classical theory. Let $\mathcal{X}$ be a coherent rigid space. For a point $x \in <\mathcal{X}>$ with associated valuation ring V_x, the point of $\mathcal{X}$ which corresponds to the height one valuation of K_x is denoted by $y = \text{sep}(x)$ and called the maximal generalization of x (y corresponds to the minimal prime ideal containing an ideal of definition). Let $[\mathcal{X}]$ be the subset of $<\mathcal{X}>$ consisting of height one points. Then we give $[\mathcal{X}]$ the quotient topology by surjection $\text{sep} :<\mathcal{X}>\longrightarrow [\mathcal{X}]$ (caution: the section corresponding to the natural inclusion $[\mathcal{X}] \rightarrow <\mathcal{X}>$ is not continuous). This space $[\mathcal{X}]$ has an advantage that it is much nearer to our topological intuition. For example

Proposition.

$[\mathcal{X}]$ is a compact Hausdorff space. Basis of closed sets is $\{\text{sep}(\mathcal{U})\}$, $\mathcal{U}$ a quasi-compact open subset ($\text{sep}^{-1}(\text{sep}(\mathcal{U})) = \overline{\mathcal{U}}$, where $^{-}$ denotes the closure).

holds. Especially there is ample supply of $\mathbf{R}$-valued functions on $[\mathcal{X}]$. Dually, a basis of open sets is obtained as follows : First we define the notion of tubes. For a model X' of $\mathcal{X}$ and a closed set C of X' $T_C = (\text{sp}^{-1}(C))^{int}$ (*int* denotes the interior), is called the tube of C. In fact, tube of C is defined as $\varinjlim_n \text{sp}^{-1}(U_n)$, where U_n is the open set of the blowing up by $(\mathcal{I}_C)^n + \mathcal{I}$ where the inverse image of $\mathcal{I}$ generates the exceptional divisor. T_C is the complement of $\overline{\text{sp}^{-1}(X' \setminus C)}$. For a tube $T = T_C$, $\text{sep}^{-1}\text{sep}(T) = T$ holds, and hence $\text{sep}(T)$ is an open set of $[\mathcal{X}]$, which is not compact in general. Images of tubes form a basis of open sets in $[\mathcal{X}]$. For most cohomological questions both topological space give the same answer:

Proposition.

For a sheaf $\mathcal{F}$ on $<\mathcal{X}>$, $R^q sep_* \mathcal{F} = 0$ if $q > 0$. For a sheaf $\mathcal{G}$ on $[\mathcal{X}]$, $sep_* \text{sep}^{-1} \mathcal{G} = \mathcal{G}$.

The proposition includes $H^q(\bar{\mathcal{U}}, \mathcal{F}) = H^q(\mathcal{U}, \mathcal{F}|_{\mathcal{U}})$ ($= H^q([\mathcal{U}], \mathcal{G})$) for a sheaf $\mathcal{F} = \text{sep}^{-1}(\mathcal{G})$ on $\bar{\mathcal{U}}$. Note that this does not apply to coherent sheaves. This is quite important in the theory of overconvergent isocrystals of Berthelot.

§2. Comparison Theorems in rigid etale cohomology

Here fundamental comparison theorems for rigid-etale cohomology are discussed. The origin for the study of rigid-etale theory is Drinfeld's work on p-adic upper half plane [D]. Most results here has an application for the study of modular varieties. The results, with many overlaps, are obtained by Berkovich for his analytic spaces (not rigid analytic one) over height one valuation fields. The relation between both approaches will be discussed elsewhere.

We want to discuss etale cohomologies of rigid-analytic spaces. It is sometimes more convenient to use a variant of rigid-geometry, defined for henselian schemes instead of formal schemes. For the affine case it is defined as follows. We take an affine henselian couple $(S, D) = (\mathrm{Spec}\ A, \tilde{I})$: $D \subset S$ is a closed subscheme with $\pi_0(S' \times_S D) = \pi_0(S')$ for any finite S-scheme S'(hensel lemma). As an example, if S is $\mathcal{I}_D$-adically complete, (S, D) is a henselian couple. Then to each open set $D \cap D(f) = \mathrm{Spec}\ A[1/f]/I[1/f]$, $f \in A$, we attach the henselization of $A[1/f]$ with respect to $I[1/f]$. This defines a presheaf of rings on D. This is in fact a sheaf, and defines a local ringed space $\mathrm{Sph}\, A$, called the henselian spectrum of A (as a topological space it is D, like a formal spectrum). General henselian schemes are defined by patching. See [Cox], [Gre], [KRP] for the details. We fix an affine henselian (or formal) couple (S, D). Put $U = S \setminus D$. We consider rigid geometry over S, i.e. rigid geometry over the henselian scheme attached to S. Of course we can work with formal schemes. Note on GAGA-functors: For a locally of finite type scheme X_U over U, there is a GAGA-functor which associates a general rigid space X_U^{rig} to X_U (X^{rig} is not necessarily quasi-compact, nor quasi-separated): Here are examples.

a) For X_U proper over U, $(X_U)^{rig} = (X^h)^{rig}$(resp. $(X^f)^{rig}$). Here X is a relative compactification of X_U over S, the existence assured by Nagata. Especially the associated rigid space is quasi-compact (and separated) in this case.

b) In general $(X_U)^{rig}$ is not quasi-compact, as in the complex analytic case. $(\mathbf{A}^1_U)^{rig}$ is an example. It is the complement of ∞_U^{rig} in $(\mathbf{P}^1_U)^{rig}$). This is associated with a locally of finite type formal (or henselian) scheme over S. c) GAGA-functor is generalized to the case of relative schemes of locally of finite presentation over a rigid space.

