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P-adic Symmetric Spaces: The Unitary Group acting on the Projective Plane

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Introduction

We study the group $SU_3(L)$ acting on the projective plane P_L^2. Here L is a quadratic extension of some complete non-archimedean local field K. We vieuw $SU_3(L)$ as an algebraic group defined over K. Let Y be the analytic space consisting of the points $x \in P_L^2$ which are stable for every maximal K-split torus T in $G(K) = SU_3(L)$. We construct a pure $G(K)$-invariant affinoid covering of Y. The components of the reduction of Y with respect to this covering are all proper. This is equivalent to giving a formal scheme $\mathcal{X}$ over the ring K^0 of the integers in K with generic fibre $\mathcal{X} \otimes K = Y$ and with as its closed fibre the reduction of Y.

The number of orbits of $SU_3(L)$ on the set of components of the reduction of Y is not finite. In particular the quotient Y/Γ is not proper for a discrete co-compact subgroup $\Gamma \subset SU_3(L)$.

1 Some Background Information

1.1 Real Symmetric Spaces

Let G be a *connected semisimple non-compact linear algebraic group* defined over the field of real numbers R. Let $\mathbf{C}^{max}$ be a *maximal compact subgroup* of $G(R)$. The maximal compact subgroups of $G(R)$ are all conjugated. The *symmetric space* belonging to $G(R)$ is the space $G(R)/\mathbf{C}^{max}$.

Let X be a *projective homogeneous variety* for $G(C)$, where C denotes the field of complex numbers. So $X = G(C)/P(C)$, where $P(C) \subset G(C)$ is a *parabolic subgroup*. Let $\mathbf{C}$ be a *compact subgroup* of $G(R)$. Then the space $Y := G(R)/\mathbf{C}$ is called a *flag domain* for $G(R)$ if there exists an embedding $Y \hookrightarrow X$, for some X as above, such that Y is an open analytical subspace.

A symmetric space is called *hermitian* if it is also a flag domain, i.e. embeddable in some projective homogeneous variety. Next we state some properties that are useful for defining p-adic symmetric spaces.

A flag domain is an open $G(R)$-orbit $Y \subset X = G(C)/P(C)$ such that every discrete cocompact subgroup $\Gamma \subset G(R)$ acts discontinuously on Y and Y/Γ is a *compact complex analytical space* .

If Y is a hermitian symmetric space then the parabolic subgroup $P(C) \subset G(C)$ is a *maximal parabolic subgroup.*

Let $\Gamma \subset G(R)$ be a discrete subgroup with finite co-volume (i.e. $vol(G(R)/\Gamma) < \infty$) and let Y be a hermitian symmetric space. Then there exists a compactification of Y/Γ such that the compactification is a *projective algebraic variety* . In particular when $\Gamma \subset G(R)$ is co-compact then Y/Γ itself is a projective variety. For flag domains which are not symmetric spaces, Y/Γ is not an algebraic variety in general.

1.2 Definition of p-adic Symmetric Spaces

Let $K \supset Q_p$ be a finite extension of the field of p-adic numbers Q_p. Let G be an *absolutely simply connected semisimple linear algebraic group* defined over K. Let $X = G/P$ be a *projective homogeneous variety* and $P \subset G$ a *parabolic subgroup.* Let $\Gamma \subset G(K)$ be a *discrete co-compact subgroup.*

A p-adic analytical space Y is called a *symmetric space* for $G(K)$ if it satisfies the following four conditions:

1) Y is an open $G(K)$-invariant subspace of some projective homogeneous variety $X = G/P$.

2) Y/Γ can be compactified to some proper analytical variety Z

3) P is a maximal parabolic subgroup

4) Z is (the analytification of) an algebraic variety.

Conditions 1 and 2 together define p-adic flag domains. There exist p-adic flag domains satisfying condition 3. For example one has the flag domains for $SL_n(K)$ which are contained in the Grassmann variety $Gr(i, n)$ if $g.c.d.(i, n) = 1$. There exist p-adic flag domains satisfying condition

4. In this case the flag domain $Y \subset G/P$ is a flag domain for $SL_n(K)$ and one has a SL_n-equivariant projection $\varphi : G/P \mapsto P^{n-1}$ such that $Y = \varphi^{-1}(\Omega_{n-1})$, where Ω_{n-1} denotes Drinfeld's symmetric space P_K^{n-1}-$\{K$-rational hyperplanes$\}$.

1.3 A Construction of p-adic Flag Domains

There is a construction giving flag domains for groups G as above (See [vdPV] and [Vo]). The construction works as follows:

Take an ample line bundle $\mathcal{L}$ on a projective homogeneous variety X. Take a G-linearization of this line bundle $\mathcal{L}$. This induces a T-linearization of $\mathcal{L}$. Here $T \subset G$ is a maximal K-split torus. Let $X^s(T, \mathcal{L})$ denote the set of points which are stable for T with respect for this T-linearization. Let $X^{ss}(T, \mathcal{L})$ denote the set of semistable points. Then the set $Y := \bigcap_{g \in G(K)} g \cdot X^s(T, \mathcal{L})$ consisting of the points stable for all maximal K-split tori in G is a flag domain for $G(K)$ if $X^s(T, \mathcal{L}) = X^{ss}(T, \mathcal{L})$. The only symmetric spaces one finds this way are Drinfeld's symmetric spaces P_K^{n-1}-$\{$ K-rational hyperplanes$\}$ for $SL_n(K)$.

The flag domains Y one finds this way all have the property that Y/Γ is proper for any discrete co-compact subgroup $\Gamma \subset G(K)$. Furthermore if the complement of the set of stable points $X^s(T, \mathcal{L})$ in X has codimension larger than or equal to two in X then Y/Γ has no meromorphic functions except for the constants. This makes it somehow interesting to study the cases where $codim(X - X^s(T, \mathcal{L})) = 1$ even when the sets of stable and semistable points are not the same.

1.4 Rigid Analytic Geometry

Since p-adic analytic geometry is not so well known it is probably a good idea to say a little bit about it. For more information on the subject we refer to [FvdP] and [BGR].

