代数幾何学シンポジウム記録
1994年度 pp.148-163

A GEOMETRIC INTERPRETATION OF THE SPACE OF CONFORMAL BLOCKS IN ABELIAN CONFORMAL FIELD THEORY

YUJI SHIMIZU

Department of Mathematics
Faculty of Sciences
Kyoto University

§. Introduction

I will report on an interpretation of the space of conformal blocks with abelian gauge symmetry as a localization on an analog of Jacobian variety in supergeometry, cf.[S].

These space of conformal blocks are introduced by Tsuchiya and Ueno [U]. They established the factorization property and hence the Kirchhoff rule (or the Verlinde formula).

My aim is to realize the gauge condition they used as the process of taking coinvariants on some space, which turns out to be an analog of Jacobian variety in supergeometry introduced by Skornyakov for a certain supercurve associated to a(n ordinary) curve and a theta characteristic.

The contents of this note is an enlarged version of the oral report. So I start with a few words about conformal field theory. Then I recall the idea of localization of representations on a "flag manifold" in some situations. Next I recall conformal field theory with abelian gauge symmetry, especially the definition of conformal blocks. After a discussion of supergeometry and an analog of Jacobian variety introduced by Skornyakov for supercurves, the main result is stated.

§1. Conformal field theory

Conformal field theory is a 2-dimensional quantum field theory. It is well-studied in connection with superstring theory and integrable lattice models. In relation with algebraic geometry, one of the interests is to study the space of conformal blocks, i.e., correlators of primary fields. In WZNW model, it is relevant to the space of generalized theta functions.

A model of conformal field theory is fixed by a family of representations of a (chosen) chiral algebra. (We skipped the explanation of chirality.) Mathematically, a chiral algebra might be best-explained by a vertex operator (super)algebra. We don't go into the detail of this topic here. As an example of chiral algebras, we consider the following three Lie algebras.

Virasoro algebra : $Vir = \mathbb{C}((t))\frac{d}{dt} \oplus \mathbb{C} \cdot c.$

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

Here c is a central element of Vir. The bracket is defined as

$$[f(t)\frac{d}{dt}, g(t)\frac{d}{dt}] = \{f'(t)g(t) - f(t)g'(t)\}\frac{d}{dt} + \mathrm{Res}_{z=0}(f'''(t)g(t)dt) \cdot c$$

Minimal series representations [BPZ] including unitary ones are well-studied.

(untwisted) **affine Lie algebra** : $\widehat{\mathfrak{G}} = \mathfrak{G} \otimes \mathbb{C}((t)) \oplus \mathbb{C} \cdot K$.

Here $\mathfrak{G}$ is a (finite-dimensional) simple Lie algebra and K is the central element of $\widehat{\mathfrak{G}}$. The bracket is defined as :

$$[X \otimes f(t), Y \otimes g(t)] = [X, Y] \otimes f(t)g(t) + (X|Y)\mathrm{Res}_{z=0}(f'(t)g(t)dt) \cdot K$$

where $(|)$ is a Cartan-Killing form on $\mathfrak{G}$ (normalized as $(\theta|\theta) = 2$, θ the highest root).

In the Wess-Zumino-Novikov-Witten model, one treats the integrable highest weight representations for a fixed positive integer ℓ ("level"). The space of conformal blocks (for the case no currents inserted) on a smooth projective curve C over $\mathbb{C}$ (with a reference point P) is defined to be $L(0)_{\mathfrak{G}(C-P)}$, the space of coinvariants of the basic representation of level ℓ $L(0)$ by a (kind of parabolic) subalgebra

$$\mathfrak{G}(C - P) = \mathfrak{G} \otimes H^0(C - P, \mathcal{O})$$

of $\widehat{\mathfrak{G}}$. Precisely speaking, one needs a formal local parameter at P to consider $\mathfrak{G}(C - P)$ as a subalgebra of $\widehat{\mathfrak{G}}$.

By several people (Faltings, Beauville-Laszlo,...), it is shown that the space of conformal blocks is isomorphic to the space of generalized theta functions $H^0(\mathcal{M}_G, \mathcal{O}(\Theta)^{\otimes \ell})$ (the Verlinde isomorphism).

affinized abelian Lie algebra : $\widehat{\mathfrak{G}}$ for the one-dimensional Lie algebra $\mathfrak{G} = \mathbb{C}$.

This is almost a Heisenberg algebra. Let us denote this algebra by $Heis$. It has a unique irreducible representation $\mathcal{F}$ (the Fock representation). Then one may define the space of conformal blocks for $Heis$ as the space $\mathcal{F}_{\mathfrak{G}(C-P)}$ as in the case of affine Lie algebra. Unfortunately, one doesn't have a finite dimenasional vector space in this way.

We will return to the right definition in §3.

§2. Method of localization

Localization of representations is a standard procedure in a geometric theory of representations of reductive Lie algebras. It is utilized by Beilinson-Bernstein and Brylinski-Kashiwara in their solution to the Kazhdan-Lusztig conjecture. This idea has been used in conformal field theory, e.g. in [BMS],[KNTY],[BS],[TUY]. We recall this method in the original setting, the Virasoro case and the affine case.

2.1 Beilinson-Bernstein theory [B]

Let G be a semi-simple algebraic group over $\mathbb{C}$, B its Borel subgroup. Then the totality of Borel subgroups in G forms the flag manifold X and is isomorphic to G/B. X can be defined equivalently as the totality of Borel subalgebras in $LieG$. Because of the left G-action on X, there is a Lie algebra homomophism :

$$\mathfrak{G} \to T_{X,x} \quad (x \in X).$$

This map extends to an algebra homomorphism

$$U(\mathfrak{G}) \to \mathcal{D}_{X,x},$$

where $\mathcal{D}_X$ denotes the sheaf of rings of linear differential operators on X with holomorphic coeffients and $\mathcal{D}_{X,x}$ its stalk at x, or to

$$U(\mathfrak{G}) \to \Gamma(X, \mathcal{D}_X)$$

The kernel can be described explicitly.

We can change the ring of scalars from $U(\mathfrak{G})$ to $\mathcal{D}_X$ by this homomorphism and get a functor ("localization functor")

$$U(\mathfrak{G})\text{-}mod_{triv} \longrightarrow \mathcal{D}_X\text{-}mod$$

from the category of $U(\mathfrak{G})$-modules with trivial infinitesimal character to the category of $\mathcal{D}_X$-modules. We have also a variant for $U(\mathfrak{G})$-modules with infinitesimal character λ and $\mathcal{D}_X^\lambda$-modules where $\mathcal{D}_X^\lambda$ denotes the sheaf of linear differential operators acting on the local sections of an invertible sheaf $\mathcal{O}(\lambda)$ on X determined by weight λ.

