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COMPLEX STRUCTURES ON PARTIAL COMPACTIFICATIONS OF CLASSIFYING SPACES D/Γ OF HODGE STRUCTURES

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§1. Preliminaries.

We recall first the definition of a (polarized) Hodge structure of weight w. Fix a free $\mathbf{Z}$-module $H_{\mathbf{Z}}$ of finite rank. Set $H_{\mathbf{Q}} := \mathbf{Q} \otimes H_{\mathbf{Z}}$, $H = H_{\mathbf{R}} := \mathbf{R} \otimes H_{\mathbf{Z}}$ and $H_{\mathbf{C}} := \mathbf{C} \otimes H_{\mathbf{Z}}$, whose complex conjugation is denoted by σ. Let w be an integer. A *Hodge structure of weight* w on $H_{\mathbf{C}}$ is a decomposition

$$(1.1) \qquad H_{\mathbf{C}} = \bigoplus_{p+q=w} H^{p,q} \quad \text{with} \quad \sigma H^{p,q} = H^{q,p}.$$

$F^p := \bigoplus_{p' \geq p} H^{p',q'}$ is called a *Hodge filtration*, and $H^{p,q}$ is recovered by $H^{p,q} = F^p \cap \sigma F^q$. The integers

$$(1.2) \qquad h^{p,q} := \dim H^{p,q}$$

are called the *Hodge numbers*.

A *polarization* S for a Hodge structure (1.1) of weight w is a non-degenerate bilinear form on $H_{\mathbf{Q}}$, symmetric if w is even and skew-symmetric if w is odd, such that its $\mathbf{C}$-bilinear extension, denoted also by S, satisfies

$$(1.3) \qquad \begin{aligned} &S(H^{p,q}, \sigma H^{p',q'}) = 0 \quad \text{unless} \quad (p,q) = (p',q'), \\ &i^{p-q} S(v, \sigma v) > 0 \quad \text{for all} \quad 0 \neq v \in H^{p,q}. \end{aligned}$$

For fixed S and $\{h^{p,q}\}$, the classifying space D for Hodge structures and its 'compact dual' $\check{D}$ are defined by

$$(1.4) \qquad \begin{aligned} \check{D} &:= \{\{H^{p,q}\} \mid \text{Hodge structure on } H_{\mathbf{C}} \text{ with } \dim H^{p,q} = h^{p,q}, \\ &\qquad\qquad \text{satisfying the first condition in (1.3)}\}, \\ D &:= \{\{H^{p,q}\} \in \check{D} \mid \text{satisfying also the second condition in (1.3)}\}. \end{aligned}$$

These are homogeneous spaces under the natural actions of the groups

$$(1.5) \qquad G_{\mathbf{C}} := \mathrm{Aut}(H_{\mathbf{C}}, S), \quad G = G_{\mathbf{R}} := \{g \in G_{\mathbf{C}} \mid gH_{\mathbf{R}} = H_{\mathbf{R}}\},$$

respectively. Taking a reference point $r \in D$, one obtains identifications

$$(1.6) \qquad \check{D} \simeq G_{\mathbf{C}}/I_{\mathbf{C},r}, \quad D \simeq G/I_r,$$

where $I_{\mathbf{C},r}$ and I_r are the isotropy subgroups of $G_{\mathbf{C}}$ and of G at $r \in D$, respectively. It is a direct consequence of the definition that

$$(1.7) \qquad G \simeq \begin{cases} O(2h,k), \\ \mathrm{Sp}(2h,\mathbf{R}), \end{cases} \quad I_r \simeq \begin{cases} U(h^{w,0}) \times \cdots \times U(h^{t+1,t-1}) \times O(h^{t,t}) & \text{if } w = 2t, \\ U(h^{w,0}) \times \cdots \times U(h^{t+1,t}) & \text{if } w = 2t+1, \end{cases}$$

where $k := \sum_{|j|\le[t/2]} h^{t+2j,t-2j}$ and $h := (\dim H - k)/2$ if $w = 2t$, and $h := \dim H/2$ if $w = 2t+1$. It is an important observation that I_r is compact, but not maximal compact in general. Hence D is a symmetric domain of Hermitian type if and only if

$$\begin{aligned} &w = 2t+1; && h^{p,q} = 0 \text{ unless } p = t+1, t. \\ &w = 2t; && h^{p,q} = 1 \text{ for } p = t+1,\ t-1,\ h^{t,t} \text{ is arbitrary,} \\ \textbf{(1.8)}\quad & && h^{p,q} = 0 \text{ otherwise; or} \\ & && h^{p,q} = 1 \text{ for } p = t+a,\ t+a-1,\ t-a+1,\ t-a \\ & && \text{for some } a \ge 2,\ h^{p,q} = 0 \text{ otherwise.} \end{aligned}$$

We denote

(1.9) $$\Gamma := \{g \in G \,|\, gH_{\mathbf{Z}} = H_{\mathbf{Z}}\}.$$

Then Γ acts on D properly discontinuously because the isotropy subgroup I_r is compact and Γ is discrete in G.

A reference Hodge structure $r = \{H_r^{p,q}\} \in D$ induces a Hodge structure of weight 0 on the Lie algebra $\mathfrak{g}_{\mathbf{C}} := \operatorname{Lie} G_{\mathbf{C}}$ by

(1.10) $$\mathfrak{g}_{\mathbf{C}}^{s,-s} := \{X \in \mathfrak{g}_{\mathbf{C}} \,|\, XH_r^{p,q} \subset H_r^{p+s,q-s} \text{ for all } p, q\}.$$

One can define the associated Cartan involution θ_r on $\operatorname{Lie} G := \mathfrak{g}$ induced by

(1.11) $$\theta_r(X) := \sum_s (-1)^s X^{s,-s} \text{ for } X = \sum_s (-1)^s X^{s,-s} \in \mathfrak{g}_{\mathbf{C}} = \bigoplus_s \mathfrak{g}_{\mathbf{C}}^{s,-s}.$$

We take the standard generators for the Lie algebras $\mathfrak{sl}_2(\mathbf{R})$ and $\mathfrak{su}(1,1)$ which are related by the Cayley transformation $\operatorname{Ad} c_1$, where

(1.12) $$c_1 := \exp\left(\frac{\pi i}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix},$$

as follows:

(1.13) $$\begin{array}{cccccc} \mathfrak{sl}_2(\mathbf{R}) & \ni & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\ \operatorname{Ad} c_1 \downarrow & & \downarrow & \downarrow & \downarrow \\ \mathfrak{su}(1,1) & \ni & \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, & \frac{1}{2}\begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}, & \frac{1}{2}\begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}. \end{array}$$

Remark(1.14). $i \in \mathfrak{h} :=$ (upper-half plane) $\simeq \mathrm{SL}_2(\mathbf{R})/U(1)$ corresponds to a Hodge structure $\mathbf{C}^2 = H_i^{1,0} \oplus H_i^{0,1}$ with $H^{1,0} = \mathbf{C}\begin{pmatrix} i \\ 1 \end{pmatrix}$. The Hodge structure on $\mathfrak{g}_{1\mathbf{C}} := \mathfrak{sl}_2(\mathbf{C})$ induced by $i \in \mathfrak{h}$ coincides with the canonical decomposition by the standard 'H-element' $\frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (cf., e.g., [Sa, II. §7]):

$$\mathfrak{g}_{1\mathbf{C}} = \mathfrak{g}_{1\mathbf{C}}^{1,-1} + \mathfrak{g}_{1\mathbf{C}}^{0,0} + \mathfrak{g}_{1\mathbf{C}}^{-1,1} = \mathfrak{p}_- + \mathfrak{k}_{\mathbf{C}} + \mathfrak{p}_+ = \mathbf{C}\frac{1}{2}\begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix} + \mathbf{C}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \mathbf{C}\frac{1}{2}\begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}.$$

From now on, we assume that $w > 0$ and all Hodge structures of weight w satisfy $H^{p,q} = 0$ unless $p,\ q \ge 0$.