The basic building blocks of p-adic analytic geometry are *affinoid spaces.* They are somewhat like affine spaces in algebraic geometry. The basic example of an affinoid space is the *p-adic unit ball* $B_n := \{(x_1, \ldots, x_n) \in (K^{alg})^n \| x_i| \leq 1\}/Gal(K^{alg}/K)$, where K^{alg} denotes the algebraic closure of K. Assocated with B_n is a ring of power series converging on B_n. It is the *affinoid algebra* $T_n := K < x_1, \ldots, x_n >:= \{\sum a_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n} | \lim_{|\alpha| \to \infty} |a_\alpha| = 0\}$.

Now B_n is the set of *maximal ideals* of T_n. General affinoid algebras are of the form T_n/I, where $I \subset T_n$ is an ideal. So general affinoid spaces are of the form $Sp(T_n/I) = \{x \in B_n | \forall f \in I, f(x) = 0\}$.

Let A be an affinoid algebra and let $Sp(A)$ be the corresponding affinoid space. For $f \in A$ and $x \in Sp(A)$ we denote by $f(x)$ the image of f in A/x. Since A/x is a finite extension of K and K is complete, the valuation $| \ |$ of K extends uniquely to a valuation of A/x. Hence $|f(x)|$ is well defined. On A we have a (semi-)norm, called the *spectral (semi-)norm* $\| \ \|$ defined by $\|f\| := \sup_{x \in Sp(A)} |f(x)|$. The spectral semi-norm is a norm if there are no nilpotent elements $\neq 0$ in A.

Let K^0 denote the ring of integers of K, i.e. $K^0 := \{x \in K | |x| \leq 1\}$. Let $A^0 \subset A$ denote the K^0-module $A^0 := \{f \in A | \|f\| \leq 1\}$ and let $A^{00} \subset A^0$ be the K^0-module $A^{00} := \{f \in A | \|f\| \leq 1\}$. We call $\bar{A} := A^0/A^{00}$ the *reduction* of A and *spec*$(\bar{A})$ the *reduction* of $Sp(A)$. One has a *reduction map* $R : Sp(A) \longrightarrow Spec(\bar{A})$. The image $R(m)$ of a maximal ideal $m \subset A$ is the maximal ideal $(m \bigcap A^0)/A^{00}$.

Let $\pi \in K$ be a non-zero element such that $|\pi| < 1$ and let A be the affinoid algebra $A := K < z_1, \ldots, z_n > /I$, where I is some ideal of
$K < z_1, \ldots, z_n >$. Then one has:
$A^0 = \lim_{\leftarrow} A^0/\pi^s A^0 = \lim_{\leftarrow}(K^0[z_1, ..., z_n]/I)/\pi^s(K^0[z_1, ..., z_n]/I)$. The formal affine scheme $Spf(A^0) \subset spec(K^0[z_1, ..., z_n]/I$ is the subspace defined by the ideal generated by π. The map $Spf(A^0) \longrightarrow Spf(K^0)$ has $Spf(A^0) \bigotimes K$ as its generic fibre and *spec*$(\bar{A})$ as its closed fibre. The closed points in the generic fibre correspond to the points of the affinoid space $Sp(A)$ and the closed fibre corresponds with the reduction *spec*$(\bar{A})$ of $Sp(A)$. Since the reduction $\bar{A}$ of A is reduced this gives us a correspondance between affinoid spaces $Sp(A)$ over K and reduced formal affine spaces $Spf(A^0)$ over K^0.

Next we define a pure affinoid covering of a rigid analytic space X. A *pure affinoid covering* $\{X_j\}_{j \in J}$ is a covering of X by affinoid spaces X_j, $j \in J$ such that:

1)for each $j \in J$, X_j intersects only a finite number of X_i

2)if $X_j \bigcap X_i \neq \emptyset$ then there exists an open affine subvariety A_{ij} in the reduction $\bar{X}_j$ of X_j such that $X_i \bigcap X_j = R_j^{-1}(A_{ij})$ and is an open affinoid subspace of X_j with A_{ij} as its reduction. Here $R_j : X_j \longrightarrow \bar{X}_j$ denotes the reduction map.

A pure affinoid covering of X is a covering of X such that the reductions of the affinoid spaces glue together nicely. In particular we can glue the formal affine schemes associated with the affinoid spaces together into a formal scheme with as its generic fibre the analytic space X and as its closed fibre the reduction of X with respect to this covering.

2 A Pure Affinoid Covering

2.1

Let $L \subset K$ be an algebraic extension of degree 2 and let $^-$ denote the generator of the Galois group $Gal(L/K)$. In this paragraph we look at $SU(L)$ acting on the projective homogeneous variety P^2_L. The unitary form is given by $x_1\bar{y}_2 + x_2\bar{y}_1 + x_0\bar{y}_0$ w.r.t. a basis e_0, e_1, e_2 of P^2_L.

We vieuw SU_3 as a group defined over K, i.e. $G(K) = SU_3(L)$. Now $G(K)$ acts on a variety $\tilde{X}$ defined over K such that $\tilde{X} \otimes L$ consists of two connected components each isomorphic to P^2_L. The Galois group $Gal(L/K)$ permutes the two components. We take one connected component $X \cong P^2_L$.

A maximal K-split torus $T \subset G$ has the form $diag(1, t, \bar{t}^{-1})$. Let us look at the usual G-linearization of $\mathcal{O}(1)$. The set of stable points $X^s(T, \mathcal{O}(1))$ is given by $x_1x_2 \neq 0$. The set of semistable points $X^{ss}(T, \mathcal{O}(1))$ consists of the points x with $x_1x_2 \neq 0$ or $x_0 \neq 0$. Note that the complement of the set of stable points has codimension 1 in P^2_L. It consists of the two lines given by $x_1 = 0$ and $x_2 = 0$.