For generic λ, the above functor gives an equivalence of categories between finitely generated $U(\mathfrak{G})$-modules and coherent $\mathcal{D}_X^\lambda$-modules. (One can be more precise about the condition on λ.)

2.2 The case of Virasoro algebra

The role of flag manifold for the Virasoro Lie algebra Vir is played by the moduli space $X = \mathcal{M}_{g,1}^{(\infty)}$ of 1-pointed algebraic curves of genus g with a formal local parameter at the marked point.

Its ($\mathbb{C}$-valued) point $x = (C, Q, t)$ consists of a smooth projective curve C (over $\mathbb{C}$), a (closed) point $Q \in C$, and a formal local parameter at Q,i.e., a $\mathbb{C}$-algebra isomorphism $t : \widehat{\mathcal{O}}_{C,Q} \simeq \mathbb{C}[[z]]$ (z is an indeterminate). One can also consider N-point version $\mathcal{M}_{g,N}^{(\infty)}$.

One has a surjective homomorphism of Lie algebra :

$$Vir/\{c=0\} = \mathbb{C}((t))\frac{d}{dt} \to T_{X,x}$$

through a calculation of $H^1(C, T_C(-mQ))$, cf.[KNTY]. This homomorphism lifts to the following :

$$Vir \to D_{X,x}^{d,\leq 1}$$

or equivalently to

$$U(Vir) \to \Gamma(X, D_X^d).$$

Here d denotes the determinant line bundle $\det R\pi_*\mathcal{O}$ for the universal curve $\pi : \mathcal{C} \to X$ and the sheaf $\mathcal{O}_\mathcal{C}$. D_X^d denotes the sheaf of differential operators acting on the local section of the invertible sheaf d and $D_X^{d,\leq 1}$ the part of operators of order ≤ 1.

Manin was led to consider the moduli space of curves $\mathcal{M}_g$ as (a quotient space of) flag manifold for the Virasoro algebra by his explanation of the critical dimension in string theory. Then the above space was found independently by Beilinson and Kontsevich, cf.[BS]. Beilinson and Feigin consider the localization of minimal series representations of Vir, cf.[BFM].

2.3 The case of affine Lie algebra

The role of flag manifold for an affine Lie algebra $\widehat{\mathfrak{G}}$ is played by the moduli space $X = \mathcal{M}_G^{(\infty)}$ of principal G-bundles on a fixed curve C rigidified at the given points Q_i with respect to formal local parameter t_i $(1 \leq i \leq N)$. Here G denotes the adjoint group for $\mathfrak{G}$. X can be understood either as an algebraic stack or as a coarse moduli scheme of stable bundles (an open part of the former).

One has a surjective homomorphism of Lie algebra :

$$\widehat{\mathfrak{G}}/\{K = 0\} = \mathfrak{G} \otimes \oplus_i \mathbb{C}((t_i)) \to T_{X,x}$$

This homomorphism lifts to the following :

$$\widehat{\mathfrak{G}} \to D_{X,x}^{d,\leq 1},$$

or equivalently to

$$U(\widehat{\mathfrak{G}}) \to \Gamma(X, D_X^d).$$

In this situation d denotes the determinant line bundle $\det R\pi_*\mathcal{P}$ for the universal curve $\pi : \mathcal{C} \to X$ and the universal bundle $\mathcal{P}$ associated to the adjoint representation of $\mathfrak{G}$. The meaning of D_X^d and $D_X^{d,\leq 1}$ are the same as in the case of Vir.

One can obtain the projective connection of [TUY] in this framework, cf.[BK].

§3. Abelian conformal field theory

Conformal field theory with gauge symmetry of $U(1)$-currents are studied by many people, e.g.[IMO],[KNTY],[ACKP],[KSU1,2].

Here we follow Ueno [U] for the N-point version of [KNTY].

3.1 Fock space

We want to look the Fock space representation more closely, cf.§1.

Let us name $a(n) = t^n \in Heis$ for $n \in \mathbb{Z}$. Then one has the commutation relation :

$$[a(m), a(n)] = m\delta_{m+n,0}.$$

Then the Fock space representation $\mathcal{F}(p)$ is the module mono-genereated by the (highest weight) vector $|p\rangle$ $(p \in \mathbb{C})$ over $Heis$ with only the following relation :

$$\begin{aligned} a(n)|p\rangle &= 0 \quad n > 0 \\ a(0)|p\rangle &= p|p\rangle \\ K|p\rangle &= |p\rangle \end{aligned}$$

Note the Heisenberg commutation relation :

$$[a(n), a(-n)] = n.$$

Hence the notation $Heis$.

Since $\mathcal{F}(p)$ is freely generated over $Heis_- := \oplus_{n\leq 0}\mathbb{C}a(n)$, $\mathcal{F}(p)$ is isomorphic to the polynomial ring $\mathbb{C}[a(-1), a(-2), \cdots]$ which we realize as is well-known :

$$\mathcal{F}(p) = \mathbb{C}[t_1, t_2, \cdots]e^{pt_0}$$

with $|p\rangle = e^{pt_0}$.

The action is given by

$$\begin{aligned} a(n) =& \frac{\partial}{\partial t_n} \qquad (n \geq 0) \\ a(-n) =& nt_n \qquad (n \geq 1) \\ K =& 1 \end{aligned}$$

Put

$$a(z) = \sum_{n\in\mathbb{Z}} a(n)z^{-n-1}.$$

This is a formal expression with an indeterminate z, but it has a meaning of linear operator acting on $\mathcal{F}(p)$ with an indeterminate z of $z \in \mathbb{C}$. It is usually called the current operator.