Definition(1.15) (cf. [Sc, p.258]). *An* SL_2*-representation* $\rho : \mathrm{SL}_2(\mathbf{R}) \to G$ *is horizontal at* $r = \{H_r^{p,q}\} \in D$ *if* $\rho_*\left(\frac{1}{2}\begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}\right) \in \mathfrak{g}_{\mathbf{C}}^{-1,1}$ *(see (1.10)). When this is a case, we call the pair* (ρ, r) *an* SL_2*-orbit.*

Remark(1.16). It is clear that (ρ, r) is an SL_2-orbit if and only if $\rho_* : \mathfrak{sl}_2(\mathbf{R}) \to \mathfrak{g}$ is a morphism of Hodge structures of type $(0,0)$ with respect to the Hodge structures induced by $i \in U$ and $r \in D$, respectively. A horizontal SL_2-representation ρ induces an equivariant horizontal map $\tilde{\rho} : \mathbf{P}^1 \to \check{D}$ with $\tilde{\rho}(i) = r$:

$$\begin{array}{ccc} \mathrm{SL}_2(\mathbf{C}) & \xrightarrow{\rho} & G_{\mathbf{C}} \\ \downarrow & & \downarrow \\ \mathbf{P}^1 & \xrightarrow{\tilde{\rho}} & \check{D} \end{array}$$

This is a generalization to the present context of the notion of '(H_1)-homomorphism' in the case of symmetric domains of Hermitian type (cf., e.g., [Sa, II. (8.5), III. §1]).

Let (ρ, r) be an SL_2-orbit and $\tilde{\rho} : \mathbf{P}^1 \to \check{D}$ the associated horizontal equivariant map. We set

$$\textbf{(1.17)} \qquad Y := \rho_*\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ N_+ := \rho_*\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ N_- := \rho_*\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ c := \rho(c_1).$$

We denote by $H(Y;\lambda)$ the λ-eigen space of the action of Y on H, and set

$$\textbf{(1.18)} \qquad W(Y)_{w-j} := \bigoplus_{\lambda \geq j} H(Y;\lambda).$$

Lemma(1.19). *Let* (ρ, r) *be an* SL_2*-orbit. Then, in the above notation,* $\lim_{\mathrm{Im}\, z \to \infty} \exp(-zN_+) \cdot \tilde{\rho}(z) = c^{-1} \cdot r \in \check{D}$. *The corresponding filtration, denoted by* F_∞, *together with* $W(Y)$, *determines the limiting* S*-polarized split mixed Hodge structure.*

Proof. $\tilde{\rho}(z) = \tilde{\rho}(i + (z-i)) = \tilde{\rho}\left(\exp\left((z-i)\cdot\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)\cdot i\right) = \exp((z-i)\cdot N_+)\cdot r$, hence $\exp(-zN_+)\cdot\tilde{\rho}(z) = \exp(-iN_+)\cdot r = \tilde{\rho}\left(\exp\left(-i\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)\cdot i\right) = \tilde{\rho}(0)$. On the other hand, $c^{-1}\cdot r = \tilde{\rho}(c_1^{-1}\cdot i) = \tilde{\rho}(0)$.

The second assertion follows from [Sc, (6,16)] and [U, (2.11), see also(2.12)]. (N, L in [Sc, (6.16)] correspond to N_+, N_- in our present notation, respectively.) □

§2. Line bundles $L(W)$.

Let W_{w-1} be a subspace of $H_{\mathbf{Q}}$ defined over $\mathbf{Q}$ which is isotropic with respect to S, i.e., $S(u,v) = 0$ for all $u, v \in W_{w-1}$. We assume throughout this paper that

$$\textbf{(2.1)} \qquad \dim W_{w-1} = \begin{cases} 1 & \text{if } w \text{ is odd,} \\ 2 & \text{if } w \text{ is even.} \end{cases}$$

Let W_w be the anihilator of W_{w-1} in $H_{\mathbf{Q}}$ with respect to S. Then we have a filtration W of $H_{\mathbf{Q}}$:

$$0 \subset W_{w-1} \subset W_w \subset W_{w+1} := H_{\mathbf{Q}}. \tag{2.2}$$

By abuse of notation, we also use W for the filtrations induced on $H = H_{\mathbf{R}}$, $H_{\mathbf{C}}$ if it does not lead any confusion. Note that $(-1)^{w-1}$-symmetric bilinear forms on W_{w-1}, form a one dimensional vector space.

We define subgroups of G:

$$\begin{aligned} &N(W) := \{g \in G \mid gW_j = W_j \text{ for all } j\}^\circ, \\ &U(W):\ \text{the unipotent radical of } N(W), \\ &C(W):\ \text{the center of } U(W), \end{aligned} \tag{2.3}$$

where $\{\ \}^\circ$ means the connected component containing 1. The induced sub- and sub-quotient groups of Γ are denoted by

$$\Gamma_W := \Gamma \cap N(W),\ U(W)_{\mathbf{Z}} := \Gamma \cap U(W),\ C(W)_{\mathbf{Z}} := \Gamma \cap C(W),\ \overline{\Gamma}_W := \Gamma_W / C(W)_{\mathbf{Z}}. \tag{2.4}$$

Definition(2.5). *$N \in \mathfrak{c} := \operatorname{Lie} C(W)$ is positive if $N \in \mathbf{R}_{>0} \cdot N_+$ for some SL_2-orbit (ρ, r) with $W(Y) = W$ (cf. (1.17), (1.18)).*

Lemma(2.6). **(i)** $\dim C(W) = 1$.

(ii) *$C(W)$ is a normal subgroup of $N(W)$, and $Ad(g)X = det(g|W_{w-1})X$ for $g \in N(W)$, $X \in \mathfrak{c} = \operatorname{Lie} C(W)$.*

(iii) *Let $r \in D$ be a reference point. Then $C(W)_{\mathbf{C}}$ acts on $D(W) := C(W)_{\mathbf{C}} \cdot D$ freely.*

Proof. Since we assume (2.1), (i) is obvious in the case of odd w. In order to examine (i) in the case of even w, we choose a $\mathbf{Q}$-basis of $H_{\mathbf{Q}}$ according to the filtration W so that the polarization form S is represented by a matrix $S = \text{antidiagonal}(J, \Delta, J)$, where $J := \text{antidiagonal}(1, \cdots, 1)$ of rank ≥ 2, $\Delta := \pm I$. In this basis, any $X \in \mathfrak{c}$ represented by a matrix

$$X = \begin{pmatrix} 0 & 0 & A \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ where } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ is a } 2 \times 2 \text{ matrix.}$$

From ${}^tXS + SX = 0$, we can derive $d = -a, b = c = 0$ elementarily. This completes the proof of (i).

By using the above basis, (ii) can be also verified elementarily.

Let N be a positive basis of $\mathfrak{c}$. Since N is nilpotent, $\nu : \mathbf{C} \simeq C(W)_{\mathbf{C}} \to D(W) \subset \check{D}$, $z \mapsto \exp(zN)$, is an algebraic morphism. ν is not a constant map, because the isotropy subgroup I_r of G at r is compact hence it does not contain a unipotent subgroup $C(W) \simeq \mathbf{R}$. It follows that ν is quasi-finite. If $\nu(z_1) = \nu(z_2)$, $z_1, z_2 \in \mathbf{C}$, then $\exp((z_1 - z_2)N) \cdot r = r$ and so $\mathbf{Z}(z_1 - z_2) \subset \nu^{-1}(r)$, which occurs only if $z_1 = z_2$. This completes the proof. □

By Lemma (2.6.iii), the quotient $D(W)' := D(W)/C(W)_{\mathbf{C}}$ is a complex manifold and that the principal $C(W)_{\mathbf{C}}$-bundle $D(W) \to D(W)'$ is a complex affine bundle. Starting from this affine bundle, we shall construct a complex line bundle $L(W) \to D(W)'$ in the following way. Take a quotient bundle

$$D(W)/C(W)_{\mathbf{Z}} \to D(W)'. \tag{2.7}$$

Set $T(W) := C(W)_{\mathbf{C}}/C(W)_{\mathbf{Z}}$. Using the positive generator N of $\operatorname{Lie} C(W)_{\mathbf{Z}}$, we have an identification $T(W) \xrightarrow{\sim} \mathbf{C}^*$, $\exp(zN) \mapsto \exp(2\pi i z)$. Let $\mathbf{C}^* \subset \mathbf{C}$ be the natural embedding. We denote by

$$\pi : L(W) := (D(W)/C(W)_{\mathbf{Z}}) \times^{\mathbf{C}} \mathbf{C}^* \to D(W)' \tag{2.8}$$

the complex line bundle associated to the principal $\mathbf{C}^*$-bundle (2.7).

Proposition(2.9). *The action of $\overline{\Gamma}_W$ on the $\mathbf{C}^*$-bundle (2.7) extends to the action on the complex line bundle (2.8), which commutes with the action of $T(W)$. $\overline{\Gamma}_W$ acts properly discontinuously on $D(W)'$ and hence on $L(W)$.*

Proof. The first part follows easily from (2.6.ii) and an observation: $\det(\gamma|W_{w-1}) = 1$ for all $\gamma \in \Gamma_W$.