Let $Y := \bigcap_{g \in SU_3(L)} g \cdot X^s(T, \mathcal{O}(1))$ be the set of points stable for every maximal K-split torus in $G(K)$. We construct a pure $G(K)$- invariant affinoid covering of Y.

2.2 The Building of $SU_3(L)$

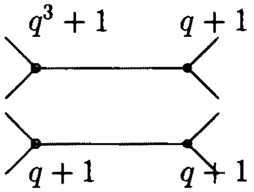
In this case the Bruhat-Tits building $\mathcal{B}$ is a tree. It can be defined by using L^0-submodules of P^2_L, where L^0 denotes the ring of integers of the field L. Actually they are equivalence classes of submodules of a vector space $V \cong L^3$ with $P(V) \cong P^2_L$. Two modules M_1 and M_2 are equivalent if and only if there exists a $\lambda \in L^*$ such that $M_1 = \lambda M_2$.

There are two types of vertices in the building. One type of vertices correspond to the $SU_3(L)$-images of the L^0-module $< e_0, e_1, e_2 >$. The other vertices correspond to the $SU_3(L)$ images of the L^0-module $< e_0, \pi e_1, e_2 >$. Here π is a generator of the maximal ideal of L^0. Two vertices of the building are joined by an edge if and only if they are the $SU_3(L)$-image of the edge joining the vertices $< e_0, e_1, e_2 >$ and $< e_0, \pi e_1, e_2 >$.

Note that the vertex $< e_0, \pi e_1, e_2 >$ has a degenerated unitary form on it when reduced modulo π. In particular we could also have represented this vertex of the building by the dual L^0-module $< e_0, e_1, \pi^{-1}e_2 >$. So the choice of modules is not unique.

The stabilizers in $SU_3(L)$ of the modules are the *maximal parahoric* (i.e. maximal compact) subgroups of $SU_3(L)$. To each maximal K-split torus $T \subset SU_3(L)$ belongs an *apartment* in the building. The vertices of the apartment belonging to T correspond to the modules that have an L^0-basis such that T acts diagonally w.r.t. this basis.

The shape of the building depends on wether the extension $L \supset K$ is ramified or not. One has:



if $L \supset K$ is unramified

if $L \supset K$ is ramified

• vertex corresponding to a degenerated module

Here q is the number of elements in the residue field of K.

In P_L^2 there are also L^0-submodules that do not correspond with vertices of the building. They are the modules $< e_0, \pi^n e_1, \pi^m e_2 >$ with $|n+m| > 1$. They correspond with segments of the building. A *finite segment* $S := [S_1, S_2]$ in the building $\mathcal{B}$ is the smallest connected part of the building containing the two points S_1 and S_2 of the building. So a finite segment is a path joining two points S_1 and S_2 and is contained in any apartment that contains both S_1 and S_2. An *infinite segment* will be either an apartment or a half-apartment in the building. We have the following lemma:

2.3 Lemma

Let M be the L^0-module $M =< e_0, \pi^n e_1, \pi^m e_2 >$ with $|n+m| > 1$. Then the stabilizer P_M in $SU_3(L)$ of M is the stabilizer P_S of the segment S joining the two vertices in $\mathcal{B}$ corresponding to the modules $< e_0, \pi^n e_1, \pi^{-n+1} e_2 >$ and $< e_0, \pi^{-m+1} e_1, \pi^m e_2 >$ if $n+m > 1$ and to the modules $< e_0, \pi^{-m} e_1, \pi^{m+1} e_2 >$ and $< e_0, \pi^{n+1} e_1, \pi^{-n} e_2 >$ if $n+m < 1$.

Proof: Let M be $M =< e_0, \pi^n e_1, \pi^m e_2 >$ and let us assume that $n+m > 1$. Let $M_i :=< e_0, \pi^i e_1, \pi^{-i} e_2 >$ and $N_i :=< e_0, \pi^i e_1, \pi^{-i+1} e_2 >$. Then $M =$

$\bigcap_{i=n-1}^{-m+1} M_i \cap \bigcap_{i=n}^{-m+1} N_i$. Hence the stabilizer P_M of the module M contains the group $H := \bigcap_{i=n-1}^{-m+1} P_{M_i} \cap \bigcap_{i=n}^{-m+1} P_{N_i}$. Clearly the group H also stabilizes the segment S joining the vertices corresponding to the modules N_n and N_{-m+1} in the building. Furthermore P_M contains an element w which permutes $\pi^n e_1$ and $\pi^m e_2$ and maps e_0 to $-e_0$. This element w maps the module N_n to N_{-m+1} and vice versa. Therefore w is also contained in P_S. It is easy to see that both P_S and P_M are generated by H and w. Therefore $P_S = P_M$. The proof for the case when $n + m < 1$ is similar.

2.4 A Pure Affinoid Covering for $X^s(T, \mathcal{O}(1))$

Each polyhedron Δ in the picture below defines an affinoid space. The union of these affinoid spaces gives a pure affinoid covering of $X^s(T, \mathcal{O}(1))$. The affinoid spaces $X_{\Delta,A}$ associated with the polyhedra Δ in the picture are as follows:

If Δ is one of the infinite polyhedra then one has

$X_{\Delta,A} := \{x \in P_L^2 \| \pi^{2n}| \leq |\frac{x_1}{x_2}| \leq |\pi^{2n-1}, |\frac{x_0}{x_1}| \leq |\pi^{-n}|\}$ or

$X_{\Delta,A} := \{x \in P_L^2 \| \pi^{2n+1}| \leq |\frac{x_1}{x_2}| \leq |\pi^{2n}|, |\frac{x_2}{x_0}| \leq |\pi^{-n}|\}$