Rigid-etale topos

For simplicity I restrict to coherent spaces.

Definition.

a) A morphism $f : \mathcal{X} \to \mathcal{Y}$ is rigid-etale if it is flat and neat ($\Omega^1_{\mathcal{X}/\mathcal{Y}} = 0$) .

b) For a rigid space $\mathcal{X}$ we define the rigid etale site of $\mathcal{X}$ the category of etale spaces over $\mathcal{X}$, where covering is etale surjective. The associated topos is denoted by $\mathcal{X}_{et}$.

For a coherent rigid space $\mathcal{X}$ the rigid-etale topos is coherent.

Categorical equivalence.

Let X be a henselian scheme which is good. Then consider the rigid henselian space $\mathcal{X} = X^{rig}$. At the same time one can complete a henselian scheme, so we have a rigid-analytic space $\mathcal{X}^{an} = (\hat{X})^{rig}$. There is a natural geometric morphism

$$(\mathcal{X})^{an}_{et} \to \mathcal{X}_{et}$$

since the completion of etale morphism is again etale, and surjections are preserved. Then the above geometric morphism gives a categorical equivalence.

It suffices to prove it for the etale site. We may restrict to coherent spaces. To show the fully-faithfulness one uses Elkik's approximation theorem [El] and some deformation theoretical argument to show morphisms are discrete. (The rigidity implies that an approximating morphism is actually the desired one.) For the essential surjectivity one can use Elkik's theorem in the affine case, since one can patch local pieces together by the fully-faithfulness. An important consequence is as follows:

Corollary.

Let $(A_i, I_i)_{i \in I}$ be an inductive system of good rings, A_i I_i-adically complete. Then $\varprojlim_{i \in I} (\mathrm{Spf}\, A_i)^{an}_{et}$ is equivalent to $(\mathrm{Sph}\, A)^{rig}_{et}$, where $A = \varinjlim_{i \in I} A_i$, which is henselian along $I = \varinjlim I_i$. Here the projective limit is the 2-projective limit of toposes defined in SGA 4 .

Since the above ring A is not I-adically complete in general (completion does not commute with inductive limit), the above equivalence gives the only way to calculate the limit of cohomology groups, especially calculation of fibers. This is the technical advantage of the introduction of henselian schemes. Moreover if we regard an affine formal scheme $X = \mathrm{Spf}\, A$ as a henselian scheme, i.e. $\tilde{X} = \mathrm{Sph}\, A$ with natural morphism $X \to \tilde{X}$ as ringed spaces, the induced geometric morphism $X^{rig}_{et} \to \tilde{X}^{rig}_{et}$ is a categorical equivalence so the "local" cohomological property of rigid analytic spaces is deduced from that of hensel schemes.

GAGA and comparison for cohomology

Let (S, D) be an affine henselian couple, X_U a finite type scheme over U. Then one has a geometric morphism

$$\epsilon : (X_U)_{et}^{rig} \to X_{et}$$

defined as follows: For an etale scheme Y over X_U, one associates Y^{rig}. Since GAGA-functor is left exact, and surjections are preserved, a morphism of sites is defined and gives ϵ. By the definition, $\epsilon^* F = F^{rig}$ for a representable sheaf F on X (we have used that F^{rig} is a sheaf on $(X_U)_{et}^{rig}$). By abuse of notation we write $F^{rig} = \epsilon^* F$ for a sheaf F on $(X_U)_{et}$. Note that the morphism ϵ is not coherent, i.e. some quasi-compact object (such as an open set of X_U) is pulled back to a non-quasi compact object.

Theorem.

For a torsion abelian sheaf $\mathcal{F}$ on $(X_U)_{et}$, the canonical map

$$H_{et}^q(X_U, \mathcal{F}) \simeq H_{et}^q((X_U)^{rig}, \mathcal{F}^{rig})$$

is an isomorphism. The equivalence also holds in the non-abelian coefficient case, i.e. ind-finite stacks.

This especially includes Gabber's formal vs algebraic comparison theorem. The above theorem itself was claimed by Gabber in early 80's.

To deduce this form of comparison from the following form, Gabber's affine analogue of proper base change theorem [Ga] is used (if (S, D) is local, we do not have to use it). For the application to etale cohomology of schemes, see [Fu]. Especially regular base change theorem, conjectured in SGA4, is proved there (this is also a consequence of Popescu-Ogoma-Spivakovsky smoothing theorem).

Corollary (Comparison theorem in proper case).

For $f : X \to Y$, proper morphism between finite type schemes over U, and a torsion abelian sheaf $\mathcal{F}$ on X, the comparison morphism

$$(R^q f_* \mathcal{F})^{rig} \to R^q (f^{rig})_* \mathcal{F}^{rig}$$

is an isomorphism. Especially, for $\mathcal{F}$ constructible , $R^q(f^{rig})_* \mathcal{F}^{rig}$ is again (algebraically) constructible (non-abelian version is also true).

There is another (more primitive) version which includes nearby cycles. We will state the claim, with a brief indication of the proof. X a scheme, $i : Y \hookrightarrow X$ a closed subscheme with $U = X \setminus Y$. $j : U \hookrightarrow X$. Let $T_{Y/X} = \mathcal{X}_{et}$, $\mathcal{X} = (X^h|_Y)^{rig}$. (It is the analogue of (deleted) tubular neighborhood of Y in X). For any etale sheaf $\mathcal{F}$ on U one associates, by a patching argument, an object of $T_{Y/X}$ which we write as $\mathcal{F}^{rig}$ (" restriction of $\mathcal{F}$ to the tubular neighborhood). Note that there is a geometric morphism $\alpha_X : T_{Y/X} \to Y_{et}$ (' fibration over Y").