In order to prove the second part, we use the $\mathbf{C}^*$-bundle (2.7). Given a compact subset $A' \subset D(W)'$. Put $A := \pi^{-1}(A')$. Take a neighborhood V_a of $a \in A \cap (D/C(W)_{\mathbf{Z}})$ satisfying that the closure $\overline{V}_a$ is compact and contained in $D/C(W)_{\mathbf{Z}}$. Then $\{\pi(V_a)\,|\, a \in A \cap (D/C(W)_{\mathbf{Z}})\}$ is an open covering of A' and so we can choose a finite subset $\{\pi(V_{a_i})\,|\, 1 \le i \le n\}$ which covers A'. Set $V := \bigcup_{1 \le i \le n} N(W)^1 \cdot \overline{V}_{a_i}$, where

$$N(W)^1 := \{g \in N(W) \,|\, \det(g|W_{w-1}) = 1\}. \tag{2.10}$$

Then, by construction, we see that $V \subset D/C(W)_{\mathbf{Z}}$, $\pi(V) \supset A'$ and that the restriction $\pi : V \to D(W)'$ is a proper map. Since $\Gamma_W \subset N(W)^1$ whose action preserves the fiber coordinate of (2.8), we see that

$$\{\gamma \in \overline{\Gamma}_W \,|\, \gamma \cdot A' \cap A' \neq \emptyset\} = \{\gamma \in \overline{\Gamma}_W \,|\, \gamma \cdot (A \cap V) \cap (A \cap V) \neq \emptyset\}.$$

The latter set is finite because the action of $\overline{\Gamma}_W$ on $D/C(W)_{\mathbf{Z}}$ is properly discontinuous and $A \cap V \subset D/C(W)_{\mathbf{Z}}$ is a compact subset. This proves that $\overline{\Gamma}_W$ acts on $D(W)'$ properly discontinuously. The assertion on the action on $L(W)$ follows from this easily. □

§3. Boundary components $B(W, p)$.

Let $\{h^{p,q}\}$ be a set of Hodge numbers in (1.2). For a filtration W in (2.2), we set

$$n_\lambda := \operatorname{gr}^W_{w-\lambda}. \tag{3.1}$$

We recall a definition in [U, (2.15)]:

Definition(3.2). *A set $p = \{p_\lambda^{a,b}\}$ of non-negative integers is called a set of primitive Hodge numbers belonging to $\{h^{p,q}, n_\lambda\}$ if it satisfies the following conditions.*

(0) *The indices a, b and λ are non-negative integrers satisfying $a + b = w - \lambda$.*

(i) $\sum_{a+b=w-\lambda} p_\lambda^{a,b} = n_\lambda - n_{\lambda+2}$ *for all λ.*

(ii) $p_\lambda^{b,a} = p_\lambda^{a,b}$ *for all a, b, λ.*

(iii) $h^{s,t} = h^{s+1,t-1} - \sum_{0\le\lambda\le t-1} p_\lambda^{s+1,t-1-\lambda} + \sum_{0\le\lambda\le s} p_\lambda^{s-\lambda,t}$ *for all s, t with $s + t = w$.*

Under the assumption (2.1), only the following sets of primitive Hodge nembers are possible.

(3.3) Case $w = 2t + 1$. The possibility is unique.

$$p_1^{t,t} = 1,$$
$$p_0^{a,b} = \begin{cases} h^{a,b} - 1 & \text{if } a = t+1,\ t, \\ h^{a,b} & \text{otherwise.} \end{cases}$$

Case $w = 2t$. There are $t + 1$ possible cases.

(3.4) For each $s = w,\ w-1, \cdots, t+1$,

$$p_1^{s,w-s-1} = p_1^{w-s-1,s} = 1,$$
$$p_0^{a,b} = \begin{cases} h^{a,b} - 1 & \text{if } a = s+1,\ s,\ w-s,\ w-s-1, \\ h^{a,b} & \text{otherwise.} \end{cases}$$

(3.5) For $s = t$,

$$p_1^{t,t-1} = p_1^{t-1,t} = 1,$$
$$p_0^{a,b} = \begin{cases} h^{a,b} - 1 & \text{if } a = t+1,\ t-1, \\ h^{a,b} - 2 & \text{if } a = t, \\ h^{a,b} & \text{otherwise.} \end{cases}$$

Definition(3.6). *Given a filtration W in (2.2) and a set $p = \{p_\lambda^{a,b}\}$ of primitive Hodge numbers belonging to $\{h^{p,q}, n_\lambda\}$, the corresponding boundary component is a classifying space of the gradedly polarized mixed Hodge structures on $W_w H_{\mathbf{C}}$: $B = B(W,p) := \{F \mid F$ is a filtration on $W_w H_{\mathbf{C}}$ satisfying the condition (3.7) below$\}$.*

(3.7) $\mathrm{gr}_w^W F$ *(resp. $\mathrm{gr}_{w-1}^W F$) is a Hodge structure of weight w (resp. $w-1$) with Hodge type $\{p_0^{a,b}\}$ (resp. $\{p_1^{a,b}\}$) and polarized by the bilinear form induced by S on $\mathrm{gr}_w^W F$ (resp. positive $(-1)^{w-1}$-symmetric bilinear forms on W_{w-1}).*

We consider polarization forms on W_{w-1} are equivalent if they differ only by a positive multiplicative constant.

Proposition(3.8). *There is an $N(W)$-equivariant embedding $B(W,p) \hookrightarrow D(W)'$.*

Proof. We shall first construct a map $\varphi : B \to D'$, where $B := B(W,p)$, $D' := D(W)'$. Let $F \in B$. In the present case, the weight length is one, hence we have the Hodge-Deligne decomposition

$$W_w H_{\mathbf{C}} = \bigoplus P_\lambda^{a,b}, \quad P_\lambda^{a,b} := F^a \cap \sigma F^b \cap W_{w-\lambda} H_{\mathbf{C}},$$

where the summation is taken over $a+b=w-\lambda$, $\lambda=0,1$. We want to extend this to an S-polarized split mixed Hodge structure on $H_{\mathbf{C}}$ uniquely up to modulo $C(W)_{\mathbf{C}}$-action. Setting $P_{0,\mathbf{C}} := \bigoplus_{a,b} P_0^{a,b}$, we have a splitting over $\mathbf{R}$ of $W_{w-1}H_{\mathbf{C}} \subset W_w H_{\mathbf{C}}$. In case $w=2t+1$, our assertion follows immediately from the fact that $P_{-1,\mathbf{C}} := P_{-1}^{t+1,t+1}$ should be perpendicular to $P_{0,\mathbf{C}}$ with respect to S. Similarly, in case $w=2t$, $P_{-1,\mathbf{C}} := P_{-1}^{s+1,w-s} + P_{-1}^{w-s,s+1}$ is distinguished up to modulo $C(W)_{\mathbf{C}}$-action by the same condition, where s is the integer satisfying $p_1^{s,w-s-1}=1$ in the given set of primitive Hodge numbers. Moreover, the summands $P_{-1}^{a+1,w-a}$ ($a=s$, $w-s-1$) are distinguished up to modulo $C(W)_{\mathbf{C}}$-action by the condition that $P_{-1}^{a+1,w-a}$ should be perpendicular to $P_1^{a,w-a-1} + P_{-1}^{a+1,w-a}$ with respect to S. Now let $P_{-1}^{t+1,t+1}$ in case $w=2t+1$ and $P_{-1}^{s+1,w-s}$, $P_{-1}^{w-s,s+1}$ in case $w=2t$ be representatives among the above constructions. These deta determine a splitting $P_1 \oplus P_0 \oplus P_{-1}$ over $\mathbf{R}$ of the filtration $W_{w-1} \subset W_w \subset W_{w+1}$, where $P_1 := W_{w-1}$, $P_\lambda := P_{\lambda,\mathbf{C}} \cap H$ ($\lambda = 0,-1$). This, in turn determines a real semi-simple element $Y \in \mathfrak{g}$ so that P_λ is the λ-eigen space of Y. On the other hand, the nilpotent element N is determined as the positive generator of $\operatorname{Lie} C(W)_{\mathbf{Z}}$. Since $[Y,N]=2N$ by construction, we have a representation $\rho : \mathrm{SL}_2(\mathbf{R}) \to G$ (not necessarily rational). Transforming the $P_\lambda^{a,b}$ by the Cayley element $c=\rho(c_1)$ in (1.17), we get the Hodge-$(Z,X_\pm)$ decomposition: $\bigoplus Q_\lambda^{a,b+\lambda} := \bigoplus cP_\lambda^{a,b}$. Then we know that $H^{p,q} := \bigoplus_\lambda Q_\lambda^{p,q}$ determines an element $r \in D$ where ρ is horizontal (see [U, (3.4) and its proof]). We now define a map

$$\textbf{(3.9)} \qquad \varphi : B \to D' \quad \text{by} \quad F \mapsto C(W)_{\mathbf{C}} \cdot r.$$