If Δ is a triangle then one has

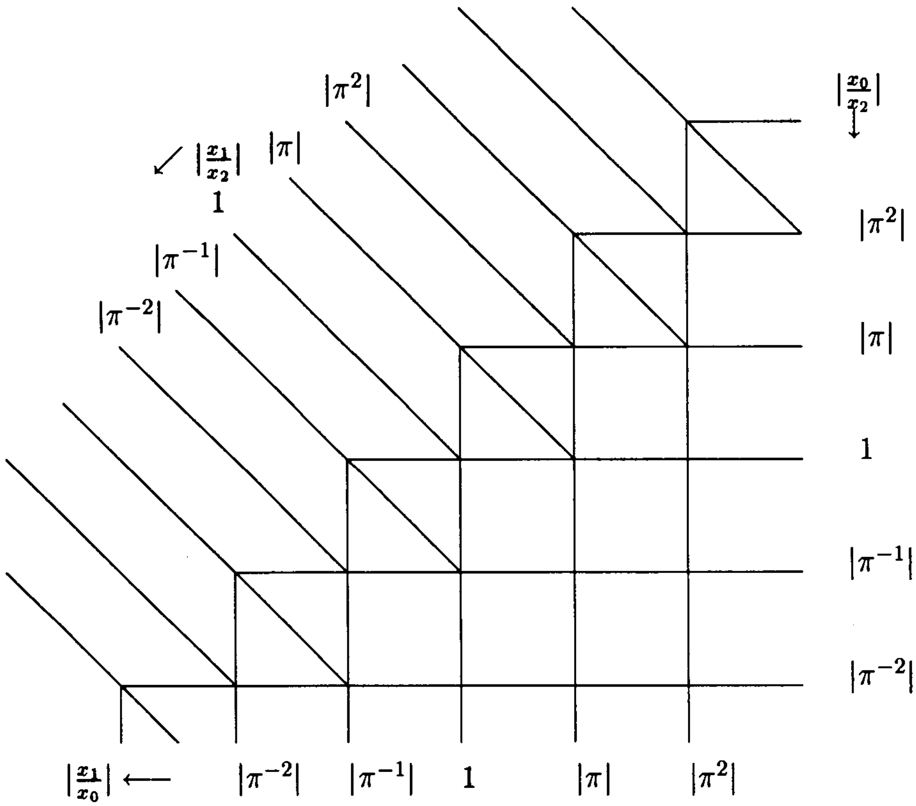
$X_{\Delta,A} := \{x \in P_L^2 \| \pi^n| \leq |\frac{x_1}{x_0}| \leq |\pi^{n-1}|, |\pi^n| \leq |\frac{x_0}{x_2}| \leq |\pi^{n-1}|, |\frac{x_1}{x_2}| \leq |\pi^{2n-1}|\}$ or

$X_{\Delta,A} := \{x \in P_L^2 \| \pi^n| \leq |\frac{x_1}{x_0}| \leq |\pi^{n-1}|, |\pi^n| \leq |\frac{x_0}{x_2}| \leq |\pi^{n-1}|, |\frac{x_1}{x_2}| \geq |\pi^{2n-1}|\}$.

If $X_{\Delta,A}$ is one of the squares then

$X_{\Delta,A} := \{x \in P_L^2 \| \pi^n| \leq |\frac{x_1}{x_0}| \leq |\pi^{n-1}|, |\pi^m| \leq |\frac{x_0}{x_2}| \leq |\pi^{m-1}|\}$, with $n - m \geq 0$.

The scaling in the picture below is logarithmic.



The vertices of the polyhedra correspond to L^0-modules $< e_0, \pi^n e_1, \pi^m e_2 >$ with $n + m \geq -1$. In particular the triangles correspond with three modules that define a chamber in the building. They are $SU_3(L)$-images of: $< e_0, e_1, e_2 >$, $< e_0, \pi e_1, e_2 >$, $< e_0, e_1, \pi^{-1} e_2 >$. The infinite polyhedra correspond with two modules that together also define a chamber. They are $SU_3(L)$-images of: $< e_0, e_1, e_2 >$ and $< e_0, \pi e_1, e_2 >$ or of $< e_0, e_1, e_2 >$ and $< e_0, \pi^{-1} e_1, e_2 >$. The squares correspond to four modules. They are: $< e_0, \pi^n e_1, \pi^m e_2 >$, $< e_0, \pi^{n+1} e_1, \pi^m e_2 >$, $< e_0, \pi^n e_1, \pi^{m+1} e_2 >$, $< e_0, \pi^{n+1} e_1, \pi^{m+1} e_2 >$. Where $n + m \geq 0$.

To each polyhedron Δ we associate the compact subgroup P_Δ of $SU_3(L)$ that leaves the modules associated with Δ invariant. So we have $P_\Delta = \bigcap P_M$, where M is in the set of modules corresponding with the vertices of the polyhedron Δ. In particular to the triangles and the infinite polyhedra we associate the stabilizer of a chamber (i.e. edge) of the building. The modules associated with the squares correspond all with segments in the building. These segments are contained in the longest segment. The stabilizer of the longest segment permutes the other segments corresponding with the modules associated with vertices of the square. Therefore we associate to the square a subgroup of the stabilizer of this longest segment. It is the group denoted by H in the proof of the previous lemma.

One easily proofs the following:

2.5 Lemma

1) *For the finite polyhedra Δ we have for all $x \in X_{\Delta,A}$:*
$|\frac{g^* x_i}{x_i}(x)| \leq 1$, $\forall g \in P_\Delta$, $i = 0, 1, 2$.
2) *For the infinite polyhedra Δ we have for all $x \in X_{\Delta,A}$:*
$|\frac{g^* x_i}{x_i}(x)| \leq 1$, $i = 1, 2$. $|\frac{g^* x_0 g^* x_0}{x_1 x_2}| \leq 1$, $\forall g \in P_\Delta$

2.6 A Pure Affinoid Covering of Y

Let $X_{\Delta,A}$ denote the affinoid space associated with a polyhedron Δ and apartment A. Then $X_\Delta^\sharp := \bigcap_{g \in P_\Delta} X_{\Delta, gA} = \bigcap_{g \in P_\Delta} g \cdot X_{\Delta,A} \subset X_{\Delta,A}$ is an open affinoid subspace. It follows from the lemma above that $X_\Delta^\sharp$ is obtained by taking the inverse image of the reduction map of an open subset of the reduction of $X_{\Delta,A}$.