3.2 Free fermion

The infinite direct sum

$$\mathcal{F} = \oplus_{p\in\mathbb{Z}}\mathcal{F}(p)$$

can be understood in terms of free fermion, i.e., a Clifford algebra $Clif$. It is an associative algebra with generators

$$\psi_\mu, \psi_\mu^\dagger \quad (\mu \in \mathbb{Z} + \frac{1}{2})$$

satisfying the following defining relations :

$$\begin{aligned} [\psi_\mu, \psi_\nu]_+ &= [\psi_\mu^\dagger, \psi_\nu^\dagger]_+ = 0 \\ [\psi_\mu, \psi_\nu^\dagger]_+ &= \delta_{\mu,-\nu} \end{aligned}$$

In order to explain the action of $Clif$ on $\mathcal{F}$, let us introduce the vertex operators :

$$V_{-1}(z) = \sum_{\mu\in\mathbb{Z}+\frac{1}{2}} \psi_\mu z^{-\mu-\frac{1}{2}}, V_{+1}(z) = \sum_{\mu\in\mathbb{Z}+\frac{1}{2}} \psi_\mu^\dagger z^{-\mu-\frac{1}{2}}$$

Then $Clif$ acts on $\mathcal{F}$ via the following relation :

$$V_k(z) = exp\{k\sum_{n\geq 1} \frac{a(-n)}{n}z^n\}e^{kt_0}e^{ka(0)\log z}exp\{-k\sum_{n\geq 1}\frac{a(n)}{n}z^{-n}\}$$

for $k = \pm 1$. This can be neatly written down as

$$V_{\pm 1}(z) =: e^{\pm\phi(z)} :$$

where : : denotes the normal ordering (cf.[KNTY]) and $\phi(z)$ is given by :

$$\phi(z) = t_0 + a(0)\log z + \sum_{n\neq 0} \frac{a(n)}{n} z^{-n}$$

(so that $\frac{d}{dz}\phi(z) = a(z)$).

Conversely, one can express *Heis* in terms of *Clif* on the Fock space $\mathcal{F}$:

$$a(z) =: V_{+1}(z)V_{-1}(z) :$$

This is the so-called boson-fermion correspondence [DJKM].

Finally we need the dual space for the next paragraph. Put

$$\mathcal{F}^\dagger(p) = Hom_{\mathbb{C}}(\mathcal{F}(p), \mathbb{C})$$
$$\mathcal{F}^\dagger = \oplus_{p\in\mathbb{Z}} \mathcal{F}^\dagger(p)$$

Then we have a natural pairing

$$\mathcal{F}^\dagger \times \mathcal{F} \to \mathbb{C}; \qquad (\langle\psi|, |\phi\rangle) \mapsto \langle\psi|\phi\rangle.$$

3.3 Conformal blocks

Let us recall the definition of the space of conformal blocks for the Fock space $\mathcal{F}$ according to Ueno [U].

Let $\mathfrak{X} = (C; Q_1, \cdots, Q_n; z_1, \cdots, z_n)$ be an n-pointed smooth curve (over $\mathbb{C}$) of genus g with formal local parameters z_i at Q_i .

For each point Q_i, we attach the representation $\mathcal{F}$. So we consider the tensor product $\mathcal{F}^{\otimes N}$.

There is a natural pairing

$$\mathcal{F}^{\dagger\otimes N} \times \mathcal{F}^{\otimes N} \to \mathbb{C}$$

induced from the pairing between $\mathcal{F}^\dagger$ and $\mathcal{F}$.

Definition The space of conformal blocks $\mathcal{V}^\dagger(\mathfrak{X})$ is the subspace of $\mathcal{F}^{\dagger\otimes N}$ consisting of vectors $\langle\psi|$ satisfying the following conditions :

$$(1) \qquad \sum_{j=1}^{n} Res_{z_j=0}(\langle\psi|\rho_j(a(z_j))|\phi\rangle g(z_j)dz_j) = 0$$

for the Laurent expansion $g(z_j)$ of any $g \in H^0(C, \mathcal{O}_C(*\sum Q_j))$ and

$$(2) \qquad \sum_{j=1}^{n} Res_{z_j=0}(\langle\psi|\rho_j(V_{\pm 1}(z_j))|\phi\rangle h(z_j)dz_j) = 0$$

for any $h \in H^0(C, \omega_C^{\otimes\frac{1}{2}}(*\sum Q_j))$ and $h(z_j)$ is its Laurent expansion of h at Q_j.

Here $\rho_j(?)$ means that ? acts on the j-th factor in the tensor product $\mathcal{F}^{\otimes N}$. ω_C denotes the dualizing sheaf of C and we have chosen its square root $\omega_C^{\otimes\frac{1}{2}}$. To put it the other way, we have chosen an $N = 1$ superconformal structure on C, cf.§4.2.

We define the dual of the above space of conformal blocks to be

$$\mathcal{V}(\mathfrak{X}) = Hom_{\mathbb{C}}(\mathcal{V}^\dagger(\mathfrak{X}), \mathbb{C}).$$

So it can be identified with the quotient of $\mathcal{F}^{\dagger\otimes N}$ modulo the relation generated by $H^0(C, \mathcal{O}_C(*\sum Q_j))$ via $a(z)$ and $H^0(C, \omega_C^{\otimes\frac{1}{2}}(*\sum Q_j))$ via $V_{\pm 1}(z)$.

This suggests it is a kind of the space of coinvariants and our purpose is to find a context where it is so.

Remark 1) The above condition (2) means that $\{\langle\psi|\rho_j(V_{\pm 1}(z_j))|\phi\rangle(dz_j)^{1/2}\}$, $(j = 1, \cdots, N)$ are the Laurent expansions of an element of $H^0(C, \omega_C^{\otimes\frac{1}{2}}(*\sum Q_j))$ at Q_j with respect to the formal local parameter z_j.

Similarly for the condition (1).

The condition (1) is the usual gauge condition, cf.[KNTY,7.1,2)].

2) The conformal blocks are defined for a general (even) level M in [U]. This M corresponds to the level of theta functions through the following theorem.

3) One can adapt the above definition in the case C is assumed to be n-pointed stable as well.

The main theorem in [U] is the following :

Theorem [U,§1]. *For a stable N-pointed curve having at least one marked point on each irreducible component, one has*

$$dim_{\mathbb{C}}\mathcal{V}^\dagger(\mathfrak{X}) = 1$$

For a smooth curve C, it is argued that one has a canonical isomorphism

$$\mathcal{V}^\dagger(\mathfrak{X}) \simeq H^0(J(C), \mathcal{O}(\Theta))$$

where $J(C)$ denotes the Jacobian of the curve C. The theorem is stated for an even level M in [U] replacing $\mathcal{O}(\Theta)$ by $\mathcal{O}(M\Theta)$.

Remark The above theorem generalizes [KNTY,7.7], giving the precise dimension of the space of conformal blocks. This was possible due to the factorization property for the conformal blocks [U,2.5]. The formulation parallels the one in [TUY] and it should construct a projectively flat connection on the sheaf of conformal blocks on the moduli space of stable curves (at least over smooth curves), cf.[BK,BFM].

There was also an attempt to generalize [KNTY] and was partially successful because only the gauge condition by $U(1)$-current was considered there, cf.[SU].