Next we define a map

$$\textbf{(3.10)} \qquad \psi : \varphi(B) \to B \quad \text{by} \quad \exp(zN)\cdot F \mapsto F \cap W_w H_{\mathbf{C}}.$$

This is well-defined. Indeed, if $\exp(z'N)\cdot F' = \exp(zN)\cdot F$ then $F' = g\cdot F$ for $g := \exp((z-z')N)$. Since $W_w = \operatorname{Ker} N$, we see that $g|W_w$ is identity and so

$$F' \cap W_w H_{\mathbf{C}} = g\cdot F \cap W_w H_{\mathbf{C}} = g(F \cap W_w H_{\mathbf{C}}) = F \cap W_w H_{\mathbf{C}}.$$

We claim now that $\psi\varphi$ is identity. Indeed, let $F \in B$ and F_∞ the Hodge filtration associated to the S-polarized split mixed Hodge structure $\{P_\lambda^{a,b} \mid a+b=w-\lambda,\ \lambda = 1,0,-1\}$ constructed above. Then the filtration F_r corresponding to $r \in D$ is $F_r = cF_\infty$ by definition. On the other hand, $cF_\infty = \exp(iN)\cdot F_\infty$ in the present situation. This follows immediately from an observation that the restriction $\rho(\mathrm{SL}_2(\mathbf{C}))|P'$, where $P' := P_1^{a-1,w-a} + P_{-1}^{a,w-a+1}$, $a-1=t$ in case $w=2t+1$, and $a-1=s$, $w-s-1$ in case $w=2t$, yields a 2-dimensional irreducible representation of $\mathrm{SL}_2(\mathbf{C})$ hence we have $c = \frac{1}{\sqrt{2}}\exp(iN)$ on $P_{-1}^{a,w-a+1}$.

It is obvious that ψ is $N(W)$-equivariant. □

Let (ρ, r) be an SL_2-orbit, Y in (1.17), and $W = W(Y)$ in (1.18). We assume that W is defined over $\mathbf{Q}$. We denote

$$\textbf{(3.11)} \qquad G_Y := \{g \in G \mid gYg^{-1} = Y\}.$$

In the notation of (2.8), we set

$$\tilde{r} := (r \bmod C(W)_{\mathbf{Z}}) \in L(W), \quad b := \pi(\tilde{r}) \in D(W)'. \tag{3.12}$$

Then, by (1.19), we have $cF_\infty = F_r = \exp(iN_+) \cdot F_\infty$, and hence $b \in B = B(W,p)$ under the identification of (3.8).

Proposition(3.13). *In the above situation, we have the following.*

(i) *The orbits $G_Y \cdot b \subset N(W) \cdot b = B \subset D(W)'$ are complex submanifolds, where $B = B(W,p)$.*

(ii) *$((C(W) \rtimes G_Y) \cdot r)^\sim \to G_Y \cdot b$ and $(N(W) \cdot r)^\sim \to B$ are punctured disc bundles contained in the line bundle (2.8). $(G_Y \cdot r)^\sim \to G_Y \cdot b$ is the family of all SL_2-orbits corresponding to the pair (Y,p), and $(N(W) \cdot r)^\sim \to B$ is the family of all nilpotent orbits corresponding to the pair (W,p).*

(iii) *$N(W) \cdot r$ is open in D if and only if D is a Hermitian symmetric domain.*

Proof. We first claim that

$$\begin{aligned} &\dim_{\mathbf{R}} N(W)/I_r \cap N(W) = 2\dim_{\mathbf{C}} N(W)_{\mathbf{C}}/I_{\mathbf{C},r} \cap N(W)_{\mathbf{C}}, \\ &\dim_{\mathbf{R}}(C(W) \rtimes G_Y)/I_r \cap (C(W) \rtimes G_Y) \\ &\qquad = 2\dim_{\mathbf{C}}(C(W)_{\mathbf{C}} \rtimes G_{Y,\mathbf{C}})/I_{\mathbf{C},r} \cap (C(W)_{\mathbf{C}} \rtimes G_{Y,\mathbf{C}}), \end{aligned} \tag{3.14}$$

where I_r and $I_{\mathbf{C},r}$ are the isotropy subgroups at r of G and of $G_{\mathbf{C}}$, respectively. (3.14) can be verified elementarily by the dimension count of the corresponding Lie algebras using bases of $H_{\mathbf{C}}$ according to the mixed Hodge-$(Y, N_\pm)$ decomposition of (ρ, r) (cf. [U, §2]), hence we left it to the reader. Similarly, we can verify elementarily that $N(W)$ acts on B transitively and so we omit this verification. (3.14) shows that orbit $N(W) \cdot r$ (resp. $(C(W) \rtimes G_Y) \cdot r$) is open in $N(W)_{\mathbf{C}} \cdot r$ (resp. $(C(W)_{\mathbf{C}} \rtimes G_{Y,\mathbf{C}}) \cdot r$) in the Hausdorff topology and the latter is a closed complex submanifold of $\check{D} = G_{\mathbf{C}} \cdot r$, hence the former induces a complex submanifold $(N(W) \cdot r)^\sim$ (resp. $((C(W) \rtimes G_Y) \cdot r)^\sim$) of $D(W)/C(W)_{\mathbf{Z}}$. From this we know that the interior of the closure of $(N(W) \cdot r)^\sim$ (resp. $((C(W) \rtimes G_Y) \cdot r)^\sim$) in $L(W)$, denoted by

$$\mathcal{N} = \mathcal{N}(W,p) \quad (\text{resp. } \mathcal{S} = \mathcal{S}(Y,p)), \tag{3.15}$$

is a complex submanifold and so the intersection of $\mathcal{N}$ (resp. $\mathcal{S}$) with the zero section of the line bundle (2.8) is a complex submanifold of the zero section. Via the projection, we get the assertion (i).

Now the first part of (ii) follows from (2.6.ii) and the observations that $N(W) = N(W)^1 \exp(\mathbf{R}Y)$ (for $N(W)^1$, see (2.10)), $\exp(iyN_+) \cdot r = \exp(\log(y+1)^{1/2} Y) \cdot r$, and

$$\det(\exp(\log(y+1)^{1/2}Y)|W_{w-1}) = y + 1 > 0 \iff e^{-2\pi y} < e^{2\pi}.$$

As for the second part of (ii), the assertion on the family $(G_Y \cdot r)^\sim \to G_Y \cdot b$ follows from [U, (3.16.iii)]. Let $g \cdot r \in N(W) \cdot r$ and $F_{g \cdot r}$ the corresponding Hodge filtration. Then, by (2.6.ii),

$$\begin{aligned} \exp(iyN_+)g \cdot r &= g\exp(iy\det(g^{-1}|W_{w-1})N_+) \cdot r \\ &= g\exp(\log(y\det(g^{-1}|W_{w-1}) + 1)^{1/2}Y) \cdot r \in D \quad \text{for } y > 0. \end{aligned} \tag{3.16}$$

On the other hand, applying the argument at the end of the proof of Proposition (3.8) to the Hodge-$(Z, X_\pm)$ decomposition and the mixed Hodge-$(Y, N_\pm)$ decomposition $H_{\mathbf{C}} = \bigoplus Q_\lambda^{a,b+\lambda} = \bigoplus P_\lambda^{a,b}$ associated to (ρ, r) (cf. [U, §2]), we see that, for $P' := P_1^{a-1,w-a} + P_{-1}^{a,w-a+1}$,

$$N_+ Q_{-1}^{a,w-a} \subset N_+ cP' = N_+ P' \subset P' = cP'.$$

It follows that $N_+ F_r^a \subset F_r^{a-1}$ and hence $N_+ F_{g\cdot r}^a \subset F_{g\cdot r}^{a-1}$ by (2.6.ii). Therefore $\exp(\mathbf{C}N_+)g\cdot r$ is a nilpotent orbit in the direction of (W, p). Conversely, let (N_+, F), $F \in \check{D}$, be a nilpotent orbit, i.e., $N_+ F^a \subset F^{a-1}$ and $\exp(iyN_+)\cdot F \in D$ for $y \gg 0$. Then, by [Sc, (6.16)], (W, F) is an S-polarized mixed Hodge structure. If (W, F) has mixed Hodge type p then this determines a point of B by $F \cap W_w H_{\mathbf{C}}$ hence, by (3.8) and the first part of (ii), we have $\exp(iyN_+)\cdot F \in N(W)\cdot r$ for $y \gg 0$. This completes the proof of (ii).