Let $\mathcal{P}_B$ denote the set of polyhedra associated with the building. To get a pure affinoid covering $\{X_\Delta | \Delta \in \mathcal{P}_B\}$ of Y we take open affinoid subspaces X_Δ in $X_\Delta^\sharp$. If Δ is a triangle or an infinite polyhedron then we take $X_\Delta := X_\Delta^\sharp$.

Now take Δ to be a square. Let $M_{s,t} :=< e_0, \pi^s e_1, \pi^t e_2 >$, $s = n, n+1$, $t = m, m+1$ be the modules assocciated with Δ. Let $H_\Delta := \bigcap_{C \in S_{M_{n,m}}} P_C$ whenever the segment $S_{M_{n,m}}$ is not a vertex. If $S_{M_{n,m}}$ is a vertex S we take $H_\Delta := P_S$. Let us for $g \in H_\Delta$ denote by f_g the function $f_g(x) := \frac{g^* x_1 g^* x_2}{g^* x_0^2}(x)$. Here the x_i are the standard coordinates associated with the basis e_i, $i = 0, 1, 2$. Our definition of $X_\Delta^\sharp$ is such that any $x \in X_\Delta^\sharp$ satisfies:
$|\pi^{n+m+2}| \leq |f_g(x)| \leq |\pi^{n+m}|$ for all $g \in H_\Delta$ with $g(S_\Delta) = S_\Delta$.
A point $x \in X_\Delta^\sharp$ also satisfies:
$\forall (g \in H_\Delta) |f_g(x)| \leq |\pi^{n+m}|$.
Now we can define X_Δ for squares:
$X_\Delta := \{x \in X_\Delta^\sharp \| f_g(x)| = |\pi^{n+m}|$ for all $g \in H_\Delta$ with $g(S_\Delta) \bigcap S_\Delta = S_{M_{n,m}}\}$.

2.7 Theorem

The affinoid spaces X_Δ form a pure affinoid covering of
$Y = \bigcap_{g \in SU_3(L)} g \cdot X^s(T, \mathcal{O}(1))$.
The components of the reduction of Y with respect to this covering are proper.
The components of the reduction correspond 1-1 with the $SU_3(L)$-images of the L^0-modules $< e_0, \pi^n e_1, \pi^m e_2 >$ with $n + m \geq -1$.

Proof: The purity of the covering $\{X_\Delta | \Delta \in \mathcal{P}_B\}$ will be proved in the next paragraph. Also the fact that the covering gives all of Y will be proved in paragraph 3.

The proof that the components of the reduction are proper is essentially the same as in [vdPV]. The fact that the components correspond with modules as given above is clear from the construction.

2.8 Remark

The $SU_3(L)$ orbits of the components of the reduction are represented by the modules $< e_0, e_1, \pi^n e_2 >$, $n \geq -1$.

If Δ is a triangle and Δ' is an infinite polyhedron such that both determine the same chamber (i.e. edge) in the building, then $X_C := X_\Delta \bigcup X_{\Delta'}$ is also an affinoid space. The covering $\{X_C, X_\Delta | C$ a chamber, Δ a square $\}$ is again pure. The components of the reduction of Y with respect to this covering correspond with the $SU_3(L)$ images of the modules $< e_0, \pi^n e_1, \pi^m e_2 >$, $n + m \geq 0$.

3 Torus Invariants

3.1

In this paragraph we study the torus invariants in some detail. This will enable us to complete the proof of theorem 2.7.

We fix a chamber C_0 in a fixed apartement A_0. There is a maximal K-split torus T_0 associated to A_0. We have basis e_0, e_1, e_2 of P_L^2 such that T_0 acts diagonally and the hermitian form has the standard form. We take as C_0 the chamber defined by the L_0-modules $< e_0, e_1, e_2 >$ and $< e_0, \pi e_1, e_2 >$.
For $x \in X^s(T_0, \mathcal{O}(1))$ we define: $r_g(x) := |\frac{g^*x_1 g^*x_2}{x_1 x_2}(x)|$
For $x \in X = P_L^2$ we define $r(x)$ as follows:
$r(x) := \inf_{g \in G(K)} |\frac{g^*x_1 g^*x_2}{x_1 x_2}(x)|$ if $x \in X^s(T_0, \mathcal{O}(1))$ and
$r(x) = 0$ if this is not the case.
For the chamber $C_0 \in A_0$ we define the following analytic space:
$Z_{C_0, A_0} := \{x \in P_L^2 | |\pi| \leq |\frac{x_1}{x_2}(x)| \leq 1\}$.
We take for $g \in SU_3(L)$ $Z_{gC_0, gA_0} := g(Z_{C_0, A_0})$. Note that Z_{C_0, A_0} is not an affinoid space. The union $\bigcup_{C \in A} Z_{C,A} = X^s(T, \mathcal{O}(1))$ where T is the torus belonging to A.

3.2 Proposition

$r(x) = 0 \Longleftrightarrow \exists (g \in G(K)) g^*x_1 g^*x_2(x) = 0$

Proof: The $\Longleftarrow$ part is trivial. So let us assume that $r(x) = 0$. If $x \notin X^s(T_0, \mathcal{O}(1))$ then we can take $g = id.$, so we may assume that this is not the case. Take a sequence $g_i \in G(K)$ such that $r_{g_i}(x) \longrightarrow 0$ for $i \longrightarrow \infty$. If $r_{g_i}(x) = 0$ for some i then there is nothing to prove anymore, so we assume that $r_{g_i}(x) \neq 0$ for all i. Let C_i be the chamber with $x \in Z_{C_i, g_i A_0}$. There are two possibilities. Either there is a bounded subset of the building $\mathcal{B}$ that contains infinitely many C_i or there does not exist such a subset. We treat both cases separately.