§4. Digression on supergeometry

In order to interpret the condition (2) in the definition of the conformal blocks in the previous paragraph, we led to consider certain supersymmetry on a given curve.

Basic references for supergeometry are [M1,2,VMP,Va].

4.1 Superspace

A supercommutative ring A is a $\mathbb{Z}/2\mathbb{Z}$-graded (associative) ring

$$A = A_{\bar{0}} \oplus A_{\bar{1}}$$

whose supercommutator is always zero. The supercommutator is defined as

$$[a,b] = ab - (-1)^{\bar{a}\bar{b}}ba$$

for homogeneous elements $a \in A_{\bar{a}}, b \in A_{\bar{b}} (\bar{a}, \bar{b} = \bar{0}$ or $\bar{1})$.

A superscheme is a locally ringed space whose structure sheaf of rings is supercommutative, cf.[M1] for more details.

A smooth supercurve is a smooth superscheme (i.e. supermanifold) of dimension $1|N$. So it is a ringed space $X = (C, \mathcal{O}_X)$ with a sheaf of supercommutative ($\mathbb{Z}/2\mathbb{Z}$-graded) rings as structure sheaf and $C = X_{red} = X^{(0)}$ is a smooth (ordinary) curve. Here we put

$$\begin{aligned} X^{(i)} &= (C, \mathcal{O}_X/\mathcal{N}^{i+1}) \\ \mathcal{N} &= \{\text{nilpotents}\} (= \mathcal{O}_{X,1} + \mathcal{O}_{X,1}^2) \end{aligned}$$

for $i \geq 0$. "Smooth of dimension $1|N$" means that $\mathcal{O}_X$ is locally of the form $\Lambda^\cdot(\mathcal{E})$ where $\mathcal{E}$ is a locally free $\mathcal{O}_C$-module of rank N. In supergeometry, one might prefer the notation $Sym^\cdot(\Pi\mathcal{E})$ for $\mathcal{O}_X$, where Π is the parity changing functor.

One recovers the ordinary (i.e. even) curve in the case $N = 0$. cf. [M2, Ch.2].

4.2 Superconformal curves

We digress on superconformal curves which are natural generalization of Riemann surfaces in supergeometry and seem to be the most important supercurves.

Let $\pi\colon X \to S$ be a smooth morphism of superschemes of relative dimension $1|N$, namely a family of smooth supercurves. A superconformal structure or $SUSY_N$-structure on π is (a choice of) a locally free and locally direct $\mathcal{O}_X$-submodule of rank $0|N$ $\mathcal{T}^1$ of the relative tangent sheaf $\mathcal{T}_{X/S}$ which locally has an isotropic direct $\mathcal{O}_X$-submodule of maximal possible rank for $N = 2k$ or $2k+1$ with respect to the Frobenius form

$$\Lambda^2\mathcal{T}^1 \to \mathcal{T}^0 := \mathcal{T}_{X/S}/\mathcal{T}^1 \quad ; \quad t_1 \wedge t_2 \mapsto [t_1, t_2] \mod \mathcal{T}^1.$$

Supercurves with $N = 1,2$ superconformal structure (or $N = 1,2$ superconformal curves) are also called $N = 1,2$ super Riemann surfaces or $SUSY_N$-curves in the literature.

Let us specify the above definition in the case $S = pt$ and $N = 1,2$. So let X be a smooth supercurve of dimension $(1|N)$.

Case $N = 1$: A ($N = 1$) *superconformal structure* on X or a $SUSY_1$*-structure* is (a choice of) a locally free $\mathcal{O}_X$-submodule $\mathcal{T}^1$ of rank $0|1$ of the tangent sheaf $\mathcal{T}_X$ which is a locally direct summand and gives rise to an isomorphism

$$(\mathcal{T}^1)^{\otimes 2} \simeq \mathcal{T}_X/\mathcal{T}^1 =: \mathcal{T}^0 \quad ; \quad t_1 \otimes t_2 \mapsto [t_1, t_2] \quad \mod \mathcal{T}^1.$$

Case $N = 2$: A ($N = 2$) *superconformal structure* on X or a $SUSY_2$*-structure* is (a choice of) a pair of locally free $\mathcal{O}_X$-submodules $\mathcal{T}', \mathcal{T}''$ of rank $0|1$ of the tangent sheaf $\mathcal{T}_X$ such that i) $\mathcal{T}' \oplus \mathcal{T}''$ is direct in $\mathcal{T}_X$, ii) $[\mathcal{T}', \mathcal{T}'] \subset \mathcal{T}', [\mathcal{T}'', \mathcal{T}''] \subset \mathcal{T}''$, and that iii) one has an isomorphism

$$\mathcal{T}' \otimes \mathcal{T}'' \simeq \mathcal{T}_X/(\mathcal{T}' \oplus \mathcal{T}'') =: \mathcal{T}^0 \quad : \quad t' \otimes t'' \mapsto [t', t''] \quad (\mathrm{mod}\ \mathcal{T}' \oplus \mathcal{T}'').$$

Here $\mathcal{T}' \oplus \mathcal{T}'' = \mathcal{T}^1$.

Over a purely even base S, one has a simple description of superconformal curves. So let $\pi_0 : X_0 \to S$ be an ordinary S-curve. Then we have

Proposition [M2, 2.7]. *There is a one-to-one correspondence between*
Case $N = 1$:
a) $\{N = 1$ *superconformal* S*-curve* $\pi : X \to S$
with $X_{red} = X_{0,red}$ *and* $\mathcal{O}_{X,0} = \mathcal{O}_{X_0}$ $\}$ *up to isomorphism identical on* X_0
b) $\{(I; \alpha) | I \in Pic(X_0/S), \alpha : I \otimes I \simeq \Omega^1_{X_0/S}\}$ *up to isomorphism of* I *transforming* α

Case $N = 2$:
a) $\{$*oriented* $N = 2$ *superconformal* S*-curve* $\pi : X \to S$
with $X_{red} = X_{0,red}$ *and* $\mathcal{O}_{X,0} = \mathcal{O}_{X_0^{(1)}}$ $\}$
b) $\{(I', I''; \beta) | I', I'' \in Pic(X_0/S), \beta : I' \otimes I'' \simeq \Omega^1_{X_0/S}\}$

Here "oriented" means that the sheaves I', I'' are globally distinguishable, cf.[M2,2.6]. $X_0^{(1)}$ is the first infinitesimal neighbourhood of X_0 in $X_0 \times_S X_0$. A pair $(I', I''; \beta)$ is called as a relative theta pair.