In order to prove (iii), we shall compute $\dim D - \dim N(W)\cdot r$. Let K be a maximal compact subgroup of G containing the isotropy subgroup I_r, $G = RTK$ an Iwasawa decomposition.

Case $w = 2t+1$, i.e., (3.3). We see that

$$G = \mathrm{Sp}(2h, \mathbf{R}),\ \ K \simeq U(h),\ \ I_r \simeq U(h^{w,0}) \times \cdots \times U(h^{t+1,t}),$$
$$K_Y \simeq U(h-1),\ \ I_{r,Y} := I_r \cap G_Y \simeq U(h^{w,0}) \times \cdots \times U(h^{t+2,t-1}) \times U(h^{t+1,t} - 1)$$

Hence

$$\dim D - \dim N(W)\cdot r = \dim G/I_r - \dim N(W)/I_{r,Y} = \dim K/I_r - \dim K_Y/I_{r,Y}$$
$$= h^2 - (h-1)^2 - (h^{t+1,t})^2 + (h^{t+1,t} - 1)^2 = 2(h - h^{t+1,t}).$$

This is zero if and only if $h = h^{t+1,t}$, that is, $K = I_r$.

Case $w = 2t$. We see that

$$G = O(2h, k),\ \ K \simeq O(2h) \times O(k),$$
$$I_r \simeq U(h^{w,0}) \times \cdots \times U(h^{t+1,t-1}) \times O(h^{t,t}),$$
$$K_Y \simeq O(2h-2) \times O(k-2) \times O(2),$$

According to the subcases (3.4), (3.5), $I_{r,Y}$ is isomorphic, respectively, to

$$U(h^{w,0}) \times \cdots \times U(h^{s+1,w-s-1} - 1) \times U(h^{s,w-s} - 1) \times \cdots \times U(h^{t+1,t-1}) \times O(h^{t,t}) \times U(1),$$
$$U(h^{w,0}) \times \cdots \times U(h^{t+1,t-1} - 1) \times O(h^{t,t} - 2) \times U(1).$$

As before, we can compute $\dim D - \dim N(W)\cdot r$ to obtain

$$2(2h + k - h^{s+1,w-s-1} - h^{s,w-s} - 2) \quad \text{in case (3.4)},$$
$$2(2h + k - h^{t+1,t-1} - h^{t,t} - 1) \quad \text{in case (3.5)}.$$

These are zero if and only if

$$h = h^{s+1,w-s-1}\ (\text{or } h^{s,w-s}) = 1,\ \ k = 2h^{s,w-s}\ (\text{or } 2h^{s+1,w-s-1}) = 2 \quad \text{in case (3.4)},$$
$$h = h^{t+1,t-1} = 1,\ \ k = h^{t,t} \quad \text{in case (3.5)}.$$

Hence, $\dim D = \dim N(W)\cdot r$ if and only if $K = I_r$. This completes the proof of the proposition. □

We denote

$$\text{(3.17)}\quad \widetilde{D}_{W,p} := D/C(W)_{\mathbf{Z}} \cup \mathcal{N}(W,p) \subset L(W),\ \widetilde{D}_W := \bigcup_p \widetilde{D}_{W,p} \subset L(W),\ \widetilde{D} := \bigsqcup_W \widetilde{D}_W,$$

where the unions are taken over all sets p of primitive Hodge numbers belonging to $\{n_\lambda, h^{p,q}\}$ and all rational S-isotropic filtrations W of $H_{\mathbf{Q}}$ in (2.2) satisfying (2.1).

§4. Construction of partial compactifications $\overline{D/\Gamma}$.

We recall first the partial compactification D^{**}/Γ of Cattani-Kaplan in [CK] and its generalization into arbitrary weight [U, Appendix] whithin our present use. Under the assumption (2.1), the disjoint union D^{**} of all *rational boundary components* and the disjoint union D^* of all *rational boundary bundles*, both in the sense of [CK], coincide and it is defined by

$$\text{(4.1)}\qquad D^* := D \sqcup (\bigsqcup_{W,p} F(W,p)),\quad F(W,p) := \{\mathrm{gr}^W F \mid F \in B(W,p)\},$$

where W and p run over all rational S-isotropic filtrations (2.2) of $H_{\mathbf{Q}}$ satisfying the condition (2.1) and all sets of primitive Hodge numbers, respectively, and $B(W,p)$ is a boundary component in the sense of (3.6).

In order to introduce the Satake topology on D^*, we choose a maximal $\mathbf{Q}$-split Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ and a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ with $\mathfrak{p} \supset \mathfrak{t}$. Let $\Phi \subset \mathfrak{t}^*$ be the $\mathbf{Q}$-root system, $\Phi^+ \subset \Phi$ the positive root system with respect to some lexicographical order in $\mathfrak{t}^*$. Let $G = RTK$ be the Iwasawa decomposition, where $R := \exp(\sum_{\alpha\in\Phi^+} \mathfrak{g}_\alpha)$, $T := \exp \mathfrak{t}$ and K is the maximal compact subgroup of G with Lie $K = \mathfrak{k}$.

Let $\mathfrak{t}^+ := \{A \in \mathfrak{t} \mid \alpha(A) > 0 \text{ for all } \alpha \in \Phi^+\}$ be the Weyl chamber. We denote by $\mathcal{A}$ the set of all rational admissible elements in the closure $\overline{\mathfrak{t}^+}$ of $\mathfrak{t}^+$ in $\mathfrak{t}$. Then we see, by construction, that $\mathcal{A}$ is finite and a set of complete representatives of all $G_{\mathbf{Q}}$-conjugacy classes of rational admissible elements. Under the assumption (2.1), $\mathcal{A}$ consists of the single element $Y := \mathrm{diagonal}(1_s, 0, \cdots, 0, -1_s)$, where $s = 1$ if w is odd and $s = 2$ if w is even. Let $W(Y)$ be the weight filtration associated to Y in (1.18). For each set $p = \{p_\lambda^{a,b}\}$ of primitive Hodge numbers, we take a reference point $r_p \in D$ lying over $[K] \in G/K$, via some fixed projection $D \to G/K$, such that (Y, r_p) is an admissible pair of type p. This is possible by [U, (3.16.ii)]. We set

$$\text{(4.2)}\qquad \begin{aligned} &\tilde{r}_p := (r_p \mod C(W(Y))_{\mathbf{Z}}) \in L(W(Y)),\\ &b_p := \pi(\tilde{r}_p) \in B(W(Y),p),\quad \bar{b}_p := \mathrm{gr}^{W(Y)}(b_p) \in F(W(Y),p). \end{aligned}$$

The *Satake topology* $\tau^\Gamma(D^*)$ on D^* relative to Γ in [CK] is introduced in the following process (i)–(iii):

(i) An *open Siegel set* subject to the Iwasawa decomposition $G = RTK$ is a subset $\mathfrak{S} := \omega T_\mu K$ of G, where ω is a relatively compact open neighborhood of 1 in R, $\mu > 0$ and $T_\mu := \{t \in T \mid e^\alpha(t) > \mu \text{ for all } \alpha \in \Phi^+\}$. An *extended Siegel set* in D^* is a subset

$\mathfrak{S}^* := \bigcup_p(\mathfrak{S}\cdot r_p \cup (\mathfrak{S}\cap N(W(Y)))\cdot \bar{b}_p)$. For suitable choices of ω and μ, there exists a finite subset E of $G_{\mathbf{Q}}$ satisfying $\Gamma E\mathfrak{S}\cdot r_p = D$ and $\Gamma_{W(Y)}(E\cap N(W(Y)))(\mathfrak{S}\cap N(W(Y)))\cdot \bar{b}_p = F(W(Y),p)$ for all p. Then, as [CK, (4.28)], $\Omega^* := E\mathfrak{S}^*$ is a Γ-*fundamental set* in D^*, i.e, satisfies the following two conditions.