First we assume that there is a bounded subset $F \subset \mathcal{B}$ that contains infinitely many chambers C_i. Since F contains only finitely many chambers there is at least one chamber $C \in F$ such that $C_i = C$ for infinitely many indices i. We now restrict ourselves to the infinite sequence g_i with $C_i = C$. After replacing each g_i in this sequence by $g_i g_1^{-1}$ (and x by $g_1(x)$) if necessary, we may assume that all the g_i are contained in the Iwahori subgroup P_C. Since P_C is compact the g_i converge to an element $g \in P_C$. Clearly we have $g^*x_1g^*x_2 = 0$.

Next we assume that there is no bounded area in the building $\mathcal{B}$ containing infinitely many chambers C_i. Let H_j, $j = 1, \ldots, s$ denote the connected components of $\mathcal{B} - C_1$. There is at least one H_j containing infinitely many C_i. We choose one such component H_j and denote it by H.

We take the subsequence of g_i such that C_i is contained in H. We look at the apartements A_i determined by the g_i. If C_1 is contained in infinitely many of them then we are done, since we can restrict ourselves to these and assume that the g_i are contained in P_{C_1}. Then again we find a g with $g^*x_1g^*x_2 = 0$.

So we may assume that C_1 is contained in only finitely many of the A_i. We now take the subsequence of g_i with $C_i \in H$ and C_1 not in A_i if $i > 1$. Without loss of generality we may assume that $C_1 = C_0$ and that $A_1 = A_0$.

Using lemma 3.4 below we have a $\tilde{g}_i \in G(K)$ with $\tilde{g}_iA_0 = g_iA_0$ and $\tilde{f}_i, \tilde{h}_i \in P_{C_0}$ such that $\tilde{g}_i^*x_1 = c_1\tilde{h}_i^*x_j$ and $\tilde{g}_i^*x_2 = c_2\tilde{f}_i^*x_j$ and $c_1, c_2 \in L^*$ with $|c_1 \cdot c_2| \geq 1$. Since all $A_i \subset H$ either $j = 1$ for all i or $j = 2$ for all i. we assume that $j = 2$. Since $|\frac{\tilde{g}_i^*x_1\tilde{g}_i^*x_2}{g_i^*x_1g_i^*x_2}(x)| = 1$ we can take $g_i = \tilde{g}_i$.

There are infinitely many indices i such that $|\frac{\tilde{h}_i^*x_2}{\tilde{f}_i^*x_2}(x)| \leq 1$ or infinitely many such that $|\frac{\tilde{f}_i^*x_2}{\tilde{h}_i^*x_2}(x)| \leq 1$. Let us assume that the first is the case. Then we take the sequence of $\tilde{h}_i$ with $|\frac{\tilde{h}_i^*x_2}{\tilde{f}_i^*x_2}(x)| \leq 1$. Since $|\frac{\tilde{h}_i^*x_1}{x_1}(x)|$ is bounded we have that $r_{\tilde{h}_i}(x) \longrightarrow 0$. Since $\tilde{h}_i \in P_{C_0}$ the sequence converges to an element h in P_{C_0} with $r_h(x) = 0$. This completes the proof.

3.3 Corollary

$Y := \bigcap_{g \in SU_3(L)} g \cdot X^s(T_0, \mathcal{O}(1)) = \{x \in X = P_L^2 | r(x) \neq 0\}$

Proof: If $x \in X^s(T_0, \mathcal{O}(1))$ then there exists an element $g \in SU_3(L)$ with $r_g(x) = 0$ if and only if there exists a maximal K-split torus $T \subset G(K) = SU_3(L)$ with $x \notin X^s(T, \mathcal{O}(1))$. Now the corollary follows directly from the proposition above.

3.4 Lemma

Let A be an apartment not containing C_0. Assume that the distance $d(C_0, A) = n$. Then there exist $f, h \in P_{C_0}$ and an element $g \in G(K)$ with $gA_0 = A$ such that $g^*x_1 = c_1 h^* x_j$, $g^*x_2 = c_2 f^* x_j$ for some $j \in \{1,2\}$ and $c_1, c_2 \in L^*$ such that $|c_1 \cdot c_2| = |\pi^{-n-s_j}|$. Here $s_1 = 0$ and $s_2 = 1$. The index j depends only on the apartement A.

Proof: Let H denote the path joining C_0 with A. It consists of the chambers $C_0, C_1, \ldots, C_n$, with $C_i \bigcap C_{i+1} \neq \emptyset$, $C_n \bigcap A \neq \emptyset$ and $C_{n-1} \bigcap A = \emptyset$. Let S be the vertex $S := H \bigcap A$. There are exactly two apartments A_1, A_2 in the building that contain C_0 and have a half apartment in common with both A_0 and A. Both contain the path H.

Since P_{C_0} acts transitively on the apartments containing C_0 there are h_i in P_{C_0} such that $h_i A_0 = A_i$, $i = 1, 2$. The choice of the h_i is such that either e_1 or e_2 is fixed by both h_i. Let us assume that it is e_1.

The L_0-module assocciated to the vertex S is
$M := h_i(< e_0, \pi^m e_1, \pi^{-m} e_2 >)$, $i = 1, 2$ if $n = 2m - 1$ and
$M := h_i(< e_0, \pi^{m+1} e_1, \pi^{-m} e_2 >)$, $i = 1, 2$ if $n = 2m$.
Therefore we have:
$M =< h_i(e_0), \pi^{m(+1)} h_i(e_1), \pi^{-m} h_i(e_2) >=< h_i(e_0), \pi^{m(+1)} e_1, \pi^{-m} h_i(e_2) >$, $i = 1, 2$.
Since $S \in A$ and e_1 is not fixed by the torus belonging to A we have:
$M =< h_i(e_0), \pi^{-m} h_1(e_2), \pi^{-m} h_2(e_2) >=< f_0, \pi^{-m} f_1, \pi^{-m} f_2 >$
Here $f_i := h_i(e_2)$, $i = 1, 2$ and f_0 is a suitable representative of $h_1(e_2)^\perp \bigcap h_2(e_2)^\perp$ and therefore satisfies $h_1^* x_1 = h_2^* x_1 = 0$. Furthermore $h_1^* x_1(f_1) = h_2^* x_1(f_2) = 0$. After multiplying the f_i with suitable units in L^* the hermitian form has w.r.t. the basis $f_0, \pi^{-m} f_1, \pi^{-m} f_2$ the following shape $y_1 \bar{y}_2 + y_2 \bar{y}_1 + y_0 \bar{y}_0$ if $n = 2m - 1$ and $\pi(y_1 \bar{y}_2 + y_2 \bar{y}_1) + y_0 \bar{y}_0$ if $n = 2m$. Here $y_i = c_i h_j^* x_1$, $i, j = 1, 2$, $i \neq j$ with $c_i \in L^*$ satisfying $|c_i| = |\pi^{-m}|$.