Theorem.

For a torsion abelian sheaf $\mathcal{F}$ on U, there is an isomorphism

$$i^* R j_* \mathcal{F} \simeq R(\alpha_X)_* \mathcal{F}^{rig}.$$

If we apply this claim to a finite type scheme over a trait (or the integral closure of it in a geometric generic point), one knows that rigid-etale cohomology in the quasi compact case is just the hypercohomology of the nearby cycles:

Corollary.

Let V be a height one valuation ring, with separably closed quotient field $K = V[1/a]$. Let X be a finitely presented scheme over V, or $X = \mathrm{Spf}\, A$, A a good ring of type v) which is finitely presented over V. Let $\mathcal{F}$ be a torsion sheaf on X_K, or a torsion sheaf on Spec $A[1/a]$. *Then*

$$R\Gamma((\hat{X})^{rig}, \mathcal{F}^{rig}) = R\Gamma(X_s, i^* Rj_* \mathcal{F})$$

holds. Here $i : X_s = X \times_V (V/\sqrt{a}) \to X$ (or $i :$ Spec $A \times_V (V/\sqrt{a}) \to$ Spec A in the affine formal case) and $j : X_K \to X$ (or $j :$ Spec $A_K \to$ Spec A in the affine formal case).

The above mentioned comparison theorem follows from this theorem, using the Gabber's affine analogue of proper base change theorem. Let me give a brief outline of the proof. The underlying idea is quite topological. Put $Z = \varprojlim_{X' \in \mathcal{B}_X} X'_{et}$ ($\mathcal{B}_X$ is the set of admissible blowing ups (in the scheme sense), $T^{unr}_{Y/X} = \varprojlim_{X' \in \mathcal{B}_X} (X' \times_X Y)_{et}$ ($T^{unr}_{Y/X}$ is the analogue of tubular neighborhood of Y). The limit is taken as toposes. Then $U_{et} \overset{j^{unr}}{\to} Z \overset{i^{unr}}{\leftarrow} T^{unr}_{Y/X}$ is a localization diagram (U is an " open set" and T^{unr} is a "closed set" of Z.) Using proper base change for usual schemes (here $\mathcal{F}$ torsion is used), one shows that $R\beta_*(i^{unr*} Rj^{unr}_* \mathcal{F}) = i^* Rj_* \mathcal{F}$ ($\beta : T^{unr} \to Y_{et}$). So we want to do a comparison on $T^{unr}_{Y/X}$.
In fact, there is a morphism $\pi : T_{Y/X} \to T^{unr}_{Y/X}$ (" inclusion of deleted tubular neighborhood") such that $R\pi_* \mathcal{F}^{rig} = i^{unr*} Rj^{unr}_* \mathcal{F}$ (this formula is valid for any sheaf!). The construction is canonical. To calculate the fibers, one needs to treat a limit argument, so we take here an advantage of henselian version, not formal one.

In the non-proper case, i.e. f is of finite type but not assumed proper, the comparison is not true unless we restrict to constructible coefficients, torsion prime to residual characteristic of S. (Since the analytic topos involved is not coherent in this case, one can not use limit argument to deduce general torsion coefficient case. This is the same as C-case.) Though the author thinks that comparison is always true for finite type morphism between quasi-excellent schemes, the only known result, which is free from resolution of singularities, is the following height one case (a corresponding result for Berkovich type analytic spaces is obtained earlier in [Be]).

Comparison theorem in the non-proper case.

Let V be a height one valuation ring, with separably closed quotient field K. $f : X \to Y$ morphism between finite type schemes over K. Then

$$(R^q f_* \mathcal{F})^{rig} \to R^q (f^{rig})_* \mathcal{F}^{rig}$$

is an isomorphism for $\mathcal{F}$ constructible sheaf, torsion prime to residual characteristics of V.

This is proved by a new variant of Deligne's technique in SGA 41/2 [Th. de Finitude] , without establishing Poincaré duality. This geometric argument, more direct, reduces the claim for open immersions (evidently the most difficult case

) to a special case, i.e. open immersion of relative smooth curves over a smooth base . Moreover one can impose good conditions, such as smoothness and tameness of $\mathcal{F}$. In this case one can make an explicit calculation. Of course the finitude in the proper case, which is already stated, is used. The details will be published elsewhere.

Using the comparison theorems, it is easy to see the comparison for $\otimes^{\mathbf{L}}$, $RHom$, f^*, f_*, $f_!$. The claim for $f^!$ follows from the smooth case. For the Poincaré duality in this case, using all the results I mentioned already, there are no serious difficulties except various compatibility of trace maps. Berkovich has announced such results already for his analytic spaces.

§3. Lefschetz-Verdier trace formula and a Deligne's conjecture

In any cohomology theory where 6-operations (f^*, $f_!$, f_*, $f^!$, $RHom$, $\otimes^L$) are available, there is a trace formula for a correspondence. This formalism was established by Verdier. In the topological case we get the original Lefschetz formula.

The formula expresses the global trace as a sum of local terms, which depend only on a neighborhood near the fixed point set. Essentially we express the global trace as the intersection number of the diagonal and the correspondence. Though the formula holds quite generally, the explicit calculation of the local terms is painful, even in the classical case, if we take general sheaves as the coefficient. Goresky-MacPherson [GM] found such a good class of correspondences, called weakly hyperbolic, and they showed that there is a fairly good formula in this case. (Especially they applied it to Hecke correspondences of arithmetic quotient of symmetric space. But the computation was done in the real category and we do not know their method applies to the minimal compactification of Baily-Borel type.) Since they use Lefschetz-Verdier formula for subanalytic spaces, even in the complex analytic case, so an abstract formulation of their method which is valid in any characteristic seemed difficult.