(4.3)
$$\Gamma\Omega^* = D^*.$$

(4.4) *There exist finitely many $\gamma_\nu \in \Gamma$ such that, if $\gamma \in \Gamma$, $\gamma\Omega^*\cap\Omega^* \neq \emptyset$, then the actions of γ and γ_ν coincide on $\Omega^*\cap\gamma^{-1}\Omega^*$ for some ν.*

(ii) A topology $\tau(\mathfrak{S}^*)$ on $\mathfrak{S}^*$ is defined so that a basis of open sets is given by open subsets of $\mathfrak{S}\cdot r_p(\subset D)$ in the natural topology together with subsets

(4.5)
$$(U_\lambda V\cdot r_p \cup U\cdot \bar{b}_p)\cap\mathfrak{S}^*$$

for all p, where U runs over the pull-backs via the projection $N(W(Y)) \to F(W(Y),p)$, $g \mapsto g\cdot\bar{b}_p$, of all open sets in $F(W(Y),p)$ in the natural topology, λ is a positive real number, $U_\lambda := \{g \in U \mid e^\alpha(g) > \lambda$ for all $\alpha \in \Phi$ with $\alpha(Y) > 0\}$, V runs over neighborhoods of 1 in K. The topology $\tau(\Omega^*)$ on Ω^* is induced from $\tau(\mathfrak{S}^*)$ in the following way: the system of neighborhoods of $x \in \Omega^*$ consists of all subsets $\mathcal{U} \subset \Omega^*$ satisfying the condition that, if $x \in e\mathfrak{S}^*$ with $e \in E$, then $e^{-1}\mathcal{U}\cap\mathfrak{S}^*$ is a $\tau(\mathfrak{S}^*)$-neighborhood of $e^{-1}x \in \mathfrak{S}^*$. Then, as [CK, (4.32)], the topology $\tau(\Omega^*)$ has the following property.

(4.6) *$\tau(\Omega^*)$ is Hausdorff and the action of $\gamma \in \Gamma$ is continuous in $\tau(\Omega^*)$ in the following sense: let $x \in \Omega^*$; if $\gamma x \in \Omega^*$, then for any $\tau(\Omega^*)$-neighborhood $\mathcal{U}'$ of γx there exists a $\tau(\Omega^*)$-neighborhood $\mathcal{U}$ of x such that $\gamma\mathcal{U}\cap\Omega^* \subset \mathcal{U}'$; if $\gamma x \notin \Omega^*$, then there exists a $\tau(\Omega^*)$-neighborhood $\mathcal{U}$ of x such that $\gamma\mathcal{U}\cap\Omega^* = \emptyset$.*

(iii) By virtue of (4.3), (4.4) and (4.6), [Sa, Theorem 1′] can be applied to obtain a Satake topology $\tau^\Gamma(D^*)$ (uniquely determined) with the following four properties.

(4.7.1) *$\tau^\Gamma(D^*)$ induces $\tau(\Omega^*)$ (and also $\tau(\mathfrak{S}^*)$).*

(4.7.2) *The action of Γ on D^* is continuous.*

(4.7.3) *If $\Gamma x\cap\Gamma x' = \emptyset$ with x, $x' \in D^*$, then there exist $\tau^\Gamma(D^*)$-neighborhoods $\mathcal{U}$ of x and $\mathcal{U}'$ of x' such that $\Gamma\mathcal{U}\cap\Gamma\mathcal{U}' = \emptyset$.*

(4.7.4) *For each $x \in D^*$, there exists a fundamental system $\{\mathcal{U}\}$ of $\tau^\Gamma(D^*)$-neighborhoods of x such that $\gamma\mathcal{U} = \mathcal{U}$ for $\gamma \in \Gamma_x$, $\gamma\mathcal{U}\cap\mathcal{U} = \emptyset$ for $\gamma \notin \Gamma_x$, where Γ_x is the isotropy subgroup of Γ at x.*

In [CK], they use a closed Siegel set in stead of an open one. In both cases the arguments are parallel. In [CK, §5], they show that the Satake topology $\tau^\Gamma(D^*)$ is independent of choices of the following things: $\mathfrak{t}$, Φ^+, K, r_p, Γ, $\mathfrak{S}$, E. As Looijenga has pointed out to the author, the induced topology on D^*/Γ is not locally compact in general (cf. [CK, (4.36.i)]).

Definition(4.8). *In the notation of (3.17), a Satake topology $\tau(\widetilde{D})$ on $\widetilde{D}$ is defined in the following way.*

(i) *We first define a topology* $\tau(D \sqcup B(W(Y)))$, *where* $B(W(Y)) := \bigsqcup_p B(W(Y),p)$. *On* D, *this topology coincides with the natural one. At a boundary point* $x \in B(W(Y))$, *a fundamental system of neighborhoods is givn by*

$$U_\lambda V \cdot r_p \sqcup U \cdot b_p,$$

where U *runs over the pull-backs via the projection* $N(W(Y)) \to B(W(Y))$, $g \mapsto g \cdot b_p$, *of all neighborhoods of* x *in* $B(W(Y))$ *in the natural topology,* λ *is a positive real number,* $U_\lambda := \{g \in U \mid e^\alpha(g) > \lambda$ *for all* $\alpha \in \Phi$ *with* $\alpha(Y) > 0\}$, V *runs over neighborhoods of 1 in* K.

(ii) *We extend* $\tau(D \sqcup B(W(Y)))$ *to* $\tau(\bigsqcup_W (D \sqcup B(W)))$, *where* W *runs over all rational* S*-isotropic filtrations (2.2) of* $H_{\mathbf{Q}}$ *satisfying the condition (2.1), so that the action of* $G_{\mathbf{Q}}$ *is continuous on the latter.*

(iii) $\tau(\widetilde{D})$ *is the topology induced from* $\tau(\bigsqcup_W (D \sqcup B(W)))$.

It is easy to see that the Satake topology $\tau(\widetilde{D})$ is well-defined, and we can prove similarly as in [CK, §5] that $\tau(\widetilde{D})$ is independent of the choices of $\mathfrak{t}$, Φ^+, K, r_p.

Lemma(4.9). *The restriction of* $\tau(\widetilde{D})$ *to* $\mathcal{N}(W,p)$ *coincides with the natural topology on it for every* W *and* p, *where* $\mathcal{N}(W,p)$ *is in (3.15).*

Proof. The assertion follows immediately by Definition(4.8) and (3.16) for the SL_2-orbit (ρ, r_p) corresponding to the admissible pair (Y, r_p). □

Problem(4.10). Compare the topology $\tau(\widetilde{D}_W)$ with the natural one on $\widetilde{D}_W \subset L(W)$.

Lemma(4.11). *The natural map* $f : \widetilde{D} \to D^*/\Gamma$ *is continuous in the Satake topologies.*

Proof. Set $W = W(Y)$. By Definition(4.8) and [CK, (5.7)] and its generalization, it is enough to show that, in the notation of (3.17), the natural map

$$f_{W,p} : \widetilde{D}_{W,p} \to D^*/C(W)_{\mathbf{Z}} \tag{4.12}$$

is continuous in the Satake topologies for any p.

It is obvious that $f_{W,p}$ is continuous on $D/C(W)_{\mathbf{Z}}$. Let $x \in B(W,p)$ and $\overline{x}$ its image in $F(W,p)$. Note that a fundamental system of $\tau(D^*)$-neighborhoods of $\overline{x} \in D^*$ is given by the following sets (cf. [CK, (4.31)], [Sa, Proof of Theorem 1′]):

$$\mathcal{U} = \Gamma_{\overline{x}}\Big(\bigcup_{g \in \Gamma E,\ g\mathfrak{S}^* \ni \overline{x}} g(\tau(\mathfrak{S}^*)\text{-neighborhood of } g^{-1}\overline{x} \in \mathfrak{S}^*). \tag{4.13}$$

Hence, in order to prove the continuity of $f_{W,p}$, it is enough to show that, on $\widetilde{D}_{W,p}$, the topology $\tau_1(\widetilde{D}_{W,p})$, similarly defined as the topology $\tau(D^*/C(W)_{\mathbf{Z}})$ on $D^*/C(W)_{\mathbf{Z}}$ induced by $\tau^\Gamma(D^*)$, coincides with the topology $\tau(\widetilde{D}_{W,p})$ induced by $\tau(\widetilde{D})$.