The above correspondences are given as follows. In the case $N = 1$,

$$X_{red} = X_{0,red}, \quad \mathcal{O}_{X,0} = \mathcal{O}_{X_0}, \quad \mathcal{O}_{X,1} = \Pi I$$

and

$$\mathcal{T}^1 = \mathcal{O}_X \otimes_{\mathcal{O}_{X_{red}}} I^{\otimes -1}.$$

α gives rise to the Frobenius form. The pair $(I; \alpha)$ is a theta characteristic of the family.

In the case $N = 2$,

$$X_{red} = X_{0,red}, \quad \mathcal{O}_{X,0} = \mathcal{O}_{X_0} \oplus I' \otimes I'', \quad \mathcal{O}_{X,1} = \Pi(I' \oplus I'')$$

and

$$\mathcal{T}' = \mathcal{O}_X \otimes_{\mathcal{O}_{X_{red}}} I'^{\otimes -1}, \quad \mathcal{T}'' = \mathcal{O}_X \otimes_{\mathcal{O}_{X_{red}}} I''^{\otimes -1}$$

β gives rise to the Frobenius form.

$N = 1$ superconformal curves are studied by many authors. In particular, their moduli is studied by LeBrun-Rothstein [LBR] and Deligne [D].

4.3 Picard groups in supergeometry

There are two natural generalization of the Picard groups in supergeometry. Let X be a superscheme.

Definition 1) Let $Pic_0(X)$ denote the set of isomorphism classes of locally free $\mathcal{O}_X$-modules of rank $1|0$.

This set has a group structure as usual and is naturally isomorphic to the group $H^1(X, \mathcal{O}^*_{X,0})$.

2) Let $Pic_\Pi(X)$ denote the set of isomorphism classes of locally free $\mathcal{O}_X$-modules of rank $1|1$ with Π-symmetry. Here a Π-*symmetry* on a locally free $\mathcal{O}_X$-module $\mathcal{E}$ of rank $1|1$ is an odd endomorphism $p : \mathcal{E} \to \mathcal{E}$ with $p^2 = -id$. An isomorphism of $\mathcal{E}$ is understood to preserve the Π-symmetry.

This is merely a pointed set and is naturally isomorphic to the set $H^1(X, \mathcal{O}^*_X)$.

We call such a pair $(\mathcal{E}, p)$ a Π-invertible sheaf or Π-invertible $\mathcal{O}_X$-module.

It is instructive to know the following exact sequence

$$Pic_0(X) \to Pic_\Pi(X) \to H^1(X, \mathcal{O}_{X,1}) \to H^2(X, \mathcal{O}^*_{X,0}),$$

where the first map associates $\mathcal{L} \oplus \Pi\mathcal{L}$ to $\mathcal{L} \in Pic_0(X)$.

Skornyakov studied the Π-Picard group $Pic_\Pi(X)$ and gave basic properties [VMP, §4].

Here we should briefly recall sheaves on a superscheme X.

First all (supercommutative) rings and modules are $\mathbb{Z}/2\mathbb{Z}$-graded. $\mathcal{O}_{X,1}$ is a coherent $\mathcal{O}_{X,0}$-module and $\mathcal{O}_X$ is a coherent ring.

A left $\mathcal{O}_X$-module has a natural right $\mathcal{O}_X$-module structure consistent with the left one, cf.[M1,Ch.3.§1,4]. Thus one can form tensor products of $\mathcal{O}_X$-modules freely.

The group of homomorphisms is $\mathbb{Z}/2\mathbb{Z}$-graded :

$$Hom_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) = Hom_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})_0 \oplus Hom_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})_1$$

The first (resp. second) factor consists of even (resp. odd) homomorphisms. An automorphism is an even endomorphism which is isomorphic.

A locally free $\mathcal{O}_X$-module of rank $p|q$ is a coherent $\mathcal{O}_X$-module which is locally isomorphic to $\mathcal{O}_X^{p|q} = \mathcal{O}_X^p \oplus \Pi\mathcal{O}_X^q$. Here Π is the parity changing functor : $(\Pi(\mathcal{E}))_i = \mathcal{E}_{i+1}, i \in \mathbb{Z}/2\mathbb{Z}$.

The set of locally free $\mathcal{O}_X$-modules of rank $p|q$ up to isomorphism is in bijection with the set $H^1(X, GL(p|q; \mathcal{O}_X))$ as usual. Here $GL(p|q; \mathcal{O}_X)$ denotes the sheaf of germs of (even) automorphisms of such an $\mathcal{O}_X$-module.

Let us call a $\mathcal{O}_X$-module of rank $1|1$ as Π-invertible $\mathcal{O}_X$-module. Then one has the following :

Lemma. *For a* Π*-invertible* $\mathcal{O}_X$*-module* $(\mathcal{E}, p)$*, one has*

$$\mathcal{E}nd_\Pi(\mathcal{E}) \simeq \mathcal{O}_X \oplus \Pi\mathcal{O}_X$$
$$\mathcal{A}ut_\Pi(\mathcal{E}) \simeq \mathcal{O}_X^*.$$

Here End_π is a subgroup of $GL(1|1)$ consisting of the morphisms preserving the Π-symmetry. Similarly for Aut_Π.

4.4 **Π-Picard schemes : a superanalog of Jacobians**

One can introduce a structure of superschemes on $Pic_0(X)$, $Pic_\Pi(X)$ for a proper smooth supercurve X of dimension $(1|N)(N = 1, 2)$.

Consider the following functors from the category of superschemes $Ssch$ to Set.

$$Pic_0 \ : S \mapsto Pic_0(X \times S)/p_S^* Pic_0(S)$$
$$Pic_\Pi \ : S \mapsto H^1(S, (p_S)_*(\mathcal{O}_{X\times S}^*))$$

Here $p_S : X \times S \to S$ is the second projection.

Then the above functors are representable by some superschemes denoted as $Pic_{0,X}$, $Pic_{\Pi,X}$, cf.[S,§3.1]. The main technical tool for the construction is the obstruction theory for extending sheaves to infinitesimal neighbourhoods (in the odd direction) (the so-called "component analysis").

In the case $N = 1$, one has

$$Pic_{0,X} = Pic_C, Pic_{\Pi,X} = Pic_C \times H^1(C, \mathcal{N}).$$

where $\mathcal{N}$ is the ideal of nilpotents in $\mathcal{O}_X$ and $C = X_{red}$ is the underlying (ordinary) curve.

In the case $N = 2$, one has

$$Pic_{0,X} = Pic_C \times H^1(C, \Lambda^2(\mathcal{F})), \quad Pic_{\Pi,X} = Pic_C \times H^1(C, 1 + \mathcal{N})$$

where $\mathcal{F} = \mathcal{O}_{X,1} \simeq \mathcal{N}/\mathcal{N}^2$ is locally free of rank 2 over $\mathcal{O}_C$.