On the other hand, in characteristic positive, Deligne conjectured the following: First fix notations:

Let (X,Y,a) be a triplet with X, Y proper varieties over $k = \mathrm{Spec}\ \bar{\mathbf{F}}_q$, $a: Y \to X \times X$ a correspondence. Put $a_i = \mathrm{pr}_i \cdot a$ where pr_i denotes the i-th projection. We assume the triplet is defined over $\mathbf{F}_q$, so that we can compose a with a power of (geometric) Frobenius. (We take the following choice to define the direction of composition: A morphism $f: X \to X$ is considered as a correspondence with the second projection $=$ id. So $(\mathrm{Fr}^n \cdot a)_1 = \mathrm{Fr}^n \cdot a_1$, $(\mathrm{Fr}^n \cdot a)_2 = a_2$. For $n \geq 0$ we take a complex $K \in D_{ctf}(X, \Lambda)$ (ctf= constructible and finite tor dimension, $\Lambda = \mathbf{Z}/\ell^n$, $\mathbf{Z}_\ell$, $\mathbf{Q}_\ell$, with ℓ invertible on X), with a cohomological correspondence $c \in \mathrm{coh.cor}(\mathrm{Fr}^n \cdot a,\ K) = H^0(Y, a^!(DK \boxtimes^{\mathbf{L}} K))$($DK = RHom(K, K_X)$ is the Verdier dual). Then the cohomological correspondence define the global trace

$$\mathrm{Lef}(\mathrm{Fr}^n \cdot a, K) = \mathrm{Trace}_\Lambda(\mathrm{Fr}^n \cdot a,\ R\Gamma(X,K))$$

(this is an abuse of notation, since the homomorphism induced on the cohomology depend on c). Note that $R\Gamma$ is a perfect complex of Λ modules, so the trace is well-defined. Deligne's conjecture means

Conjecture.
For n sufficiently big (which depends only on the correspondence) , we have

$$\mathrm{Lef}(\mathrm{Fr}^n \cdot a,\ K) = \Sigma_{D \in \pi_0(\mathrm{Fix}\ \mathrm{Fr}^n \cdot a)} \mathrm{naive.loc}_D(\mathrm{Fr}^n \cdot a,\ K)$$

where naive. loc_D *means the naive local term around D. As an important (and characteristic) property, this term vanishes if the fiber of K over $a_1(D) = a_2(D)$ is zero.*

Note that this conjecture implies a kind of Lefschetz formula for open varieties: To see this, assume that X is a compactification of U ($j : U \to X$), U is stable under the given correspondence a, and $a_2|_U$ is finite. Take a complex K over U with a cohomological correspondence c_U over U. Then by a formalism of cohomological correspondences, c_U extends uniquely to a correspondence for $j_!K$. Applying the Deligne's conjecture, we will have that the trace on $R\Gamma_c(U, K)$ is the sum of naive local terms along the fixed point set inside U, since any contribution from the infinity vanishes ($j_!K|_{a_1(D)} = 0$ for $D \subset X \backslash U$). This, with an additional assumption that K is a Λ-smooth sheaf, is the original version of the conjecture. In the followings we will restrict our attention to this case.

$\dim X = 1$ case follows from [Il]. $\dim X = 2$ and $K = \mathcal{F}$ is a smooth sheaf with finite monodromy, a weaker version of the conjecture (the equality holds but the coincidence of each local term is not shown) is due to Zink[Z]. When U is smooth, admits a smooth compactification with normal crossing complement, and $\mathcal{F}$ tame smooth sheaf, Pink [P] and Shpiz [Sh] proved independently the conjecture. Moreover Pink reduced the general case to this special case, using full force of Hironaka resolution.

The reason why we seek for this kind of trace formula in the non-proper case is explained by the necessity in the Langlands correspondence [FK], [La]. To establish a reciprocity law, it is necessary to compare Arthur-Selberg trace formula and Lefschetz-Verdier trace formula for modular varieties.
A brief dictionary is as follows:

A reductive group G and it's adelized group $\Leftrightarrow$ modular variety $\mathcal{M}$ corresponding to G

Arthur-Selberg (invariant , or non-invariant) trace formula $\Leftrightarrow$ Lefschetz-Verdier trace formula

Test function $f = \prod_x f_x$ on the adelic group $G(\mathbf{A})$ with specific properties at two places v and ∞ $\Leftrightarrow$ a sheaf $\mathcal{F}$ on $\mathcal{M}_v$ determined by f_∞ and a (linear combination of)Hecke correspondence(s) determined by $f^{v,\infty}$ composed with a special correspondence determined by f_v.

Simple version of the trace formula (Kazhdan type , i.e. f_v, a spherical function, corresponds to a High power of Frobenius by the Satake transform) $\Leftrightarrow$ Deligne's conjecture for the above data for a reduction of modular varieties. (In general, we should compare stabilized trace formula (still conjectural) and the global trace on the intersection cohomology group.)

How to prove the conjecture?

We must formulate the form of trace formula which yields the desired result. The idea that such trace formula in rigid geometry will be effective came from Gabber. Let V be a height one valuation ring, with separably closed quotient field.

η = Spec K its generic point, S = Spec V, $\bar{\eta}$ the generic point, and s the closed point.
For $f : X \to S$, a morphism of finite type, put $i_X : X_s \to X$, $j_X : X_{\bar{\eta}} \to X$.
Take K in $D^b_c(X_{\bar{\eta}}, \Lambda)$, a correspondence $a : Y \to X \times_S X$, and a cohomological correspondence $c \in$ coh. cor$(a_\eta,\ K)$ lifting a_η.