If $n = 2m$ then $g \in G(K)$ defined by $g(e_i) = \pi^{-m} f_i$, $i = 1, 2$ and $g(e_0) = f_0$ satisfies the lemma. If $n = 2m - 1$ then we can take an element g given by $g(e_1) = \pi^{-m} f_1$, $g(e_2) = \pi^{-m-1} f_2$, $g(e_0) = f_0$.

When both h_i fix e_2 the proof is similar.

3.5 Lemma

If $x \in X_{\Delta,gA_0}$ *and* $r_g(x) = r(x) \neq 0$ *then* $x \in X_\Delta$.
In particular we have $Y \subset \bigcup_\Delta X_\Delta$.

Proof: It is sufficient to proof the lemma for the case $gA_0 = A_0$. If $x \in X_{\Delta,A_0}$ then we have for all $h \in P_\Delta$ $|\frac{h^*x_i}{x_i}(x)| \leq 1$, $i = 1,2$. The minimality of $r_{id}(x) \neq 0$ implies that $r_h(x) \geq 1$ for all $h \in P_\Delta$. Therefore $|\frac{h^*x_i}{x_i}(x)| = 1$ for all $h \in P_\Delta$. Hence $x \in X_\Delta$.

The second statement in the lemma follows from the fact that when $x \in Y$ one has $r(x) \neq 0$. Therefore we can find an apartment $A = gA_0$ with $r_g(x) = r(x)$. Now $x \in X_{\Delta,A}$ for some polyhedron Δ. Hence $x \in X_\Delta$.

3.6 Lemma

$x \in X_\Delta$, $\Delta \in gA_0$, $|\frac{g^*x_0^2}{g^*x_1 g^*x_2}(x)| \leq 1 \Longrightarrow x \in Y$.

Proof: Let us first assume that the polyhedron is either an infinite polyhedron or a triangle. Therefore Δ determines a chamber $C \in gA_0$. It is sufficient to prove the lemma for the case where $C = C_0$ and $gA_0 = A_0$.

Since $x \in X_\Delta$ we have $|\frac{h^*x_i}{x_i}(x)| = 1$, $i = 1,2$ for all $h \in P_{C_0}$. So for all apartments hA_0 with $h \in P_{C_0}$ we have $r_h(x) = 1$. Using lemma 3.4 we easily conclude that for the apartments gA_0 that do not contain C_0 we have $r_g(x) \geq 1$. So for all $g \in SU_3(L)$ we have $r_g(x) \geq 1$. Hence $x \in Y$.

Let us now assume that Δ is a square. Then one of the modules associated to Δ has to be of form $M := < e_0, \pi^n e_1, \pi^{-n} e_2 >$. Therefore the group H_Δ as defined in section 2.6 is $H_\Delta = P_S$, where S is the vertex corresponding to the module M. Now $x \in X_\Delta$ has the property that for all $g \in P_S$ one has $|\pi^2| \leq |f_g(x)| \leq 1$. Here f_g is as defined in section 2.6. Now using lemma 3.4 one concludes that $r(x) \neq 0$. Therefore $x \in Y$.

3.7

On the apartment A_0 we take a coordinate function y which has on the vertex corresponding with the module $< e_0, \pi^n e_1, \pi^m e_2 >$, $n + m = 0, 1$ the value $n - m$. We define a map $\varphi_{A_0} : X^s(T_0, \mathcal{O}(1)) \longrightarrow A_0$ by
$\varphi_{A_0}(x) = p \Longleftrightarrow v(\frac{x_1}{x_2}(x)) = y(p)$

Here v is the additive valuation of L, normalized in such a way that $v(\pi) = 1$. Note that $\varphi_{A_0}(t \cdot x) = t \cdot \varphi_{A_0}(x)$ for $t \in T_0$. This function φ_{A_0} cannot be extended to the set of semistable points. However we can also associate to each stable point a segment in the building. We can also do this for semistable points.

To each point $x \in X^s(T_0, \mathcal{O}(1))$ with $x_0 \neq 0$ we associate the following segment in A_0: $S_{x,A_0} := [p_1, p_2]$, where the p_i are determined by $2v(\frac{x_1}{x_0}(x)) = y(p_1)$ and $2v(\frac{x_0}{x_2}(x)) = y(p_2)$. If $x_0 = 0$ then we take $S_{x,A_0} := A_0$. Note that S_{x,A_0} is a point if and only if $|\frac{x_0^2}{x_1 x_2}(x)| = 1$. If $x_0 \neq 0$ then it follows from $\frac{x_1}{x_0} \cdot \frac{x_0}{x_2} = \frac{x_1}{x_2}$ that $\varphi_{A_0}(x)$ is exactly in the middle of S_{x,A_0}.

We can also associate a segment to the semi-stable points which are non-stable. These segments are either half-apartments or apartments. If $x = (x_0, x_1, 0)$ with $x_0, x_1 \neq 0$ then $S_{x,A_0} := \lim_{\varepsilon \to 0} S_{x(\varepsilon),A_0}$, where $x(\varepsilon) = (x_0, x_1, \varepsilon)$. For $x = (x_0, 0, x_2)$ with $x_0, x_2 \neq 0$ we take $S_{x,A_0} := \lim_{\delta \to 0} S_{x(\delta),A_0}$ where $x(\delta) = (x_0, \delta, x_2)$. For $x = (1, 0, 0)$ we take $S_{x,A_0} := A_0$. For a general apartment $A = gA_0$ and $x \in X^{ss}(gT_0g^{-1}, \mathcal{O}(1))$ we take $S_{x,A} := g(S_{g^{-1}(x),A_0})$.

