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Recent Results on Moduli of Abelian Surfaces

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In the last few years, several people have been considering the moduli of abelain surfaces with non-principal polarisations (and often a bit more structure). I intend to explain in some detail what has been done, and why, and to descibe some open problems.

1. Abelian varieties and polarisations.

For the sake of completeness, and because there has been some work on moduli of higher-dimensional abelian varieties with non-principal polarisations, I shall not restrict myself to surfaces until I have to. We shall, in fact, almost always work over the complex numbers, and accordingly I shall describe things from the point of view of complex manifolds.

A *complex torus* is a quotient $X = \mathbb{C}/\Lambda$ of the g-dimensional vector space $\mathbb{C}^g$ by a lattice Λ of rank $2g$: thus $\Lambda \cong \mathbb{Z}^{2g}$ and X is a compact complex manifold of dimension g. In general X is not an algebraic variety. The well-known Appel-Humbert theorem, which can be found in any book on abelian varieties (for instance [M] or [LB]), gives a necessary and sufficient condition for X to be algebraic: there should exist a Hermitian form H on $\mathbb{C}^g$ which is positive definite and takes integer values on Λ. If such a form exists then X is said to be an *abelian variety* and H is called a *polarisation.* The terminology is justified by the fact that "polarisation" in this sense coincides with the usual sense of "polarisation" in algebraic geometry: there is a one-to-one correspondence between such Hermitian forms and ample line bundles on X.

With respect to a suitable $\mathbb{Z}$-basis for Λ, the imaginary part $E = \operatorname{Im} H$ of H (which is, of course, an alternating bilinear form) has matrix

$$\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$$

for some diagonal $n \times n$ matrix $D = \operatorname{diag}(d_1, \dots, d_g)$. The d_i can be chosen to be positive integers such that $d_i | d_{i+1}$; subject to that, they are determined uniquely by the polarisation. The polarisation is said to be *of type* $(d_1, \dots, d_g)$, or to be a $(d_1, \dots, d_g)$*-polarisation*. A *principal polarisation* is a polarisation of type $(1, \dots, 1)$.

For practical purposes we may assume that $d_1 = 1$. If $\mathcal{L}$ is an ample line bundle determining a polarisation of type $(d_1, \dots, d_g)$, then the polarisation given by $\mathcal{L}^{\otimes a}$ is of type $(ad_1, \dots, ad_g)$, so to assume $d_1 = 1$ is just to take the smallest ample line bundle in the ray in $\operatorname{Pic} X \otimes \mathbb{R}$ generated by the polarisation.

Principally polarised abelian varieties have attracted most attention, not least because the Jacobian of a curve is an abelian variety that comes with a natural principal polarisation. However, non-principal polarisations do also arise naturally, for instance on Prym varieties.

2. An example: Horrocks-Mumford surfaces.

The Horrocks-Mumford bundle (see [HM]), which we shall denote by $\mathcal{F}$, is probably the most famous vector bundle in the world. It is a rank 2 bundle on $\mathbb{P}^4$ with many beautiful properties. The beautiful property of the Horrocks-Mumford bundle that we are interested in is this one: if $s \in \Gamma(\mathcal{F})$ is a general section then the zero set of s is an abelian surface $X_s \subseteq \mathbb{P}^4$, and $\mathcal{O}(1)$ is a $(1,5)$-polarisation. Every abelian surface in $\mathbb{P}^4$ arises in this way, and a $(1,5)$-polarisation on an abelian surface is always very ample and embeds the abelian surface in $\mathbb{P}^4$.

Suppose p is an odd prime and X is a $(1,p)$-polarised abelian surface with dual abelian variety $\hat{X}$. The polarisation induces a map $X \to \hat{X}$. The kernel of this map is a vector space over the finite field $\mathbb{F}_p$ which inherits an alternating form from E. A *level structure* is a choice of symplectic basis for this kernel.

In the case of Horrocks-Mumford surfaces, the section s determines a level structure on X_s, which arises because X_s is invariant under the action of the Heisenberg group on $\mathcal{F}$. It follows that an open subset of $\mathbb{P}\Gamma(\mathcal{F}) = \mathbb{P}^3$ parametrises $(1,5)$-polarised abelian surfaces with a level structure.

3. Moduli spaces.

Henceforth we consider abelian surfaces with a $(1,t)$-polarisation for some integer t. One expects a family of abelian surfaces with extra structure to be parametrised by a Siegel modular threefold (unless the structure is such that its existence imposes conditions on the abelian surface, in which case other moduli spaces such as Hilbert modular surfaces arise). By a Siegel modular threefold we mean a quotient of the Siegel upper half-plane of degree 2 by an arithmetic subgroup of $\mathrm{Sp}(4,\mathbb{Q})$. The following case, which generalises the case of Horrocks-Mumford surfaces, has attracted much attention.