Assume X is proper. $\mathcal{X} = (X^h)^{rig}$ (resp. $\mathcal{Y} = (Y^h)^{rig}$): the rigid space associated with $\hat{X}$ (resp. $\hat{Y}$). For a correspondence $a : Y \to X \times_S X$, $\alpha : \mathcal{Y} \to \mathcal{X} \times_{\bar{\eta}} \mathcal{X}$ the associated correspondence in the rigid-category.

Definition.

We say a is contracting near $D \subset \pi_0(\mathrm{Fix}\, a_{\bar{\eta}})$ if the following conditions are satisfied:
a) There are quasi-compact open sets $\mathcal{U} \subset \mathcal{X}$, $\mathcal{V} \subset \mathcal{Y}$ with $\mathcal{V} \subset \alpha_1^{-1}(\mathcal{U}) \cap \alpha_2^{-1}(\mathcal{U})$, $\alpha_2 : \mathcal{V} \to \mathcal{U}$ is proper.
b) There is a continuous function $\phi :< \mathcal{U} > \to \mathbf{R}$ satisfying
$\phi(x) = 0 \Leftrightarrow x \in (a_2(D))^{rig}$.
$\phi \cdot a_1(y) < \phi \cdot a_2(y)$ *if* $y \in < \mathcal{V} >,\ y \notin D^{rig}$.

First we need to introduce such a continuous function ϕ on non-Hausdorff space. It factors through the separated quotient $[\mathcal{X}]$, and the compactness of this space makes it easier to construct such ϕ.

We assume that the coefficient ring is $\Lambda = \mathbf{F}_\ell, \mathbf{Z}_\ell, \mathbf{Q}_\ell$.

Theorem.

Let (X, Y, a) be a triplet over V, K a constructible Λ-sheaf on $X_{\bar{\eta}}$. If a is contracting near D and $K|_{a_2(D)} = 0$ then the local term of K along D is zero.

The theorem is proved without using Hironaka resolution. For the application to Deligne's conjecture, let me mention the followings:

Corollary.

K = separable closure of $\mathbf{F}_q((t))$. Assume the triplet (X, Y, a) is defined over $\mathbf{F}_q$, $y \in \mathrm{Fix\,Fr}^n \cdot a$, a_2 is finite near y, of degree d. If $q^n > d$ then a is contracting near y, and hence the local term is equal to the naive local term (by universal local acyclicity due to Deligne, local term remains unchanged by extension of the base field, so it is equal to the local term over $\bar{\mathbf{F}}_q$). Especially Deligne's conjecture is true if the second projection is finite (note that Grothendieck's trace formula for a power of Frobenius is the consequence of our result. I do not know if it has been known that the each term of his formula is equal to the local term of Lefschetz-Verdier formula).

Put $x = \alpha_2(y)$. Locally we embed $\mathcal{X}$ into the unit disk D^N with coordinate $(T_1, ..., T_N)$, $x \mapsto 0$. Take a continuous function $d : \mathcal{X} \to \mathbf{R}$, which is $\sup_i\{|T_i|\}$ ("distance from 0"). Then Fr^n makes the distance smaller, $d(\mathrm{Fr}^n(p)) \leq d(p)^{q^n}$. On the other hand, a_2^{-1}, since a_2 is assumed finite of degree d at x, makes it bigger, like $d(p)^{1/d}$. Finally $\mathrm{Fr}^n\, a_1 a_2^{-1}$ makes d smaller than (constant times of) $d^{q^n/d}$. By our assumption that $q^n > d$, we know that $\alpha = \mathrm{Fr}^n \cdot a$ is contracting near y.

As is clear from the argument, Frobenius makes a distance smaller and smaller, which is the essence of Frobenius. (It is possible to show the following: If a rigid space $\mathcal{X}$ is defined over $\mathbf{F}_q$, there is a distance d on $[\mathcal{X}]$ which satisfies

$d(\mathrm{Fr}(p), \mathrm{Fr}(p')) \leq C \cdot d(p,p')^q$, where C is a positive constant.) Though the special case of Deligne's conjecture in case of a_2 finite is sufficient in most applications, in the final version of [Fu 2] the author plans to treat the general case with the affirmative answer.

We will examine a method of getting a trace formula for weakly contracting correspondences of complex analytic varieties.

Let X be a compact complex analytic manifold, $f : X \to X$ a morphism, $K \in D^b_c(X, \mathbf{C})$ has a cohomological correspondence lifting f, i.e. $f^*K \simeq K$. $x \in X$, and assume this is isolated for simplicity.

Definition.

f is weakly contracting near x iff

a) x has a decreasing neighborhood $U_1 \supset U_2 \supset \ldots \supset U_n \supset \ldots$, $\cap_n U_n = \{x\}$.

b) $f(U_n) \subset U_{n+1}$, $\bar{U}_n$ subanalytic.

In this case, we prove that the local term at x is the naive local term, i.e. the trace of the endomorphism $f_x : K_x \to K_x$ induced on fiber at x. By a general formalism, we may assume $K_x = 0$.

Claim.

a) (Local Lefschetz formula) $\mathrm{loc}_x K = \mathrm{Trace}_{\mathbf{C}}(f|_{\bar{U}_n}, K|_{\bar{U}_n})$.

b) (Continuity) $\varinjlim_n H^q(\bar{U}_n, K|_{\bar{U}_n}) = H^q(x, K_x) = 0$, *where transition maps are restrictions.*

a) is proved using the 6-operations for subanalytic and subanalytic constructible sheaves (Lefschetz formula for $(\bar{U}_n, f|_{\bar{U}_n}, K|_{\bar{U}_n})$). b) holds quite generally for the projective system of compact Hausdorff spaces.