We need the following supervariety in this situation :

$$Pic_{\Pi,X^{(1)}} = Pic_C \times H^1(X, \mathcal{F})$$

where $X^{(1)}$ sits in the infintesimal thickening (Note $\mathcal{N}^3 = 0$) :

$$X^{(0)} = X_{red} = C \subset X^{(1)} = (C, \mathcal{O}_C \oplus \mathcal{F}) \subset X^{(2)} = X$$

§5. Results

We give a natural geometric framework for conformal field theory with $U(1)$ gauge symmetry using the Π-Picard scheme §4.4 and its dressed version of certain $N = 2$ supercurves.

5.1 As I mentioned after the definition of the conformal blocks in §3.3, the problem in the spirit of the method of localization is to interpret the condition that the conformal blocks satisfy as (the dual of) coinvariants with respect to some infinitesimally homogeneous space.

We know that $H^0(C, \mathcal{O}_C(*\sum Q_j))$ can be realized on the dressed space of the Jacobian of the curve C. Since the remaining $H^0(C, \omega_C^{\otimes\frac{1}{2}}(*\sum Q_j))$ is a subspace of $Clif$ through $V_{\pm 1}(z)$, the basic strategy is to interpret it as fermionic symmetry related to C and $\omega_C^{\frac{1}{2}} = \omega_C^{\otimes\frac{1}{2}}$.

Let $\omega_C^{1/2}$ be a theta characteristic on C. It amounts to choose a $N=1$ superconformal structure overlying C §4.1. We consider the supercurve of odd dimension $N=2$

$$X = (C, S{\cdot}\Pi(\omega_C^{1/2} \oplus \omega_C^{1/2}))$$

as well as

$$X^{(1)} = (C, \mathcal{O}_C \oplus \Pi(\omega_C^{1/2} \oplus \omega_C^{1/2})).$$

See §4 for the notation.

The solution of the above problem is to consider the superscheme $Pic_{\Pi, X^{(1)}}$ whose even part is just the Jacobian Pic_C.

To go further, we have to introduce the dressed version of these (super)schemes.

5.2 Dressed Π-Picard schemes

To consider the dressed version of some moduli space is just to replace the (infinitesimal) automorphism of the object classified by its affinized version, cf.§1. We refer to [SU] for the dressed version of the Jacobian.

We briefly describe the dressed version of $Pic_{\Pi, X^{(1)}}$. For teh details, see [S].

Let $X = (C, \mathcal{O}_X)$ be the proper smooth supercurve of odd dimension $N = 2$ in \$5.1 associatd to $(C, \omega_C^{\otimes\frac{1}{2}})$.

The dressed Π-Picard group classifies dressed Π-invertible sheaves, i.e. Π-invertible sheaves with trivialization at given points.

Let $Q \in C$ be a (closed) point and $Z = (z, \theta_1, \theta_2)$ be formal local coordinates at Q, i.e. z is a formal local coordinate at Q and θ_1, θ_2 are local generating sections of $\mathcal{P} := \Pi(\omega_C^{1/2} \oplus \omega_C^{1/2})$:

$$\widehat{\mathcal{O}}_{C,Q} \simeq \mathbb{C}[[z]], \quad \widehat{\mathcal{P}}_Q \simeq \mathbb{C}[[z]]\theta_1 \oplus \mathbb{C}[[z]]\theta_2$$

Thus we have

$$\widehat{\mathcal{O}}_{X,Q} \simeq \Lambda(\mathbb{C}[[z]]\theta_1 \oplus \mathbb{C}[[z]]\theta_2)$$

Let $\mathfrak{m}_Q$ denote the maximal ideal of the (supercommutative) local ring $\mathcal{O}_{X,Q}$, which is generated by the maximal ideal of $\mathcal{O}_{C,Q}$ and odd generators θ_1, θ_2.

Definition Let k be an integer ≥ 1.

A trivialization of k-th order at a point $Q \in C$ of a Π-invertible sheaf $\mathcal{L}$ is an $\mathcal{O}_{X,Q}/\mathfrak{m}_Q^{k+1}$-isomorphism

$$\alpha\colon \mathcal{L}/\mathfrak{m}_Q^{k+1}\mathcal{L} \simeq \mathcal{O}_{X,Q}/\mathfrak{m}_Q^{k+1} \oplus \Pi(\mathcal{O}_{X,Q}/\mathfrak{m}_Q^{k+1})$$

which transforms the Π-symmetry on the left to the obvious one on the right.

Considering the projective limit of such an isomorphism, we define the formal trivialization of a Π-invertible sheaf.

Of course, we can consider these notions on $X^{(1)}$.

Let $\mathfrak{X}^{(1)} = (X^{(1)}; Q_1, \cdots, Q_N; Z_1, \cdots, Z_N)$ be a datum consisting of the supercurve $X^{(1)}$, its points and formal local coordinates at these points.

Denote by $Pic_{\Pi}^{(k)}(\mathfrak{X}^{(1)})$ (resp. $Pic_{\Pi}^{(\infty)}(\mathfrak{X}^{(1)})$) the set of isomorphism classes of all Π-invertible sheaves with trivialization of k-th order (resp. formal trivialization) at given points.

Then $Pic_{\Pi}^{(k)}(\mathfrak{X}^{(1)})$ (resp. $Pic_{\Pi}^{(\infty)}(\mathfrak{X}^{(1)})$) has a structure of superscheme which is a $\Pi_i\mathbb{G}_m^{1|1}(\mathcal{O}_{X^{(1)},Q_i}/\mathfrak{m}_{Q_i}^{k+1})$-torsor (resp. $\Pi_i\mathbb{G}_m^{1|1}(\widehat{\mathcal{O}}_{X^{(1)},Q_i})$-torsor) over $Pic_{\Pi,X^{(1)}}$. Let us denote these superschemes by $Pic_{\Pi,\mathfrak{X}}^{(k)}$, $Pic_{\Pi,\mathfrak{X}}^{(\infty)}$.

We can have similar objects on X.