Let $A_{1,p}$ be the moduli space of $(1,p)$-polarised abelian surfaces with a level structure, where p is an odd prime. The Siegel upper half-plane is

$$\mathcal{S}_2 = \{Z \in M_{2\times 2}(\mathbb{C}) \mid Z = {}^TZ, \operatorname{Im} Z > 0\}$$

and $\mathrm{Sp}(4,\mathbb{Q})$ acts, as usual, by fractional linear transformations: that is, if $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(4,\mathbb{Q})$ then $\gamma : Z \mapsto (AZ+B)(CZ+D)^{-1}$. We put

$$\Gamma_{1,p} = \left\{ \gamma \in \mathrm{Sp}(4,\mathbb{Q}) \mid \gamma - \mathbf{1} \in \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & p\mathbb{Z} \\ p\mathbb{Z} & p\mathbb{Z} & p\mathbb{Z} & p^2\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & p\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & p\mathbb{Z} \end{pmatrix} \right\}$$

which is obviously an arithmetic subgroup. Then $A_{1,p} = \Gamma_{1,p}\backslash\mathcal{S}_2$. (Strictly speaking, $A_{1,p}$ is $\Gamma'_{1,p}\backslash\mathcal{S}_2$, where $\Gamma'_{1,p}$ is a subgroup of $\mathrm{Sp}(4,\mathbb{Q})$ conjugate to $\Gamma_{1,p}$, but this makes no difference and it is convenient to work with a subgroup of $\mathrm{Sp}(4,\mathbb{Z})$ if possible.) Similarly, if $\tilde{A}_{1,p}$ is the moduli space of $(1,p)$-polarised abelian surfaces (no level structure this time), and

$$\tilde{\Gamma}_{1,p} = \left\{ \gamma \in \mathrm{Sp}(4,\mathbb{Q}) \mid \gamma - \mathbf{1} \in \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & p\mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} & p\mathbb{Z} & p\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & p\mathbb{Z} \\ \mathbb{Z} & \frac{1}{p}\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix} \right\}$$

(we can't stay inside $\mathrm{Sp}(4,\mathbb{Q})$ now), then $\tilde{A}_{1,p} = \tilde{\Gamma}_{1,p}\backslash\mathcal{S}_2$.

$A_{1,p}$ is a quasi-projective variety, but it is non-compact and, since $\Gamma_{1,p}$ has torsion no matter how large p is, it is singular. There is a natural compactification of $A_{1,p}$, the Satake compactification, but it is quite badly singular: it is better to compactify toroidally (adding divisors), and in fact in this case there is a natural choice of such a compactification. The toroidal compactification $A^*_{1,p}$ has only finite quotient singularities. It is studied in detail (along with the corresponding degenerations) by Hulek, Kahn and Weintraub in their recent book [HKW1] and the singularities are described in [HKW2].

The reason for restricting attention to the case of an odd prime is that it is then possible to describe the boundary $A_{1,p}^{*} \setminus A_{1,p}$. If the polarisation type is $(1,t)$ and t is not a prime ($t = 2$ is a special case anyway) then the boundary has many more components and the picture becomes extremely complicated, though it is possible to make a special study of $A_{1,p}^{*}$ for any t that seems particularly interesting ($t = 4$, for instance). Some results can be got by methods that do not require knowledge of the geometry of the boundary, but for most purposes we need some such information.

4. Known results.

We shall try to understand the birational geometry of $A_{1,p}$ and $A_{1,p}^{*}$.

A. The Kodaira dimension.

It follows from the remarks about the Horrocks-Mumford surfaces, above, that $\kappa(A_{1,5}^{*}) = -\infty$ and indeed that $A_{1,5}^{*}$ is rational. Recently, Manolache and Schreyer ([MS]) have shown that $A_{1,7}^{*}$ is rational. They look at the syzygies of abelian surfaces in $\mathbb{P}^6$ and produce a birational equivalence between $A_{1,7}^{*}$ and a Fano variety of type V_{22}, via polar hexagons, which is known to be rational by a theorem of Mukai.

In the other direction, Gritsenko ([G]) shows that $\kappa(A_{1,p}^{*}) \geq 0$ if $p \geq 13$ and $\kappa(A_{1,p}^{*}) \geq 1$ if $p \geq 29$, and Hulek and I proved (in [HS1]) that $\kappa(A_{1,p}^{*}) = 3$ if $p \geq 41$. Both these results are obtained by looking at modular forms for $\Gamma_{1,p}$, but in rather different ways, as I will now explain.