By the claim, it is natural to guess the local term is zero, since for big n the global trace on cohomologies will vanish (the cohomology itself tends to zero). But this is not true (consider a rotation around a point). So the contracting assumption is necessary.

We put $V_n = H^q(\bar{U}_n, K|_{\bar{U}_n})$. V_n is a finite dimensional vector space. Since $f|_{\bar{U}_n}$ factors into $\bar{U}_n \xrightarrow{f'} \bar{U}_{n+1} \xrightarrow{f''} \bar{U}_n$. We put $\beta_n = H^q(f')$, $\beta_n : V_{n+1} \to V_n$, $\gamma_n = H^q(f'')$, $\gamma_n : V_n \to V_{n+1}$ (restriction map). $\alpha_n = H^q(f|_{\bar{U}_n})$. So we have a system of finite dimensional vector spaces $V_i (i \in \mathbf{N})$, $\varinjlim_i V_i = 0$ (limit taken with respect to γ_n, $\alpha_n = \beta_n \cdot \gamma_n = \gamma_{n-1} \cdot \beta_{n-1}$.

Lemma.

For an inductive system of finite dimensional vector spaces $\{V_n\}_{n \in \mathbf{N}}$ with transition maps $\gamma_n : V_n \to V_{n+1}$, assume we are given $\beta_n : V_{n+1} \to V_n$, $\alpha_n = \beta_n \cdot \gamma_n$ which satisfy $\alpha_{n+1} = \gamma_{n-1} \cdot \beta_{n-1}$. Then if the inductive limit of $\{V_n\}_{n \in \mathbf{N}}$ is zero,

$$\mathrm{Trace}(\alpha_n) = 0 \ \forall n.$$

In fact, it follows that α_n is nilpotent: define an increasing filtration $W.$ on V_j by $W_s = \mathrm{Ker}(\gamma_{s+j} \cdot \gamma_{s+j-1} \cdot \cdot \gamma_j)$. By our assumption that the inductive limit is zero, it follows that $W_s = V_j$ for s big, and condition $\alpha_m = \beta_m \cdot \gamma_m = \gamma_{m-1} \cdot \beta_{m-1}$ for all m implies that $W.$ is preserved by α. By $\gamma_{s+j-1} \cdot \cdot \gamma_j$ Gr^W_s with α action is

identified with a subspace of $\mathrm{Ker}(\gamma_{s+j})$ with α_{s+j} action, and the latter action is zero from $\alpha_{s+j} = \beta_{s+j} \cdot \gamma_{s+j}$.
By the lemma, we have that the trace is zero, and hence the local term is zero.

We analyze the above proof to get the corresponding formula in rigid geometry. In the above proof, to get an appropriate local version of Lefschetz formula (Claim a)), we needed to introduce non-analytic subspaces such as $\bar{U}_n$, and 6-operations for subanalytic constructible sheaves are used. We need to have an analogue of it in rigid geometry (without defining "subanalytic spaces" if possible). The rest of the proof will be rather formal, with Claim b).
For claim a), we even need to think about the definition of the fixed point set. This was one motivation for defining the Zariski-Riemann space for a rigid space.

Theorem (Topological Lefschetz trace formula).

Let $U_s \subset Z_s$, $V_s \subset W_s$ be open sets, $\mathcal{U}$, $\mathcal{V}$ the rigid open subspaces defined by U_s, V_s, D a connected component of $\mathrm{Fix}\, a_{\bar{\eta}}$.

Assume $\mathcal{V} \subset \alpha_1^{-1}(\mathcal{U}) \cap \alpha_2^{-1}(\mathcal{U})$ and $\bar{\mathcal{V}} \cap \mathrm{set.Fix}\,\alpha = D^{an}$ (here $\mathrm{set.Fix}\,\alpha$ means the set theoretical fixed point set, i.e. the fixed point set as topological spaces).

Take $K \in D_{ctf}(X_{\bar{\eta}}, \Lambda)$, equipped with a cohomological correspondence c_η which lifts a_η. Assume $\alpha_2|_{\mathcal{V}} : \mathcal{V} \to \mathcal{U}$ is proper.
Then for $D \in \pi_0(\mathrm{Fix}\, a_{\bar{\eta}})$

$$\mathrm{loc}_D(a_{\bar{\eta}}, K_{\bar{\eta}}) = \mathrm{Lef}(u_s|_{\bar{U}_s}, j_*j^*R\psi_\eta(K)|_{U_s})).$$

Here $R\psi_\eta(K)$ is given the specialization c_s of c_η as the cohomological correspondence (discussed later), and Lef means the global trace defined by the correspondence.

Morally, the right hand side (global trace term) of the topological Lefschetz formula is equal to

$$\mathrm{Lef}(\alpha|_{\bar{\mathcal{U}}}, K|_{\bar{\mathcal{U}}})$$

where the cohomology is the rigid-etale cohomology (the author has not verified it completely). Since $\bar{\mathcal{U}}$ is not an analytic subset, the above formula looks like a Lefschetz formula in topology, so the naming was done, and we will be able to realize the topological proof in our abstract case using this formula.