We can also associate to each module $m =< e_0, \pi^n e_1, \pi^m e_2 >$ a segment $S_{M,A_0} := S_{x,A_0}$, where x is the point $x = (1, \pi^n e_1, \pi^m e_2)$. Note that $S_M \subseteq S_{M,A_0}$, where S_M is the segment defined in section 2.2. Generally these segments are not equal! For instance if we take a module M with $n + m = 1$ then S_M is the vertex S corresponding to this module. Whereas S_{M,A_0} consists of the two chambers contained in A_0 that contain the vertex S.

Let $Z := \bigcap_{g \in SU_3(L)} g \cdot X^{ss}(T_0, \mathcal{O}(1))$ and let Z^+ be the subspace $Z^+ := \{x \in Z |$ if $r_g(x) = r(x)$ then $|\frac{g^* x_1 g^* x_2}{g^* x_0^2}(x)| \leq 1\}$. For $x \in Z^+$ we can also define a segment S_x in the building independent of the apartment. To do this we need a lemma.

3.8 Lemma

Let $x \in Z^+$ and assume that S_{x,gA_0} is not a point. Then there exists an apartment A such that $S_{x,gA_0} \subset S_{x,A}$ and $S_{x,A}$ is maximal.

Proof: Take a chamber C in the building such that $C \bigcap S_{x,gA_0}$ contains at least two points. Now we look at the segments S_{x,fgA_0} with $f \in P_C$. From the compactness of the group P_C it follows that we can find at least one maximal segment S_{x,fgA_0} containing S_{x,gA_0}.

3.9

Let us define for $x \in Z^+$ the set of apartments which contain a maximal segment for X:
$\mathcal{M}(x) := \{A | A \text{ is an apartment with } S_{x,A} \text{ maximal and not a point }\}$.
Furthermore we define $S_x := \bigcap_{A \in \mathcal{M}(x)} S_{x,A}$.
We have for $x \in Z^+$ the following lemma:

3.10 Lemma

$A \in \mathcal{M}(x)$ if and only if $S_x \subset A$.

Proof: We fix a chamber C as follows. Take an apartment A in $\mathcal{M}(x)$. If $x \in Y$ we take C such that $\varphi_A(x) \in C$. If $x \notin Y$ then we only demand that $C \in S_x$. Now one applies lemma 3.4 to the chamber C. After some calculations which we omit here, the lemma follows.

Some more calculation yields the following:

3.11 Proposition

Let $x \in Z^+$ and $A \in \mathcal{M}(x)$. Suppose $S_{x,A} = [p_1, p_2]$, where we allow $p_i = \pm\infty$ for (half-)apartments. Let $C_i \in A$ be the chamber with $p_i \in C_i$ and such that $C_i \bigcap S_{x,A}$ contains at least two points. If $p_i = \pm\infty$ we do not define C_i. If C_i is defined and $C_i \bigcap S_{x,A}$ contains a vertex S_i corresponding with a degenerated module satisfying $2 \cdot |y(p_i) - y(S_i)| \leq |y(p_1) - y(p_2)|$ then we take $q_i := S_i$. Otherwise we take $q_i := p_i$. Then S_x is the segment $[q_1, q_2]$.

3.12 Lemma

$x \in X_\Delta \bigcap Z^+ \iff S_\Delta^- \subseteq S_x \subseteq S_\Delta^+$

Proof: Since $x \in Z^+$ the polyhedron Δ has to be a square. For a square we define $S_\Delta^+ := S_\Delta$ and S_Δ^- as being the segment S_M. Here M is the module associated with Δ that gives the shortest segment. Again the proof consists of explicit calculations that we omit.

3.13 Lemma

If $x \in X_{\Delta,A}$ and $x \notin Y$ then $x \notin X_\Delta$.

Proof: If $x \in Z - Y$ then the lemma follows from the lemmas 3.6 and 3.12. If $x \notin Z$ then one easily calculates that $x \notin X_\Delta$.

3.14 Proposition

The covering $\{X_\Delta | \Delta \in \mathcal{P}_B\}$ is pure and $\bigcup_\Delta X_\Delta = Y$.

Proof: From lemmas 3.5 and 3.13 one easily derives that $\bigcup_\Delta X_\Delta = Y$. So we have only to show that the covering is pure.

Let us fix a polyhedron Δ. Let us assume that $X_\Delta \bigcap X_{\Delta'} \neq \emptyset$. If there exists an apartment A such that both Δ and Δ' are polyhedra associated to A. Then we must have $\Delta \bigcap \Delta' \neq \emptyset$. This gives us a finite number of Δ'. If there does not exist such an apartment A then at least one of the polyhedra Δ and Δ' has to be a square. If both are squares then it follows from lemma 3.12 that one has $S^+_\Delta \bigcap S^+_{\Delta'} = S^-_\Delta = S^-_{\Delta'}$. Again this gives us a finite number of Δ'. If one of the polyhedra is not a square then this polyhedron determines a chamber C in the building. Clearly we must have that $C \bigcap S_\Delta \neq \emptyset$ if Δ' is not a square. The other case is similar. Again we get a finite number of Δ'.

Next we have to show that if $X_\Delta \bigcap X_{\Delta'} \neq \emptyset$ the intersection has property 2 of the definition given in section 1.4. If both Δ and Δ' are polyhedra associated to some apartment A then this follows directly from the fact that the intersection of $X_{\Delta,A}$ and $X_{\Delta',A}$ has this property. If there is no such apartment A then one has, if both Δ and Δ' are squares, $X_\Delta \bigcap X_{\Delta'} = \{x \in Y | S_x = S^-_\Delta = S^-_{\Delta'}\}$. Therefore also in this case the intersection has the required property. The other case is more or less similar and we leave it to the reader.

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