We many assume that the Siegel set $\mathfrak{S}$ and a finite subset $E \subset G_{\mathbf{Q}}$ satisfy $C(W)_{\mathbf{Z}}\mathfrak{S} \supset C(W)$ and $\Gamma_W(E \cap N(W))(\mathfrak{S} \cap N(W)) \cdot b_p = B(W,p)$ for all p. Set $\mathfrak{S}_W := \mathfrak{S} \cap N(W)$, $r := r_p$ and $b := b_p$. Since $\mathfrak{S}_W \exp(\mathbf{R}_{>0} \cdot Y) = \mathfrak{S}_W$, $(\mathfrak{S}_W \cdot r)^\sim \sqcup \mathfrak{S}_W \cdot b$ is an open subset of

$\mathcal{N} := \mathcal{N}(W,p)$ in the natural topology. It follows that the topology $\tau_1((\mathfrak{S}_W \cdot r)^\sim \sqcup \mathfrak{S}_W \cdot b)$, induced from $\tau_1(\mathfrak{S}_W \cdot r \sqcup \mathfrak{S}_W \cdot b)$ which is similarly defined as $\tau(\mathfrak{S}^*)$, coincides with the natural topology on $(\mathfrak{S}_W \cdot r)^\sim \sqcup \mathfrak{S}_W \cdot b \subset \mathcal{N}$. Since the action of $N(W)$ on $\mathcal{N}$ is continuous in the natural topology, the topology $\tau_1(\mathcal{N})$, similarly defined as $\tau(D^*/C(W)_{\mathbf{Z}})$, coincides with the natural topology on $\mathcal{N}$ by (4.13). Evidently the multiplication by $g \in N(W)$ from the left to $U_\lambda V$ in (4.5) does not impose any effect on the neighborhood V of 1 in K. Thus we get $\tau_1(\widetilde{D}_{W,p}) = \tau(\widetilde{D}_{W,p})$. □

Corollary(4.14). *For any $x \in B(W,p)$, there exists a Satake neighborhood $\mathcal{U}_x$ of x in $\widetilde{D}$ such that the Γ-equivalence and Γ_W-equivalence coincide on $\mathcal{U}_x \cap D/C(W)_{\mathbf{Z}}$.*

Proof. By the lemma, this follows immediately from (4.7.4). □

Lemma(4.15). *In the Satake topology, the action of $\overline{\Gamma}_W$ on $\widetilde{D}_W$ is properly discontinuous, hence the Γ_W-equivalence relation is closed on $\widetilde{D}_W$.*

Proof. Let $x \in B(W,p)$, and $\overline{x} \in F(W,p)$ its image. Let $\mathcal{U}_{\overline{x}}$ be a Satake neighborhood of $\overline{x} \in D^*$ satisfying the condition (4.7.4). By Lemma (4.11), we can take a Satake neighborhood $\mathcal{U}_x = (U_\lambda V \cdot r_p)^\sim \cup U \cdot b_p$ of $x \in \widetilde{D}_{W,p}$ contained in $f_{W,p}^{-1}(\mathcal{U}_{\overline{x}} \mod C(W)_{\mathbf{Z}})$. By Proposition (2.9), we may assume that $\{\gamma \in \overline{\Gamma}_W \,|\, \gamma U \cdot b_p \cap U \cdot b_p \neq \emptyset\}$ is finite. Since $F(W,p) = B(W,p)/U(W)$, where $U(W)$ is in (2,3), we see that the isotropy subgroup $\Gamma_{\overline{x}}$ at $\overline{x}$ is equal to $U(W)_{\mathbf{Z}} \rtimes \Gamma_x$.

For $\gamma \in U(W)_{\mathbf{Z}}$, we claim that $\gamma\mathcal{U}_x \cap \mathcal{U}_x \neq \emptyset$ if and only if $\gamma U \cdot b_p \cap U \cdot b_p \neq \emptyset$. To see this, notice that $\gamma\mathcal{U}_x \cap \mathcal{U}_x \neq \emptyset$ is equivalent to

$$\gamma(U_\lambda V \cdot r_p)^\sim \cap (U_\lambda V \cdot r_p)^\sim \neq \emptyset, \quad \text{or} \quad \gamma U \cdot b_p \cap U \cdot b_p \neq \emptyset.$$

The former implies $\gamma U_\lambda V \cap U_\lambda V I_{r_p} \neq \emptyset$, hence, by the uniqueness of the Iwasawa decomposition, we have $\gamma U_\lambda \cap U_\lambda \neq \emptyset$, and so $\gamma U \cdot b_p \cap U \cdot b_p \neq \emptyset$ as desired. This proves the 'only if' part. The converse is obvious.

Thus we see $\{\gamma \in \overline{\Gamma}_W \,|\, \gamma\mathcal{U}_x \cap \mathcal{U}_x \neq \emptyset\} = \{\gamma \in \overline{\Gamma}_{\overline{x}} \,|\, \gamma\mathcal{U}_x \cap \mathcal{U}_x \neq \emptyset\} = \{\gamma \in \overline{\Gamma}_W \,|\, \gamma U \cdot b_p \cap U \cdot b_p \neq \emptyset\}$, which is finite. This proves the lemma. □

Using the Satake neighborhoods $\mathcal{U}_x$ in (4.14), we now construct our partial compactification $\overline{D/\Gamma}$ by patching up

$$\textbf{(4.16)} \qquad \overline{\Gamma}_W \cdot \mathcal{U}_x/\overline{\Gamma}_W \overset{\text{open}}{\supset} \overline{\Gamma}_W \cdot (\mathcal{U}_x \cap D/C(W)_{\mathbf{Z}})/\overline{\Gamma}_W \overset{\text{open}}{\subset} D/\Gamma$$

for all $x \in B(W,p)$, all rational S-isotropic filtrations W of $H_{\mathbf{Q}}$ in (2.2) satisfying the condition (2.1) and all sets $p = \{p_\lambda^{a,b}\}$ of primitive Hodge numbers belonging to $\{h^{p,q}, n_\lambda\}$. In the above construction, the W can be taken over a set

(4.17) $\mathcal{W} :=$ (set of complete representatives of the $G_{\mathbf{Q}}$-orbit of $W(Y)$ mod Γ-action),

which is finite by (4.3).

Theorem(4.18). *$\overline{D/\Gamma}$ with the Satake topology is Hausdorff and carries the complex structure induced from $\widetilde{D}_W \subset L(W)$ for all $W \in \mathcal{W}$.*

Proof. By construction, $\overline{D/\Gamma} \simeq D/\Gamma \sqcup \bigsqcup_{W\in\mathcal{W},\, p} B(W,p)/\Gamma_W$ as point sets. Let Δ be the graph of the equivalence relation defined by the projection $\widetilde{D} \to \overline{D/\Gamma}$. Notice that $\overline{D/\Gamma}$ is Hausdorff if and only if the graph $\Delta \subset \widetilde{D} \times \widetilde{D}$ is closed. To see the closedness of Δ, it is enough to show the following: if x_i, $y_i \in D$, and $\gamma_i \in \Gamma$ with $y_i = \gamma_i x_i$ satisfy (x_i mod $C(W)_{\mathbf{Z}}) \to x \in B(W,p)$, ($y_i$ mod $C(W)_{\mathbf{Z}}) \to y \in B(W',p')$ in the Satake topology, then $(x,y) \in \Delta$.

By Lemma (4.11) and the Hausdorffness of D^*/Γ in [CK, (4.36.i)], the images of x and y in D^*/Γ coincide, hence lie in the same boundary components $F(W,p)/\Gamma_W$ of D^*/Γ. It follows that $W' = \delta W$ for some $\delta \in \Gamma$ and $p = p'$. Replacing y_i, y by $\delta^{-1}y_i$, $\delta^{-1}y$, it suffices to prove the assertion in the special case: x, $y \in B(W,p)$. We consider a diagram:

$$\begin{array}{ccccc} \widetilde{D}_{W,p} & \xrightarrow{f_{W,p}} & D^*/C(W)_{\mathbf{Z}} & \to & D^*/\Gamma \\ & & \cup & & \cup \\ & & F(W,p) & \to & F(W,p)/\Gamma_W \end{array}$$

Since x, y have the same image in D^*/Γ, their images in $F(W,p) \subset D^*/C(W)_{\mathbf{Z}}$ differ by a $\gamma \in \Gamma_W$. Again replacing y_i, y by $\gamma^{-1}y_i$, $\gamma^{-1}y$, we may assume that x, y have the same image $\overline{x} \in F(W,p) \subset D^*/C(W)_{\mathbf{Z}}$. Let $\mathcal{U}_{\overline{x}} \subset D^*$ be a Satake neighborhood of $\overline{x}$ satisfying the condition in (4.7.4). Then $\mathcal{V} := f_{W,p}^{-1}(\mathcal{U}_{\overline{x}}/C(W)_{\mathbf{Z}})$ is a Satake open subset of $D(W)/C(W)_{\mathbf{Z}} \cup \mathcal{N}(W,p)$ containing x, y. Therefore, x_i, y_i mod $C(W)_{\mathbf{Z}} \in \mathcal{V}$ if $i \gg 0$. In other words, x_i, $y_i \in \mathcal{U}_{\overline{x}} \cap D$ if $i \gg 0$. Now $y_i = \gamma_i x_i$, $\gamma_i \in \Gamma$, so, by the assumption on $\mathcal{U}_{\overline{x}}$, we see $\gamma_i \in \Gamma_{\overline{x}} \subset \Gamma_W$ for $i \gg 0$. Hence the first assertion follows from Lemma (4.15). The second assertion follows from Corollary (4.14) and Lemma (4.15). □

§5 Extension of period maps.