For the proof we use the specialization formalism.
For $c \in \mathrm{coh.cor}(a_{\bar{\eta}}, K_{\bar{\eta}}) = H^0(Y, a_{\bar{\eta}}^!(DK_{\bar{\eta}} \boxtimes^{\mathbf{L}} K_{\bar{\eta}})$ we have the specialization $c_s \in \mathrm{coh.cor}(a_s, R\psi_\eta(K))$. This corresponds to the specialization of a cycle class, and defined by using the commutativity of the Verdier duality with the nearby cycle functor.
For the proof of the theorem, it is important to note the specialization formalism is compatible with the change of the model: If we have another (X', Y', a') dominating (X, Y, a) with the same generic fiber, the proper push forward of the specialization with respect to (X', Y', a') is equal to (X, Y, a).
So we can restrict the specialization correspondence c_s of $R\psi_\eta(K)$ to $(U_s, V_s, a_s|_{V_s})$, and extend it to a correspondence $\bar{c}_s$ of $Rj_*j^*R\psi_\eta(K)$ lifting $(\bar{U}_s, \bar{V}_s, a_s|_{\bar{V}_s})$. This extension is possible canonically since $a_2|_{V_s}$ is assumed proper over U_s. This is also

compatible with the change of models.
So we have the expression for the global trace as

$$\text{Lef} = \text{Trace}(a_s|_{\bar{U}_s}, Rj_*j^*R\psi_\eta(K)) = \Sigma_{\tilde{D}\in\pi_0(\text{Fix } a_s\cap\bar{V}_s)}\text{loc}_{\tilde{D}}(a_s|_{\bar{U}_s}, Rj_*j^*R\psi_\eta(K)).$$

We use the following lemma:

Lemma.

There is a good model (X', Y', a') such that there is no connected component $\tilde{D}$ of $\text{Fix}\, a'_s$ which intersects with $\bar{V}_s \setminus V_s$.

In the proof of the lemma, the quasi-compactness of the Zariski-Riemann space is used: Since the claim is true on the Zariski-Riemann space by our assumption, it should be satisfied with some model.

By replacing the original model by the one constructed in the lemma, we may assume that the there is no fixed point on $\bar{V}_s \setminus V_s$. So the global trace is rewritten as

$$\Sigma_{\tilde{D}\subset\text{Fix } a_s|_{V_s}, \text{ proper}}\ \text{loc}_{\tilde{D}}(a_s, R\psi_\eta(K)).$$

The last term is equal to $\Sigma_{D\in\text{Fix } a_\eta}\text{loc}_D(a_{\bar{\eta}}, K_{\bar{\eta}})$, which is equal to $\text{loc}_D(a_{\bar{\eta}}, K_{\bar{\eta}})$ since there is only one connected component by our assumption. The last equality is checked by formalism of specialization, but intuitively explained as follows: local terms are some intersection multiplicity, coefficient of $c_{\bar{\eta}}\cdot\Delta_{X_\eta}$, and by specialization invariance of intersection number the sum should be equal to $c_s\cdot\Delta_{U_s}$, which is our claim. The intersection should be compact to make this argument rigorous. This is satisfied for our model.

For Claim b), by the continuity of topos cohomology with respect to 2-projective limit, SGA 4, it is checked in case of torsion coefficient. But note that our proof will be ineffective for non-integral coefficient rings (the above lemma is valid only for a field or integral rings). So we can not prove any statement for $\Lambda = \mathbf{Z}/\ell^n\mathbf{Z}$, $n \geq 2$. This forces us to work with $\mathbf{Z}_\ell$ or $\mathbf{Q}_\ell$ directly. But for such coefficient, i.e. ℓ-adic cohomologies, which is a continuous cohomology [Ek], [Ja] it is not calculated by injective resolution of sheaves but rather by pro-sheaves. This implies that there are no general machinery such as SGA 4 and the claim itself is false in general (in the first version of [Fu 2] this problem, peculiar to ℓ-adic case, was unnoticed). Fortunately, in our case of rigid-etale cohomology, such continuity result is true in the necessary cases:

Theorem.

Let X be a proper algebraic variety over K, K a height one valuation field which is algebraically closed, Y a closed subvariety. For an ℓ-adic constructible sheaf K, ℓ invertible in the integer ring of K,

$$\varinjlim_{\mathcal{U}} H^q(\mathcal{U}_{et},\ K^{rig}|_{\mathcal{U}}) = H^q(Y_{et}, K|_Y)$$

holds. Here $\mathcal{U}$ runs over all quasi-compact neighborhood of Y^{rig} in X^{rig}.

By an introduction of a suitable local version of the statement, we prove it by a variant of Deligne's technique. The claim is reduced to X smooth, Y smooth divisor and $K = j_!\mathcal{F}$, $\mathcal{F}$ a tame smooth sheaf on $X \setminus Y$. In this case the claim follows by a direct calculation. The details will be found in [Fu2].

Since we have claims a), b) in the abstract case, the argument in the classical case applies, and the calculation of the local terms is done for contracting correspondences.

§4. Grothendieck's absolute purity conjecture

In the following we discuss the relation between Grothendiek's absolute purity conjecture and Hironaka resolution.

Grothendieck has conjectured the following: X = Spec R, R a strictly hensel regular local ring, $D = V(f) \subset X$ a regular divisor. Then for n invertible on X

$$H^i_{et}(X\backslash D,\ \Lambda) = 0$$

if $i > 1$, $\Lambda = \mathbf{Z}/n\mathbf{Z}$. (For $i = 0,\ 1$ the group is easy to calculate.) Note that the conjecture is quite essential in the construction of cycle classes on general regular schemes. Moreover this conjecture implies the following: Assume the dimension of X is greater than 1. Then $Br(X \setminus s)_\ell = 0$. Here s denotes the closed point, Br means the Brauer group (we can take cohomological Brauer group) and ℓ is a prime invertible on X. (Gabber announced that he can prove this purity for Brauer group [Ga 2], but the proof is unfortunately unpublished except $\dim X \leq 3$.)

First we note the following: Assume the truth of the conjecture in dimension less than N. Then the truth of the conjecture for any complete local R implies the general case. Especially we can assume the excellence of R in the study of absolute purity. By this remark and previous known results, we deduce the following:

Claim.