The infinitesimal structure of the dressed Π-Picard schemes can be described as follows :

$$T_{(\mathcal{L},\alpha)}Pic_{\Pi,\mathfrak{X}^{(1)}}^{(\infty)} \simeq \mathbb{C}((z)) \otimes (\mathbb{C} \oplus \mathbb{C}\theta_1 \oplus \mathbb{C}\theta_2)/H^0(X^{(1)}, \mathcal{O}_{X^{(1)}}(*\sum_i Q_i))$$

where $(\mathcal{L}, \alpha)$ is a ($\mathbb{C}$-)point of $Pic_{\Pi,\mathfrak{X}^{(1)}}^{(\infty)}$. This is calculated by H^1 of the group of infinitesimal automorphisms of the object in question, namely, $\mathcal{E}nd_{\Pi}(\mathcal{L})(-(k+1)\sum_i Q_i)$ for k finite.

For $k = 0$, we have

$$T_{\mathcal{L}}Pic_{\Pi,X^{(1)}} \simeq H^1(C, \mathcal{O}_C) \oplus H^1(C, \mathcal{F})$$

This description shows that the dressed Π-Picard schemes are homogeneous spaces of the loop group $\Pi_i\mathbb{G}_m^{1|1}(\widehat{\mathcal{O}}_{X^{(1)},Q_i})$, infinitesimally.

In relation to the method of localization, the above homomorphism can be understood as a Lie superalgebra homomorphism

$$\Pi_{i=1}^N(Heis \oplus Clif_1) \to T_{(\mathcal{L},\alpha)}P_{\mathfrak{X}}^{\infty}$$

for $(\mathcal{L}, \alpha) \in Pic_{\Pi}^{(\infty)}(\mathfrak{X}^{(1)})$ Here we put $Pic_{\Pi,\mathfrak{X}^{(1)}}^{(\infty)} = P_{\mathfrak{X}}^{\infty}$ and denote the degree 1 part of $Clif$ by $Clif_1$:

$$Clif_1 = \oplus_{\mu \in \mathbf{Z}+\frac{1}{2}}\mathbb{C}\psi_\mu \oplus \oplus_{\mu \in \mathbf{Z}+\frac{1}{2}}\mathbb{C}\psi_\mu^\dagger.$$

The kernel of the above homomorphism is $H^0(C, \mathcal{O}_{X^{(1)}}(*\sum_i Q_i))$, which is a Lie subsuperalgebra of $\Pi_i(Heis \oplus Clif_1)$ via the formal trivialization. $H^0(C, \mathcal{O}_C(*\sum_i Q_i))$ (resp. $H^0(C, \omega^{1/2} \oplus \omega^{1/2}(*\sum_i Q_i))$ injects into $\Pi_i(Heis)$ (resp. $\Pi_i(Clif_1)$) through

$$\begin{aligned} g \mapsto & (Res_{z_i=0}(g(z_i)a(z_i)dz_i)_i \\ (h_1, h_2) \mapsto & (Res_{z_i=0}(h_j(z_i)V_{(-1)^j}(z_i)dz_i)_{i,j=1,2} \end{aligned}$$

Then we have the following :

Proposition. *The above homomorphism lifts to*

$$\mathcal{O}_{P^\infty_{\mathfrak{X}}} \otimes \Pi_{i=1}^N(Heis \oplus Clif_1) \to \mathcal{D}^{\leq 1}_{d(\mathcal{L}_{univ})}$$

where $d(\mathcal{L}_{univ})$ is the determinant line bundle pulled back to $P^\infty_{\mathfrak{X}}$. $\mathcal{D}^{\leq 1}_{d(\mathcal{L}_{univ})}$ is the degree ≤ 1 part of the ring of differential operators $\mathcal{D}_{d(\mathcal{L}_{univ})}$ acting on the sections of the invertible sheaf $d(\mathcal{L}_{univ})$.

The kernel of this homomorphism equals $\mathcal{O}_{P^\infty_{\mathfrak{X}}} \otimes H^0(C, \mathcal{O}_{X^{(1)}}(*\sum_i Q_i))$.

Recall that we have the universal (Poincaré) bundle $\mathcal{L}_{univ}$ on $C \times Pic_C$. Then we have the determinant line bundle $\det R\pi_*(\mathcal{L}_{univ}) = d(\mathcal{L}_{univ})$ on Pic_C where $\pi\colon C \times Pic_C \to Pic_C$ is the second projection.

The proof uses the method of [BS].

From this proposition, we obtain a ring homomorphism

$$\rho\colon \mathcal{O}_{P^\infty_{\mathfrak{X}}} \otimes (\otimes_i(U(Heis) \otimes Clif) \to \mathcal{D}_{d(\mathcal{L}_{univ})}$$

Given representations of $Heis \oplus Clif_1$ $M_i(i = 1, \cdots, N)$ with the same center, the localization on $Pic^{(\infty)}_{\Pi,\mathfrak{X}^{(1)}}$ of $\otimes_i M_i$ is defined to be the scalar extension by ρ

$$\Delta(\otimes_i M_i) = \mathcal{D}_{d(\mathcal{L}_{univ})} \otimes_\rho \otimes_i M_i.$$

Its fiber is just the coinvariants :

$$\Delta(\otimes_i M_i) \otimes \mathcal{O}_{P^\infty_{\mathfrak{X}}}/\mathfrak{m}_{(\mathcal{L},\alpha)} \simeq \otimes_i M_i / H^0(C, \mathcal{O}_{X^{(1)}}(*\sum_i Q_i)) \otimes_i M_i$$

where $\mathfrak{m}_{(\mathcal{L},\alpha)}$ is the ideal sheaf of $P^\infty_{\mathfrak{X}}$ at the point $(\mathcal{L}, \alpha)$.

5.3 We now apply the localization functor Δ to the Fock space representation $\mathcal{F}^{\otimes N}$. Recall that $\mathcal{F}$ becomes a representation of $Clif$ through vertex operators $V_{\pm 1}(z_j)$.

Then one of the main results in [S] is the following :

Theorem. *The space of conformal blocks $\mathcal{V}(\mathfrak{X})$ equals the fiber of the localization $\Delta(\mathcal{F}^{\dagger\otimes N})$ at any point of $Pic^{(\infty)}_{\Pi,\mathfrak{X}^{(1)}}$.*

Remark Penkov's theorem [P] on the equivalence of the category of $\mathcal{D}$-modules and that of $\mathcal{D}_{red}$-modules implies that it is enough to consider the restriction of the module $\Delta(\mathcal{F}^{\otimes N})$ to $(Pic^{(\infty)}_{\Pi,\mathfrak{X}^{(1)}})_{red} = Pic^{(\infty)}_{\{C;(Q_i);(z_i)\}}$. This last space is the dressed Picard scheme on the even part $\{C; (Q_i); (z_i)\}$ of $\mathfrak{X}^{(1)}$, cf.[SU].