Let $\varphi : \Delta^* \to D/\Gamma$ be a *period map*, i.e., a homolomorphic map with horizontal local liftings, from the punctured unit disc Δ^*. Let $\mathfrak{h} \to \Delta^*$, $z \mapsto \exp(2\pi i z)$, be the universal cover, $\widetilde{\varphi} : \mathfrak{h} \to D$ a lifting of φ, $\gamma \in \Gamma$ an element satisfying $\widetilde{\varphi}(z+1) = \gamma\widetilde{\varphi}(z)$ for all $z \in \mathfrak{h}$, N the logarithm of the unipotent part of γ, and $W(N)$ the monodromy weight filtration.

Theorem(5.1). (i) *Any period map $\varphi : \Delta^* \to D/\Gamma$ from the puncture disc with the monodromy weight filtration $W = W(N)$ satisfying the condition (2.1) extends holomorphically to $\overline{\varphi} : \Delta \to \overline{D/\Gamma}$.*

(ii) *For any boundary point $\overline{\xi} \in \overline{D/\Gamma} - D/\Gamma$, there exists a period map $\varphi : \Delta^* \to D/\Gamma$ with the property described in (i) and its holomorphic extension $\overline{\varphi} : \Delta \to \overline{D/\Gamma}$ such that $\overline{\varphi}(0) = \overline{\xi}$.*

Proof. As the proof is almost analoguous to the one in [CK], we shall write down the proof as long as it is needed. By the rational version of the SL_2-orbit theorem [Sc, (5.13), (5.19), (5.26)], there exists an SL_2-orbit (ρ, r_p) with ρ defined over $\mathbf{Q}$, such that $\rho_* \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = N$, and satisfies the property (5.2) below. Let $Y := \rho_* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Choose a maximal $\mathbf{Q}$-split Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ containing Y, and a positive root system $\Phi^+ \subset \mathfrak{t}^*$ for the adjoint action of $\mathfrak{t}$ on $\mathfrak{g}$ satisfying that any root α with $\alpha(Y) > 0$ belongs to Φ^+. Set $R := \exp(\sum_{\alpha\in\Phi^+} \mathfrak{g}_\alpha)$ and $T := \exp \mathfrak{t}$. Then the centralizer of T in G is a product TM with M $\mathbf{Q}$-anisotropic, and $P := RTM$ is a minimal $\mathbf{Q}$-parabolic subgroup of G. Let K be the maximal compact subgroup of G corresponding to the Cartan involution θ_{r_p} determined by the reference point r_p as in (1.11). Then $G = PK = RTMK$, and we have the following:

(5.2) There exist functions $r(x,y)$, $t(x,y)$, $m(x,y)$ and $k(x,y)$ defined and real analytic on a domain $\{x+iy \in \mathfrak{h} \mid y > \beta\}$ for some β and taking values in groups R, T, M and K, respectively, such that

(5.2.1) $\widetilde{\varphi}(x+iy) = r(x,y)t(x,y)m(x,y)k(x,y)\cdot r_p$.

(5.2.2) As $y \to +\infty$, the functions converge

$r(x,y) \to \exp(xN)r(\infty)$, $\exp(\log y^{-1/2}Y)t(x,y) \to 1$, $m(x,y) \to 1$, $k(x,y) \to 1$, uniformly in x, where $r(\infty) \in \exp \mathfrak{v}$ with $\mathfrak{v} := \mathrm{Im}(\mathrm{ad}_\mathfrak{g} N) \cap \mathrm{Ker}(\mathrm{ad}_\mathfrak{g} N)$.

By [CK, (6.4)], we see $\exp \mathfrak{v} \subset U(W)$. (Since $N^2 = 0$ in the present case, the proof is easier.) φ factors through $\Delta^* \to D/C(W)_\mathbf{Z}$, denoted also by φ, by an abuse of the notation. We now claim

(5.3) $\lim_{t\to 0}\varphi(t) = r(\infty)\cdot b_p \in D/C(W)_\mathbf{Z} \cup \mathcal{N}(W,p)$ in the Satake topology, where $b_p \in B(W,p)$ is induced from r_p as in (3.12).

In order to set the situation where we have introduced the Satake topology, we choose a maximal compact subgroup K' of G whose associated Cartan involution acts on $\mathfrak{t}$ by multiplication by -1. Then, as in the proof of [U, (3.16.ii)], there exists $g \in G_Y$ such that $K' = (\mathrm{Int}\, g)K$. $g \in G_Y$ splits according to the decomposition $G = PK$, hence we may assume moreover $g \in P \cap G_Y$. Set $r'_p := g\cdot r_p \in D$ and $b'_p := g \cdot b_p \in B = B(W,p)$. We are thus in the situation after (4.1). Then (5.3) follows if we show

(5.4) in the notation of (4.8), for the pull-back U' via the projection $N(W) \to B, h \mapsto h\cdot b'_p$, of any neighborhood of $\xi' := gr(\infty)\cdot b_p$ in B, any $\lambda > 0$ and any neighborhood V' of 1 in K', there exists $\beta > 0$ such that $g\cdot\widetilde{\varphi}(x+iy) \in U'_\lambda V' \cdot r'_p$ for all $y > \beta$ and $|x| < 1$.

Indeed, (5.4) implies $\widetilde{\varphi}(x+iy) \in g^{-1}U'_\lambda V' \cdot r'_p$ for all $y > \beta$ and $|x| < 1$. It is easy to see that this, in turn, yields, $\varphi(t) \in ((g^{-1}U')_{\lambda_0\lambda}V'\cdot r'_p)^\sim$ for $0 < |t| < e^{-2\pi\beta}$, where $\lambda_0 := \min\{e^\alpha(g^{-1}) \,|\, \alpha \in \Phi \text{ with } \alpha(Y) > 0\}$. Since $((g^{-1}U')_{\lambda_0\lambda}V'\cdot r'_p)^\sim \cup (g^{-1}U')\cdot b'_p$ is a Satake neighborhood of $g^{-1}\xi' = r(\infty)\cdot b_p$ in $D/C(W)_\mathbf{Z} \cup \mathcal{N}(W,p)$, which can be taken arbitrarily small, we get (5.3).

Now we shall prove (5.4). Set $g = r_0 t_0 m_0$, $r_0 \in R$, $t_0 \in T$ and $m_0 \in M$. Then, from (5.2.1), $R \lhd P$ and $M \subset K$, we see

$$g\widetilde{\varphi}(x+iy) = r'(x,y)t(x,y)k'(x,y)\cdot r'_p, \quad \text{where}$$
$$r'(x,y) := gr(x,y)g^{-1}r_0(t(x,y)m'(x,y))r_0^{-1}(t(x,y)m'(x,y))^{-1} \in R,$$
$$k'(x,y) := m'(x,y)gk(x,y)g^{-1} \in K',$$
$$m'(x,y) := m_0 m(x,y) m_0^{-1} \in M.$$

It follows from (5.2.2) that, as $y \to +\infty$, the following converge uniformly in x :

$$m'(x,y) \to 1, \quad r'(x,y) \to g\exp(xN)r(\infty)g^{-1}, \quad k'(x,y) \to 1.$$

Hence there exists $\beta > 0$ such that $r'(x,y)t(x,y) \in U'_\lambda$ and $k'(x,y) \in V$ for all $y > \beta$ and $|x| < 1$. (5.4) is proved, and this completes the proof of (i).

In order to prove (ii), we take the lifting $\xi \in B(W,p)$ of $\overline{\xi}$ with $W \in \mathcal{W}$ (see (4.17)). Then, by Proposition(3.13.ii), there exists a nilpotent orbit $(N,\widetilde{F})$ such that $\pi(\widetilde{F}) = \xi$, where N is the positive generator of $C(W)_{\mathbf{Z}}$ and $\widetilde{F} \in \mathcal{N}(W,p)$. Then for some $\beta > 0$, $\nu : \{z \in \mathbf{C} \mid \mathrm{Im}\, z > \beta\} \to \mathcal{N}(W,p) \subset \widetilde{D}_{W,p}$, $z \mapsto \exp(zN)\cdot\widetilde{F}$, is a holomorphic map with horizontal liftings and, by (4.9), $\nu(z) \to \xi$ as Im $z \to +\infty$. Hence $\varphi(t) :=$ (projection) $\circ$ $\nu((1/2\pi i)\log t + i\beta) \in D/\Gamma$ is the desired period map. □

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