If f is a modular form of weight $3k$ for $\Gamma_{1,p}$ then $f(Z)(d\tau_1 \wedge d\tau_2 \wedge d\tau_3)^{\otimes k}$ (where $Z = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau 3 \end{pmatrix} \in \mathcal{S}_2$) gives a k-fold differential form, possibly with some poles, on a desingularisation of $A_{1,p}$, at least if k is sufficiently divisible. It extends, possibly with logarithmic poles, to the boundary of a desingularisation of $A_{1,p}^{*}$. In [HS1] the method is to obtain a large supply of modular forms of high weight (which is easy) and then count the conditions that the corresponding differential form must satisfy in order to extend without poles. It turns out that the dimension of the space of modular forms of weight $3k$ is about $\frac{1}{640}p^5k^3$ and the number of conditions is about $8p^4k^3$, so if $p \geq 41$ we really do get order k^3 pluricanonical forms. In [G], on the other hand, Gritsenko produces cusp forms of weight 3 for $\Gamma_{1,t}$ (actually for $\tilde{\Gamma}_{1,t}$). This is very difficult to do but the resulting canonical (not pluricanonical!) form extends automatically to the whole of a desingularisation of

$A^*_{1,t}$, so $p_g(A^*_{1,t}) > 0$. In particular no information about the boundary is needed and it is therefore possible to get results about most t: in some cases where all the prime factors of t are small (for instance $t = 30$) the method fails to produce any forms.

In this context we should also note that O'Grady, in [O'G], proved that A^*_{1,p^2} is of general type if $p \geq 17$.

Nothing at all is known about $\kappa(A^*_{1,11})$ or about $\kappa(A^*_{1,3})$, but it would be astonishing if $A^*_{1,3}$ were not rational. I suspect, on no evidence whatever, that $\kappa(A^*_{1,11}) = 3$, though it would be much more fun if it were not.

B. Hodge and Betti numbers.

Let A_p be a desingularisation of $A^*_{1,p}$ (there is a fairly obvious choice of desingularisation to make). It is quite easy to see that the first Betti number $b_1(A_p)$ is zero. Gritsenko's results give lower bounds on $h^{3,0}$ and Zintl has calculated the Euler characteristic (and is calculating the other Chern numbers). The Euler characteristic of $A^*_{1,p}$ is very negative, so b_3 is large for large p. Apart from that, nothing is known. I have hopes that b_2 and b_3 can be calculated by reducing mod q and using the Weil conjectures (Lee and Weintraub, in [LW], carried out such a program for principally polarised abelian surfaces of level 2), but the technical difficulties are considerable. For small p we might be able to calculate the Hodge numbers via modular forms, but we would probably only be able to get sharp enough bounds to do this if some of them vanished, which is unlikely to be the case for large p.

C. Miscellaneous.

Hulek and I have recently shown, in [HS2], that A_p is simply connected. (I believe that this should be true for most, perhaps even all, Siegel modular varieties, except, obviously, curves.) This (rather easy) result raises the possibility of A_{11} (or even, for all we know, A_3) being a Calabi-Yau 3-fold. Zintl's calculation shows that $b_3(A^*_{1,11})$ is quite large (about 50), which proves nothing but suggests that $A^*_{1,11}$ is probably not rational (there would have to be a lot of surfaces that could be contracted and if there were we would expect to have seen them by now).

The proof that A_p is simply connected runs like this: $\pi_1(A_p)$ is a quotient of $\Gamma_{1,p}$. The principal congruence subgroup $\Gamma(p^2)$, that is, $\{\gamma \in \mathrm{Sp}(4,\mathbb{Z}) \mid \gamma \equiv 0 \,(\mathrm{mod}\, p^2)\}$, is contained in $\Gamma_{1,p}$ and it follows from a result of Knöller that

$$\Gamma(p^2) \subseteq \mathrm{Ker}\left(\phi : \Gamma_{1,p} \longrightarrow \pi_1(A_p)\right).$$

The element

$$M_0 = \begin{pmatrix} \mathbf{1} & \mathbf{U} \\ \mathbf{0} & \mathbf{1} \end{pmatrix},$$

where $\mathbf{U} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, is also in this kernel: to prove this one finds a loop in $\phi(M_0)$ and constructs an explicit null homotopy, contracting the loop via a boundary point. Then it can be shown by direct calculation that the smallest normal subgroup of $\Gamma_{1,p}$ containing both $\Gamma(p^2)$ and M_0 that is invariant under conjugation by $\tilde{\Gamma}_{1,p}$ is $\Gamma_{1,p}$ itself. As $\mathrm{Ker}\,\phi$ has all these properties, it follows that $\pi_1(A_p) = 1$.

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