We can relate the space of conformal blocks with the space of global sections of the determinant line bundle. Let us restrict $Pic_{\Pi,X^{(1)}}$ etc. over the component of degree $g-1$. In order to carry it out, let us descend the $\mathcal{D}_{d(\mathcal{L}_{univ})}$-module $\Delta(\mathcal{F}^{\otimes N})$ on $Pic^{(\infty)}_{\Pi,\mathfrak{X}^{(1)}}$ to Pic^{g-1}_C and then integrate on Pic^{g-1}_C.

Denote the natural projection by $r\colon Pic^{(\infty)}_{\Pi,\mathfrak{X}^{(1)}} \to Pic_{\Pi,X^{(1)}}$. Then $\Delta(\mathcal{F}^{\otimes N})$ descends to

$$\Delta_N = r_*(\Delta(\mathcal{F}^{\otimes N}))^{\Pi_i \mathbf{G}_m^{1|1}(\widehat{\mathcal{O}}_{X,Q_i})}|_{Pic_C^{g-1}}$$

This amounts to taking invariants with respect to the change of formal trivialization of $\mathcal{L}$. We also used the relation $(Pic_{\Pi,X^{(1)}})_{red} = Pic_C$.

It is known that the fibers of $d(\mathcal{L}_{univ})$ injects into the Fock space $\mathcal{F}^{\otimes N}$ through the semi-infinite exterior product, cf.[SU]. Then we have :

$$d(\mathcal{L}_{univ}) \hookrightarrow \Delta_N$$

As its dual we have :

$$d(\mathcal{L}_{univ})^{-1} \leftarrow \Delta_N^*$$

Taking the global sections, we obtain a surjection

$$H^0(Pic_C^{g-1}, d(\mathcal{L}_{univ})^{-1}) \leftarrow H^0(Pic_C^{g-1}, \Delta_N^*)$$

Remember that $d(\mathcal{L}_{univ})$ on Pic_C^{g-1} is nothing but the dual $\mathcal{O}(-\Theta)$ of the theta divisor, cf.[Sz]. We also note that Δ_N^* is the module whose fiber is the space of conformal blocks $\mathcal{V}^\dagger(\mathfrak{X})$ since taking invariants is cancelled by taking r_*.

Comparing the dimension, one obtains

Theorem. *We have a canonical isomorphism*

$$H^0(Pic_C^{g-1}, \mathcal{O}(\Theta)) \simeq \mathcal{V}^\dagger(\mathfrak{X})$$

Remark One can develop the story in the relative situation.

References

[ACKP] E.Arbarello, C.De Concini, V.Kac, C.Procesi, *Moduli spaces of curves and representation theory*, Commun. Math. Phys. **117** (1988), 1–36.

[B] A.Beilinson, *Localization of representations of reductive Lie algebras*, Proceedings of the ICM '83, Warsaw, pp. 699–710.

[BFM] A.Beilinson, B.Feigin, B.Mazur, *Introduction to algebraic field theory on curves*, A half of the first draft (1991).

[BK] A.Beilinson, D.Kazhdan, *Projective flat connections*, Preliminary draft (1990?).

[BS] A.Beilinson, V.Schechtman, *Determinant bundles and Virasoro algebras*, Commun. Math. Phys. **118** (1988), 651–701.

[DJKM] E.Date, M.Jimbo, M.Kashiwara, T.Miwa, *Transformation groups for soliton equations*, Proceedings of RIMS Symposium, Non-linear Integrable Systems - Classical Theory and Quantum Theory, World Scientific, Singapore, 1983, pp. 39–119.

[D] P.Deligne, *Letter to Yu.I.Manin*, Sept. 25 (1987).

[IMO] N.Ishibashi, Y.Matsuo, H.Ooguri, *Soliton equations and free fermions on Riemann surfaces*, Mod. Phys. Lett. **A2** (1987), 119–?.

[KNTY] N.Kawamoto, Y.Namikawa, A.Tsuchiya, Y.Yamada, *Geometric realization of conformal field theory on Riemann surfaces*, Commun. Math. Phys. **116** (1988), 247–308.

[KM] F.F.Knudsen, D.Mumford, *The projectivity of the moduli space of the stable curve I*, Math. Scand. **39** (1976), 19–55.

[KSU1] T.Katsura, Y.Shimizu, K.Ueno, *New bosonization and conformal field theory over* $\mathbb{Z}$, Commun. Math. Phys. **121** (1988), 603–622.

[KSU2] T.Katsura, Y.Shimizu, K.Ueno, *Formal groups and conformal field theory over* $\mathbb{Z}$, Advanced Studies in Pure Math. **19** (1989), 347–366.

[LBR] C.LeBrun, M.Rothstein, *Moduli of super Riemann surfaces*, Commun. Math. Phys. **117** (1988), 159–176.

[M1] Yu.I.Manin, *Gauge field theory and complex geometry*, Springer-Verlag, 1988.

[M2] Yu.I.Manin, *Topics in Noncommutative Geometry*, Princeton University Press, 1991.

[P] I.B.Penkov, *D-modules on supermanifolds*, Invent. Math. **71** (1983), 501–512.

[S] Y.Shimizu, *Abelian conformal field theory and* $N = 2$ *superconformal curves*, J. Math. Kyoto Univ. (1995) (to appear).

[SU] Y.Shimizu, K.Ueno, *Abelian conformal field theory and some D-modules on the moduli space of curves*, Manuscript (1991); *D-modules on the moduli space of curves associated with abelian CFT*, Proceedings of Algebraic Geometry Kinosaki Symposium 1991, 1992.

[Sz] L.Szpiro (ed.), Séminaire sur les pinceaux arithmétiques : la conjecture de Mordell, Astérisque, vol. 127, 1985.

[TUY] A.Tsuchiya, K.Ueno, Y.Yamada, *Conformal field theory on family of stable curves with gauge symmetries*, Adv. Stud. Pure Math. **19** (1989), 459–566.

[U] K.Ueno, *On abelian conformal field theory*, Proceedings of Algebraic Geometry Kinosaki Symposium 1992, 1993, pp. 55–62; *On conformal field theory*, to appear in Proc. of Durham Conference 1993.

[Va] A.Yu.Vaintrob, *Deformation of complex superspaces and coherent sheaves on them*, J. of Soviet Math. **51** (1990), 2140–2188.

[VMP] A.A.Voronov, Yu.I.Manin, I.B.Penkov, *Elements of supergeometry*, J. of Soviet Math. **51** (1990), 2069–2083.

[W] E.Witten, *Quantum field theory, Grassmannians, and algebraic curves*, Commun. Math. Phys. **113** (1988), 529–600.