Absolute purity is true in the following cases:
a) $\dim X \leq 2$ ([Ga 3] *)*.
b) X *is of equicharacteristic (use relative purity, SGA4 XVI 3.7, over a field).*
c) X *is the henselization of a finite type scheme over* $\mathbf{Z}$ *or* $\mathbf{Z}_p$*, and any prime divisor of* n *is bigger compared with the dimension of* X ([Thom] *)*.

We try to explain how this conjecture is related to the birational geometry of X. In fact, our approach is similar to Hironaka's proof of " non-singular implies rational " in the continuous coefficient case. In his proof his strong form of resolution of singularity was used, and we will try to do the same thing in the discrete coefficient case. But it turns out that the spectral sequence involved are too complicated in the naive approach, so we will use log-structures of Fontaine-Illusie-Kato to avoid the difficulty.

The form of embedded resolution we want to use is the following:
For a pair $(X,\ Y)$, where X is a quasi-excellent regular scheme and Y is a reduced normal crossing divisor, we define a good blowing up $(X',\ Y')$ by X' is the blowing up of X along D, where D is a regular closed subscheme of X which cross normally with Y. (The last condition implies that etale locally we can find a regular parameter system $\{f_j\}$, $1 \leq j \leq n$ such that Y is defined by $\prod_{i=1}^m f_i = 0$ and D is defined by $\{f_j = 0,\ j \in J\}$ for a subset J of $\{1,,...n\}$.) $Y' =$ total transform of Y_{red}.

We say $\pi : (X',\ Y') \to (X,\ Y)$ is a good modification if π is a composition of good blowing ups. The point is we can control normal crossing divisors.

Conjecture (Theorem of Hironaka in characteristic 0 [H]).

Let $\mathcal{C}_{X,Y}$ be the category of all good modifications of (X, Y), and $\mathcal{B}_{X,Y}$ the category of proper modifications of X which becomes isomorphic outside Y. Then $\mathcal{C}_{X,Y}$ is cofinal in $\mathcal{B}_{X,Y}$.

Note that it is even not clear that $\mathcal{C}_{X,Y}$ is directed. Since any element in $\mathcal{B}_{X,Y}$ is dominated by admissible blowing ups, this conjecture is equivalent to the existence of a good modification which makes a given admissible ideal invertible.

So the conjecture is a strong form of simplification of coherent ideals, which is shown by Hironaka in characteristic zero. It is easy to see the validity of conjecture in dimension 2, but I do not know if it is true in dimension 3.

The implication of the conjecture in rigid geometry is the following: We define the tame part $T_{Y/X}^{tame}$ of $T_{Y/X} = \mathcal{X}_{rig-et}$ by

$$T_{Y/X}^{tame} = \varprojlim_{(X',\ Y')\in \mathcal{B}_{X,\ Y}} Y'_{log-et}$$

Here we give X' the direct image log-structure from $X'\backslash Y'$, and Y' the pullback log-structure. The limit is taken in the category of toposes. Since Y' is normal crossing, the behavior is very good. By the conjecture, we can determine the points of this tame tubular neighborhood (note that the topos has enough points by Deligne's theorem) .

Lemma.

Let $\epsilon : T_{Y/X}^{tame} \to T_{Y/X}^{unr}$ be the canonical projection (defined using the conjecture). Then for a point x of $T_{Y/X}^{unr}$, which corresponds to strictly hensel valuation ring $V = V_x^{sh}$, the fiber product $T_{Y/X}^{tame} \times_{T_{Y/X}^{unr}} (\mathrm{Sph}\, V)^{unr}$ is equivalent to $(\mathrm{Sph}\, V)^{tame}$.

So the points above x is unique up to non-canonical isomorphisms, which corresponds to the integral closure of V in the maximal tame extension of the fraction field of V. Using this structure of points we have

Proposition.

For any torsion abelian sheaf $\mathcal{F}$ on $T_{Y/X}^{tame}$ order prime to residual characteristics, we have

$$R\alpha_*\alpha^*\mathcal{F} = \mathcal{F}.$$

Here α denotes the projection from $T_{Y/X}$.

This is just the fiberwise calculation (α is cohomologically proper), using that the Galois cohomology of henselian valuation fields without any non-trivial Kummer extension. (This part is completely the same as one dimensional cases.) Then our theorem is the following:

Theorem.

The conjecture implies Grothendieck's absolute purity conjecture.

To see this, we use comparison theorem first.

$$R\Gamma(X\backslash Y,\ \Lambda) = R\Gamma(T_{Y/X},\ \Lambda)$$

By the proposition, this is equal to $R\Gamma(T^{tame}_{Y/X},\ \Lambda)$. So we want to calculate this cohomology. Since the topos T^{tame} is defined as a 2-projective limit, we have

$$H^q(T^{tame}_{Y/X},\ \Lambda) = \varinjlim_{(X',\ Y')\in\mathcal{C}_{X,Y}} H^q(Y'_{log-et},\ \Lambda)$$

So we conclude by the following lemma:

Lemma.
For a good modification $\pi : (X',\ Y') \to (\tilde{X},\ \tilde{Y})$

$$R\pi'_*\Lambda = \Lambda,$$

where $\pi' : Y'_{log-et} \to \tilde{Y}_{log-et}$.

In fact, this is a consequence of the absolute purity conjecture. To prove the lemma, we may assume that π is a good blowing up. In this case we use proper base change theorem in log-etale theory, and reduce the claim to equicharacteristic cases. Especially to the relative purity theorem over a prime field.

The details will be found elsewhere [Geometric Ramification Theory], in preparation.

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